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SOME GROUPS WITH COMPUTABLE CHERMAK–DELGADO LATTICES

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Dedicated to John Cossey, in celebration of his 70th birthday

Abstract

Let G be a finite group and let $H \leq G$. We refer to $|H||C_G(H)|$ as the *Chermak–Delgado measure of* H with respect to G. Originally described by Chermak and Delgado, the collection of all subgroups of G with maximal Chermak–Delgado measure, denoted $\mathcal{CD}(G)$, is a sublattice of the lattice of all subgroups of G. In this paper we note that if $H \in \mathcal{CD}(G)$ then H is subnormal in G and prove that if K is a second finite group then $\mathcal{CD}(G \times K) = \mathcal{CD}(G) \times \mathcal{CD}(K)$. We additionally describe the $\mathcal{CD}(G \wr C_p)$ where G has a nontrivial centre and p is an odd prime and determine conditions for a wreath product to be a member of its own Chermak–Delgado lattice. We also examine the behaviour of centrally large subgroups, a subset of the Chermak–Delgado lattice.

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1. Introduction

Chermak and Delgado [1] defined a family of functions from the set of subgroups of a finite group into the set of positive integers. They then used these functions to obtain a variety of results, including a proof that every finite group G has a characteristic abelian subgroup N such that $|G : N| \leq |G : A|^2$ for all abelian $A \leq G$.

In [5], Isaacs focused on one member of this family, which he referred to as the *Chermak–Delgado measure*. Isaacs showed for a fixed group G that the subgroups with maximal measure form a sublattice within the lattice of subgroups of G, which he referred to as the *Chermak–Delgado lattice* of G. After observing a paucity of groups which were members of their own Chermak–Delgado lattice, it seemed natural to investigate their existence. Thus, in this paper we study the Chermak–Delgado lattice of direct products and wreath products. We prove that its members are always subnormal in G and find special conditions in which $G \wr H$ is in its own Chermak–Delgado lattice. As a by-product of our efforts, we show that every 2-group can be embedded as a subnormal subgroup of a group that is a member of its Chermak–Delgado lattice.

Moreover, in a recent article Glauberman studied some large subgroups of the Chermak–Delgado lattice [4]. We show that this collection of subgroups behaves nicely in direct products and wreath products $G \wr C_p$, where C_p is the cyclic group of odd order p.

Throughout the paper we use the following familiar notation. For n a positive integer we use S_n to denote the symmetric group on n points and A_n to denote the alternating subgroup of S_n. We use C_n, D_n, and Q_n to represent the cyclic, dihedral, and quaternion group of order n (respectively, and for applicable values of n). If D is a direct product with G as one of its factors then π_G will represent the natural projection map from D onto G. If D is the direct product of multiple copies of G with itself, then we use G_i to represent the ith factor in D and π_i to represent the projection map from D onto G_i.

2. Preliminaries

Define the *Chermak–Delgado measure* of a subgroup H with respect to a finite group G with $H \leq G$ as

$$m_G(H) = |H||C_G(H)|.$$

From the definition, it is clear that the groups discussed in this paper are necessarily finite. The next two lemmas are straightforward to prove using just the definition of $m_G(H)$ and recollections about centralisers from introductory group theory courses.

LEMMA 2.1. *If $H \leq G$ then $m_G(H) \leq m_G(C_G(H))$, and if the measures are equal then $H = C_G(C_G(H))$.*

LEMMA 2.2. *If $H, K \leq G$ then $m_G(H)m_G(K) \leq m_G(\langle H, K\rangle)m_G(H \cap K)$. Moreover, equality occurs if and only if $\langle H, K\rangle = HK$ and $C_G(H \cap K) = C_G(H)C_G(K)$.*

The full details of the proofs of these lemmas can be found in [5, Section 1.G]. For any finite group G, let $\mathfrak{M}_G$ denote the maximal measure over all subgroups in G and let the set of all subgroups $H \leq G$ with $m_G(H) = \mathfrak{M}_G$ be denoted by $\mathcal{CD}(G)$. From Lemmas 2.1 and 2.2 we obtain the following theorem.

THEOREM 2.3. *For a finite group G the set $\mathcal{CD}(G)$ is a sublattice within the lattice of subgroups of G, and for all H, K in $\mathcal{CD}(G)$ we have $\langle H, K\rangle = HK$. Moreover, if $H \in \mathcal{CD}(G)$ then $C_G(H) \in \mathcal{CD}(G)$ and $H = C_G(C_G(H))$.*

The lattice described in Theorem 2.3 will be referred to as the *Chermak–Delgado lattice of G*. Clearly $\mathcal{CD}(G)$ is a sublattice within the lattice of subgroups of G. For large or complex groups G, it can be a challenge to determine $\mathcal{CD}(G)$ by hand. However, the calculations for small groups and abelian groups are refreshingly easy.

(1) Let G be abelian. If $H \leq G$ then $m_G(H) = |H||G|$. Therefore the only subgroup of maximal measure is G, and $\mathcal{CD}(G) = \{G\}$.

(2) Let $G = S_4$; then $m_G(G) = 24 = m_G(\mathrm{Z}(G))$. With a little work, one can show that the measure of any other subgroup of S_4 is less than 24—for example, $m_G(A_4) = 16$. Hence $\mathcal{CD}(S_4) = \{S_4, 1\}$.

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(3) Let $G = S_3$; then $m_G(G) = m_G(Z(G)) = 6$. On the other hand, the subgroup A_3 is abelian and is also its own centraliser. Thus $m_G(A_3) = 9$. The subgroups of order two in G are also their own centralisers, therefore these subgroups have measure 4. Hence $\mathcal{CD}(G) = \{A_3\}$.

(4) Consider D_8, the dihedral group of order eight. There are five subgroups of D_8 with measure 16: D_8, $Z(D_8)$, and the three subgroups of order four. All other subgroups have smaller measure. Hence $\mathcal{CD}(D_8)$ is also the lattice of normal subgroups of D_8.

(5) One can also show that $\mathcal{CD}(Q_8)$ is isomorphic to the lattice of normal subgroups of Q_8, which happens to be isomorphic to the lattice of normal subgroups of D_8.

Observe, since both D_8 and S_3 can be represented as subgroups of S_4, that there is not a straightforward relation between $\mathcal{CD}(U)$ and $\mathcal{CD}(G)$ when $U \le G$. Of course, one notices that if $U \le G$ then

$$\begin{aligned} \mathfrak{M}_U &= m_U(V) \quad \text{for some } V \le U \\ &= |V||C_U(V)| \\ &\le |V||C_G(V)| \\ &\le m_G(V) \\ &\le \mathfrak{M}_G. \end{aligned}$$

The next result is due to Wielandt, and can be found as [5, Theorem 2.9]. Isaacs refers to the result as a 'Zipper Lemma' and we continue that reference here.

Theorem 2.4 (Zipper Lemma). *Suppose that $S \le G$ where G is a finite group, and assume that $S \lhd\lhd H$ for every proper subgroup H of G that contains S. If S is not subnormal in G then there is a unique maximal subgroup of G that contains S.*

The Zipper Lemma makes way for the use of induction with regard to the Chermak–Delgado lattice. Another important fact regarding $U \in \mathcal{CD}(G)$ is given in the following proposition.

Proposition 2.5. *Let $U \in \mathcal{CD}(G)$ for a finite group G. If $S < G$ with both $U \le S$ and $UC_G(U) \le S$ then $U \in \mathcal{CD}(S)$.*

Proposition 2.5 is easy to see—when $UC_G(U) \le S$ then $C_G(U) = C_S(U)$. In fact, not only is $U \in \mathcal{CD}(S)$ but also $C_G(U)$ and $UC_G(U)$ are in $\mathcal{CD}(S)$. This useful proposition, together with the Zipper Lemma, is enough to prove the following result.

Theorem 2.6. *Let G be a group. If $U \in \mathcal{CD}(G)$ then $U \lhd\lhd G$.*

Proof. Assume that, for every proper subgroup U of G, if $X \in \mathcal{CD}(U)$ then $X \lhd\lhd U$. Now let $U \in \mathcal{CD}(G)$. We show that $V = UC_G(U) \lhd\lhd G$, which is sufficient for the theorem since $U \trianglelefteq V$.

If $V = G$ our conclusion holds, so assume that $V < G$. For every $S < G$ with $V \le S$, we know that $U \le S$ and therefore $V \in \mathcal{CD}(S)$ by Proposition 2.5. By

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induction $V \lhd\lhd S$. If V is not subnormal in G then we may apply the Zipper Lemma, resulting in the existence of a unique maximal subgroup M of G that contains V. Notice, by the previous few sentences, that $V \lhd\lhd M$.

Let $x \in G$; since $V^x \in \mathcal{CD}(G)$, we have $VV^x \in \mathcal{CD}(G)$ as well. If $VV^x = G$ then there exist $v, v_0 \in V$ such that $vv_0^x = x$, and careful multiplication shows that $x = v_0 v \in V$. Thus if $VV^x = G$ then $V = G$. We assumed $V < G$, though, so $VV^x < G$. There exists a proper maximal subgroup N of G that contains VV^x; however, since $V < N$ and M is the unique maximal subgroup containing V, we know that $N = M$.

One can repeat the use of the Zipper Lemma on V^x and determine that M^x is the unique maximal subgroup of G containing V^x. Yet M contains V^x, so $M = M^x$ for all $x \in G$. Hence $M \trianglelefteq G$. Since subnormality is transitive, $V \lhd\lhd G$ as desired. □

A trivial consequence of Theorem 2.6 is that the Chermak–Delgado lattice of any simple group S is $\{\mathrm{Z}(S), S\}$ (of course $\mathrm{Z}(S) = S$ when S is abelian). Another easy consequence of Theorem 2.6 is the expansion of (2) and (3) (following Theorem 2.3): given a symmetric group S_n, for $n \geq 5$, we know that the only possible subgroups in $\mathcal{CD}(S_n)$ are 1, A_n and S_n. Since the measure of A_n will be less than that of S_n, we know that $\mathcal{CD}(S_n) = \{1, S_n\}$; therefore the Chermak–Delgado lattice of any symmetric group is completely determined.

One might question whether Theorem 2.6 can be strengthened, that is, whether all subgroups in $\mathcal{CD}(G)$ are actually normal in G. The answer, demonstrated by the next example, is negative.

Example 2.7. Let G be as follows:

$$\begin{aligned} G = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = [a, b] = [b, c] = [b, d] \\ = [c, d] = [a, c]b = [a, d]c = 1\rangle. \end{aligned}$$

This presentation is convenient for computations, though G actually is a 2-generator group. A few calculations (done by hand or with GAP [3]) show that $X = \langle a, b\rangle$ is a member of $\mathcal{CD}(G)$. One can show that d does not normalise X, therefore $X \lhd\lhd G$ with defect greater than 1. There are a few other subgroups in $\mathcal{CD}(G)$ that are not normal, such as $\langle b, da\rangle$ and $\langle b, da^3\rangle$, though showing by hand that these subgroups are not normal is tedious.

Having shown that the members of $\mathcal{CD}(G)$ are subnormal, one continues by asking about the Chermak–Delgado lattice of a direct product. Before proceeding, though, we introduce a subset of the Chermak–Delgado lattice.

In [4] Glauberman defines the notion of a *centrally large subgroup* and shows, among other things, that a subgroup U is centrally large exactly when $U \in \mathcal{CD}(G)$ and $\mathrm{Z}(U) = C_G(U)$. We denote the set of centrally large subgroups of G by $\mathcal{CL}(G)$. Note that $\mathcal{CL}(G)$ is closed under joins and contains the largest element in $\mathcal{CD}(G)$.

In addition to describing $\mathcal{CD}(G \times H)$ for finite groups G and H, we also describe $\mathcal{CL}(G \times H)$. We utilise the following basic fact about centralisers in direct

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products, the proof of which follows directly from the mechanics of conjugation in a direct product.

Lemma 2.8. *Let G and H be groups. If $U \le G \times H$ then $C_{G\times H}(U) = C_G(\pi_G(U)) \times C_H(\pi_H(U))$.*

Theorem 2.9. *For any finite groups G and H, the lattices $\mathcal{CD}(G \times H)$ and $\mathcal{CD}(G) \times \mathcal{CD}(H)$ are equal and $\mathcal{CL}(G \times H) = \mathcal{CL}(G) \times \mathcal{CL}(H)$.*

Proof. Let $U \le G \times H$. We have the following inequality, with the second step due to Lemma 2.8:

$$\begin{aligned} m_{G\times H}(U) &= |U||C_{G\times H}(U)| \\ &= |U||C_G(\pi_G(U)) \times C_G(\pi_H(U))| \\ &\le |\pi_G(U) \times \pi_H(U)||C_G(\pi_G(U)) \times C_G(\pi_H(U))| \\ &\le |\pi_G(U)||C_G(\pi_G(U))||\pi_H(U)||C_H(\pi_H(U))| \\ &\le m_G(\pi_G(U))m_H(\pi_H(U)). \end{aligned}$$

Equality occurs *exactly* when $U = \pi_G(U) \times \pi_H(U)$. Therefore, the subgroups of $G \times H$ with maximal measure are exactly those direct products $X \times Y$ where $X \in \mathcal{CD}(G)$ and $Y \in \mathcal{CD}(H)$. This gives $\mathcal{CD}(G \times H) = \mathcal{CD}(G) \times \mathcal{CD}(H)$.

Now suppose that $U \in \mathcal{CL}(G \times H)$; then $U \in \mathcal{CD}(G \times H)$. For $X \in \{G, H\}$ we have $\pi_X(U) \in \mathcal{CD}(X)$. Let $g \in C_G(\pi_G(U))$. The element $(g, 1)$ centralises U, so its projection g is in $C_G(\pi_G(U))$. Therefore $C_G(\pi_G(U)) \le \pi_G(U)$; we can similarly prove the same with respect to H. Hence $\pi_X(U) \in \mathcal{CL}(X)$ for $X = G, H$.

Alternatively, assume that $\pi_X(U) \in \mathcal{CL}(X)$ for $X \in \{G, H\}$. Then $\pi_X(U) \in \mathcal{CD}(X)$ and hence $U \in \mathcal{CD}(G \times H)$. Moreover, $\mathrm{Z}(U \cap X) = C_X(\pi_X(U))$ for both values of X, so $\mathrm{Z}(U) = C_{G\times H}(U)$ after using Lemma 2.8. Therefore $U \in \mathcal{CL}(G \times H)$. Hence $\mathcal{CL}(G \times H) = \mathcal{CL}(G) \times \mathcal{CL}(H)$, as desired. □

3. Wreath products

This section discusses our attempts to describe the Chermak–Delgado lattice of a wreath product. As a byproduct of our efforts we show that every finite 2-group G can be embedded in a finite 2-group E such that $E \in \mathcal{CD}(E)$, while noting that there are 2-groups E such that $E \notin \mathcal{CD}(E)$.

Let G and H be finite groups with H a permutation group of degree n acting on a set Ω. The *wreath product of G by H*, denoted $G \wr H$, is the semidirect product $B \rtimes H$ where $B = G^{\Omega}$ is the group of all functions $f : \Omega \to G$ under pointwise multiplication. The subgroup B is referred to as the *base* of W. If $h \in H$ and $f \in B$ then

$$f^h(\omega) = f(\omega h^{-1})$$

for $\omega \in \Omega$.

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We focus on wreath products $G \wr H$ where $H \cong C_n$ for some positive integer n, so $\Omega = \{1, 2, \ldots, n\}$. As in the case with direct products, we start by examining centralisers.

Proposition 3.1. *Let G be a nontrivial group and set $W = G \wr C_n$ where $C_n = \langle\sigma\rangle$ is the cyclic group of order n. Let B be the base group of W. If, for some $f \in B$, the element $f\sigma \in W - B$ commutes with an element $b \in B$ then*

$$b(i) = b(1)^{f(1)f(2)\cdots f(i-1)} \quad \text{for } 1 < i \leq n.$$

Thus all $b(i)$ are in some orbit of $\langle f(1), f(2), \ldots, f(n)\rangle$. Furthermore, $b(1) \in C_G(f(1)f(2)\cdots f(n))$ and hence $\pi_1(C_B(f\sigma)) \cong C_G(f(1)f(2)\cdots f(n))$.

Proof. Suppose that $f \in B$, and further suppose that there exists $b \in B$ such that $f\sigma \in W - B$ commutes with b. Notice that

$$f\sigma b = bf\sigma \iff b^{\sigma^{-1}} = b^f.$$

In particular, for $1 < i \leq n$,

$$b^{\sigma^{-1}}(i) = b(i\sigma) = b^f(i) = b(i)^{f(i)}.$$

Plugging in a few values for i, we see that $b(2) = b(1)^{f(1)}$ and

$$b(3) = b(2)^{f(2)} = (b(1)^{f(1)})^{f(2)} = b(1)^{f(1)f(2)}.$$

Continuing in this way, we conclude that $b(i) = b(1)^{f(1)f(2)\cdots f(i-1)}$ for all i with $1 < i \leq n$.

In fact, since $n\sigma = 1$ we also see that

$$b(1) = b(n\sigma) = b(n)^{f(n)} = b(1)^{f(1)f(2)\cdots f(n)},$$

hence $b(1)$ commutes with $f(1)f(2)\cdots f(n)$ in G and $\pi_1(C_B(b\sigma)) \cong C_G(f(1)f(2)\cdots f(n))$ as described in the statement of the proposition. □

This proposition can generalise straightforwardly to more general H, but since the notation quickly becomes cumbersome and we do not apply such a generalisation here, we refer the reader to [7]. Proposition 3.1 is enough to establish some facts about $C_W(B)$ and $Z(W)$, allowing us to better calculate $m_W(B)$ and $m_W(W)$.

Proposition 3.2. *Let $W = G \wr C_n$ with G a nontrivial group and base group B. The centraliser in W of B is $Z(B)$; consequently $m_W(B) = m_B(B) = |G|^n|Z(G)|^n$.*

Proof. Set $C_n = \langle\sigma\rangle$. Suppose that an element $z \in W$ centralises B. If there exists $f \in B$ such that $z = f\sigma \notin B$ then Proposition 3.1 redefines the structure of B, namely telling us that $B \cong C_G(f(1)f(2)\cdots f(n))$. Yet by its definition B cannot be isomorphic to a subgroup of G. Hence z must be an element of B, yielding $C_W(B) \leq B$. Thus $C_W(B) = Z(B)$. Therefore $m_W(B) = |B||Z(B)| = |G|^n|Z(G)|^n$ as claimed. □

Combining Proposition 3.2 with [5, Exercise 3A.9], which states that elements commuting with the generator of C_n must be diagonal, we have the following description of $Z(W)$.

Proposition 3.3. *Let G be a nontrivial group, set $W = G \wr C_n$, and let B represent the base of W. The centre of W is equal to the diagonal of $\mathrm{Z}(B)$, and consequently $m_W(W) = n|G|^n|\mathrm{Z}(G)|$.*

The next proposition is a straightforward consequence of our calculations in Propositions 3.2 and 3.3. The result implies that even when $G \in \mathcal{CD}(G)$ and $H \in \mathcal{CD}(H)$, the wreath product $W = G \wr H$ need not be a member of $\mathcal{CD}(W)$.

Proposition 3.4. *Let G be a group and let $W = G \wr C_n$ for an integer $n \geq 2$. If $|\mathrm{Z}(G)| \geq 2$ or $n > 2$ then $W \notin \mathcal{CD}(W)$.*

Proof. Let $z = |\mathrm{Z}(G)|$. We first calculate the measures of W and B using Propositions 3.3 and 3.2:

$$m_W(W) = n|G|^n|\mathrm{Z}(G)| = n|G|^n \cdot z$$

and

$$m_W(B) = |G|^n|\mathrm{Z}(G)|^n = |G|^n \cdot z^n.$$

Thus $m_W(W) < m_W(B)$ if and only if $z > n^{1/n-1}$. One easily confirms that this latter expression is strictly decreasing for integers $n \geq 2$. When $n = z = 2$ or when $|\mathrm{Z}(G)| = 1$, we have $m_W(W) \geq m_W(B)$. Otherwise, though, $m_W(W) < m_W(B)$ and thus $W \notin \mathcal{CD}(W)$. □

Observe from the proof of Proposition 3.4 that when $|\mathrm{Z}(G)| = n = 2$ then $m_W(W) = m_W(B)$. We saw an example of this situation, D_8, where $W \in \mathcal{CD}(W)$. Therefore, in light of $\mathcal{CD}(D_8)$ and Proposition 3.4, we are interested in two questions.

(1) If $|\mathrm{Z}(G)| = 2$, will $W = G \wr C_2$ be a member of $\mathcal{CD}(W)$?
(2) If $W = G \wr C_n$ with $|\mathrm{Z}(G)| > 2$ or $n > 2$, will $\mathcal{CD}(W) = \mathcal{CD}(B)$?

In the remainder of this section we address both of these questions. Let us first note that if G is not in its own Chermak–Delgado lattice then W need not be in $\mathcal{CD}(W)$. The first nonabelian group G with $\mathrm{Z}(G) \cong C_2$ and $G \notin \mathcal{CD}(G)$ is D_{12}, the dihedral group of order 12.

Example 3.5. Let $G = D_{12}$. First we show that $G \notin \mathcal{CD}(G)$. Let r be an element of order six; then $\langle r\rangle = C_G(\langle r\rangle)$. Hence $m_G(\langle r\rangle) = 6^2 = 36$. Yet $m_G(G) = 12 \cdot 2 = 24$, so $G \notin \mathcal{CD}(G)$.

Let $W = G \wr C_2$; then $m_W(W) = 6^2 \cdot 2^4$. Let U be the subgroup of the base of W isomorphic to $\langle r\rangle \times \langle r\rangle$. Observe that $U \leq C_W(U)$ and therefore $m_W(U) \geq |U|^2 = 6^4$. Since $m_W(U) > m_W(W)$, we know that $W \notin \mathcal{CD}(W)$.

Thus, with regard to question (1), we show that if $|\mathrm{Z}(G)| = 2$ *and* $G \in \mathcal{CD}(G)$ then $W = G \wr C_2$ is in $\mathcal{CD}(W)$ and $\mathcal{CD}(B) \leq \mathcal{CD}(W)$ as lattices. To attain this answer and to address question (2), we examine $\mathcal{CD}(W)$ by considering $m_W(U)$ for $U \in \mathcal{CD}(W)$. There are four cases, depending upon whether or not $U \leq B$ or $C_W(U) \leq B$. The next lemma describes a reduction in calculating the order of U; it is a direct consequence of the Isomorphism Theorems [2, Theorem 3.18].

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Lemma 3.6. *Let G be a nontrivial group, $W = G \wr C_p$ for some prime p, and B be the base of W. If $U \leq W$ then*

$$|U : B \cap U| = \begin{cases} 1 & \text{if } U \leq B, \\ p & \text{if } U \nleq B. \end{cases}$$

We use Lemma 3.6 in the proof of the following result, the key observation for calculating the Chermak–Delgado measure of a subgroup in a wreath product.

Proposition 3.7. *Let G be a nontrivial group and let $W = G \wr C_p$ for a prime p. Suppose that B is the base of W and let $U \leq W$.*

(1) *If $U \leq B$ and $C_W(U) \nleq B$ then $|U| = |\pi_1(U)|$ and $|C_W(U)| = p|C_G(\pi_1(U))|^p$.*
(2) *If $U \nleq B$ and $C_W(U) \nleq B$ then $|U| = p|\pi_1(U \cap B)|$ and $|C_W(U)| = p|\pi_1(C_B(U))|$.*

Proof. Let $U \leq B$ and suppose that $C_W(U) \nleq B$. After applying Lemma 3.6 to $C_W(U)$, we see that $|C_W(U)| = p|C_B(U)|$; moreover, $C_W(U)/C_B(U) \cong W/B \cong \langle\sigma\rangle$ and there must exist $f \in B$ such that $C_W(U) = C_B(U)\langle f\sigma\rangle$.

Proposition 3.1 then applies to $U \leq B$ and $f\sigma \in C_W(U)$, so that if $u \in U$ there exists a $g \in C_G(f(1)f(2)\cdots f(p-1))$ with

$$u(i) = g^{f(1)f(2)\cdots f(i-1)} \quad \text{for each } i \in \Omega.$$

Thus $\pi_i(U) = (\pi_1(U))^{f(1)f(2)\cdots f(i-1)}$ for $2 \leq i \leq p$. Therefore U is a 'diagonal-type' subgroup and $|U| = |\pi_1(U)|$, as claimed. Moreover, the description of U from Proposition 3.1 implies that $|U| = |\pi_1(U)| \leq |C_G(f(1)f(2)\cdots f(p-1))|$.

Lemma 2.8 states that $C_B(U) = \prod_{i=1}^{p} C_G(\pi_i(U))$. Given the structure of U, we can establish that $\pi_1(U) \cong \pi_2(U)^{f(1)}$ and, similarly, $\pi_i(U) = (\pi_1(U))^{f(1)f(2)\cdots f(i-1)}$ for all i with $3 \leq i \leq p$. Therefore $C_G(\pi_1(U)) \cong C_G(\pi_i(U))$ for all i with $2 \leq i \leq p$, and $|C_B(U)| = |C_G(\pi_1(U))|^p$.

Now suppose that neither U nor $C_W(U)$ is a subgroup of B. Then Lemma 3.6 tells us that $|U| = p|U \cap B|$ and $|C_W(U)| = p|C_B(U)|$. Yet $U \cap B \leq B$ and $C_W(U \cap B)$ contains $C_W(U)$, and hence $C_W(U \cap B) \nleq B$. Applying part (1) to $U \cap B$ we have $|U \cap B| = |\pi_1(U \cap B)|$. Thus

$$|U| = p|\pi_1(U \cap B)|,$$

as desired.

Let $X = C_W(U)$. Note that $X \nleq B$ and we established $|X| = p|X \cap B|$. Since $U \leq C_W(X)$ we know that $C_W(X) \nleq B$. Apply the argument of the last paragraph to X; thus $|X| = p|\pi_1(X \cap B)|$. Since $X \cap B = C_B(U)$, we therefore have shown that $|C_W(U)| = p|\pi_1(C_B(U))|$. □

Theorem 3.8. *Let $G \in \mathcal{CD}(G)$ and suppose that $|Z(G)| = 2$. Let $W = G \wr C_2$. The group W is a member of $\mathcal{CD}(W)$ and $\mathcal{CD}(B) \leq \mathcal{CD}(W)$, as lattices.*

Proof. First we calculate the measures of W and B, using Propositions 3.3 and 3.2. This gives

$$m_W(W) = 2^2|G|^2 = m_W(B) = m_B(B).$$

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We will show for all $U \in \mathcal{CD}(W)$ that $m_W(U) \leq 2^2|G|^2$, thus determining that W has maximal measure and implying $W \in \mathcal{CD}(W)$. To do this we will first consider $U \leq B$ and then turn our attention to $U \nleq B$.

If $C_W(U) \leq B$ then $C_W(U) = C_B(U)$. Thus $m_W(U) = m_B(U)$. Since $G \in \mathcal{CD}(G)$, we know by Theorem 2.9 that $B \in \mathcal{CD}(B)$. Therefore

$$m_W(U) = m_B(U) \leq m_B(B) = m_W(W).$$

If, on the other hand, when $U \leq B$ we also have $C_W(U) \nleq B$ then Proposition 3.7 yields that $|U| = |\pi_1(U)|$. This, together with the information about the centraliser of U from Proposition 3.7, yields

$$\begin{aligned} m_W(U) &= |U||C_W(U)| \\ &\leq |\pi_1(U)| \cdot 2 \cdot |C_G(\pi_1(U))|^2 \\ &\leq 2 \cdot m_G(\pi_1(U)) \cdot |C_G(\pi_1(U))|. \end{aligned}$$

Yet $G \in \mathcal{CD}(G)$; thus $m_G(\pi_1(U))$ is less than $2|G|$. Also, the centraliser of $\pi_1(U)$ clearly has order no more than $|G|$. This allows $m_W(U) \leq 2^2|G|^2$. Therefore if $U \leq B$ then $m_W(U) \leq m_W(W)$.

Now suppose that $U \nleq B$. If $C_W(U) \leq B$ then we know already that $m_W(C_W(U)) \leq m_W(W)$, by the preceding paragraphs. Yet $U \in \mathcal{CD}(W)$, so $m_W(U) = m_W(C_W(U))$. Hence we need only examine the case where $C_W(U) \nleq B$.

In this case, Proposition 3.7 tells us that $|U| = 2|\pi_1(U \cap B)|$ and $|C_W(U)| \leq 2|\pi_1(C_B(U))|$. Notice that $C_B(U) \leq C_B(U \cap B)$, and therefore

$$\pi_1(C_B(U)) \leq \pi_1(C_B(U \cap B)).$$

It is a straightforward argument to show that

$$\pi_1(C_B(U \cap B)) \leq C_G(\pi_1(U \cap B)).$$

Therefore

$$\begin{aligned} m_W(U) &= |U||C_W(U)| \\ &\leq 2 \cdot |\pi_1(U \cap B)| \cdot 2 \cdot |\pi_1(C_G(U \cap B))| \\ &\leq 2^2 \cdot |\pi_1(U \cap B)||C_G(\pi_1(U \cap B))| \\ &\leq 2^2 m_G(\pi_1(U \cap B)). \end{aligned}$$

Yet $G \in \mathcal{CD}(G)$, so we can conclude that

$$m_W(U) \leq 2^3|G|.$$

Since $|G| \geq 2$, this yields the desired result for $U \nleq B$.

To finish the proof, let $U \in \mathcal{CD}(B)$. Then $m_B(U) = m_B(B)$, yet we have established that this latter quantity equals $\mathfrak{M}_W$. Hence $U \in \mathcal{CD}(W)$, as well. □

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The proof of Theorem 3.8 establishes that when W is as described then $\mathcal{CD}(W)$ contains at least $\mathcal{CD}(B)$ and new maximal and minimal elements (W and $Z(W)$, respectively). There may even be other elements of $\mathcal{CD}(W)$ that are not in $\mathcal{CD}(B)$. And, as a corollary of Theorem 3.8, we have the result mentioned at the start of the section.

Corollary 3.9. *If G is a 2-group then there exists a 2-group E with $E \in \mathcal{CD}(E)$ such that G can be embedded as a subgroup of E.*

Proof. The group G can be embedded as a subgroup of S_n for some n. Let E be the Sylow 2-subgroup of S_n that contains G. Recall that E is a direct product whose factors are iterated wreath products of C_2. Each of the iterated wreath products is contained in its Chermak–Delgado lattice, by Theorem 3.8. Thus $E \in \mathcal{CD}(E)$ by Theorem 2.9. □

Corollary 3.9 is not trivial, in the sense that there are 2-groups which are not in their own Chermak–Delgado lattice. In fact, there are 2-groups G with $Z(G) = 2$ such that $G \notin \mathcal{CD}(G)$. We provide one example here.

Example 3.10. Let G be the Sylow 2-subgroup of the general linear group of $n \times n$ matrices with entries in the field of order two. It is known that G is isomorphic to the group of upper triangular matrices over the field of order two and that $|Z(G)| = 2$.

Let A be an abelian subgroup of maximal rank in G. In [6] it is shown that $|A| = 2^{xy}$ where x is the greatest integer less than or equal to $n/2$ and y is the smallest integer greater than or equal to $n/2$; thus $m_G(A) \geq 2^{2xy}$. On the other hand, $m_G(G) = 2^{n(n-1)/2} \cdot 2$. When $n = 5$, we have $x = 2$ and $y = 3$ so it is easy to see that $m_G(A) > m_G(G)$.

Recall question (2): If $W = G \wr C_n$ where $|Z(G)| > 2$ or $n > 2$, will $\mathcal{CD}(W) = \mathcal{CD}(B)$? We address this question only in the case where n is a prime number. The techniques to give an affirmative answer for this restricted question are along the same lines as what we have done so far in this section. We begin with a lemma.

Lemma 3.11. *Let p be a prime number, G be a group with $Z(G) > 1$, and $W = G \wr C_p$ with base B. If $W \not\cong D_8$ then for every $U \in \mathcal{CD}(W)$ either $U \leq B$ or $C_W(U) \leq B$.*

Proof. We prove the contrapositive of the lemma, supposing that there exists $U \in \mathcal{CD}(W)$ with $U \nleq B$ and $C_W(U) \nleq B$. Then Proposition 3.7 yields:

$$\begin{aligned}
\mathfrak{M}_W &= |U||C_W(U)| \\
&\leq p^2 |\pi_1(U \cap B)||\pi_1(C_B(U))| \\
&\leq p^2 |\pi_1(U \cap B)||\pi_1(C_B(U \cap B))| \\
&\leq \mathfrak{M}_G \cdot p^2.
\end{aligned}$$

At the same time, though, we know that $\mathfrak{M}_B = (\mathfrak{M}_G)^p \leq \mathfrak{M}_W$. Thus

$$(\mathfrak{M}_G)^p \leq \mathfrak{M}_G \cdot p^2.$$

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The usual algebra tactics allow us to rearrange the inequality: $\mathfrak{M}_G \le (p^2)^{1/(p-1)}$. Yet this last expression is the square of a function that is strictly decreasing on integers $n \ge 2$; hence its maximum value is when $p = 2$. Additionally, $m_G(G) \le \mathfrak{M}_G$ and therefore

$$|G||\mathrm{Z}(G)| \le (p^2)^{1/(p-1)} \le 4.$$

Since $|\mathrm{Z}(G)| \ge 2$, the above can only occur when $|\mathrm{Z}(G)| = |G| = p = 2$. In this case, though, $W \cong D_8$. □

Theorem 3.12. *Let p be a prime, G be a group with $\mathrm{Z}(G) > 1$, and $W = G \wr C_p$ with base B. If $|\mathrm{Z}(G)| > 2$ or $p > 2$ then for every $U \in \mathcal{CD}(W)$ both $U \le B$ and $C_W(U) \le B$. Thus in this case $\mathcal{CD}(W) = \mathcal{CD}(B)$ and similarly, $\mathcal{CL}(W) = \mathcal{CL}(B)$.*

Proof. We prove the contrapositive of the theorem. Assume that there exists $U \in \mathcal{CD}(W)$ and at least one of U or $C_W(U)$ is not a subgroup of B. If both U and $C_W(U)$ are not subgroups of B then Lemma 3.11 tells us that $W \cong D_8$. In this case $|\mathrm{Z}(G)| = p = 2$, so the theorem holds.

Suppose that exactly *one* of U or $C_W(U)$ is not a subgroup of B. As $U \in \mathcal{CD}(W)$, we know that $U = C_W(C_W(U))$. Therefore we may assume, without loss of generality, that $U \le B$ and $C_W(U) \nleq B$.

Proposition 3.7 implies that

$$\begin{aligned}\mathfrak{M}_W &= |U||C_W(U)| \\ &= |\pi_1(U)||C_G(\pi_1(U))|^p \cdot p \\ &= \mathfrak{M}_G \cdot |C_G(\pi_1(U))|^{p-1} \cdot p.\end{aligned}$$

Again we know that $\mathfrak{M}_B = (\mathfrak{M}_G)^p \le \mathfrak{M}_W$ and additionally $|C_G(\pi_1(U))| \le G$. Hence $(\mathfrak{M}_G)^p \le \mathfrak{M}_G \cdot |G|^{p-1} \cdot p$ and therefore $\mathfrak{M}_G \le |G| \cdot p^{1/(p-1)}$. Then $|G||\mathrm{Z}(G)| \le \mathfrak{M}_G \le |G| \cdot p^{1/(p-1)}$ and therefore $|\mathrm{Z}(G)| \le p^{1/(p-1)}$. This familiar expression is strictly decreasing on integers $n \ge 2$, as before. Therefore $|\mathrm{Z}(G)| \le 2$ and, given the hypotheses of the theorem, $|\mathrm{Z}(G)| = p = 2$.

Therefore if $|\mathrm{Z}(G)| > 2$ or $p > 2$ then for every $U \in \mathcal{CD}(W)$ we know that $U \le B$ and $C_W(U) = C_B(U)$. Thus $m_W(U) = m_B(U)$. It is always true that $\mathfrak{M}_B \le \mathfrak{M}_W$, so in this case $U \in \mathcal{CD}(B)$ and $\mathcal{CD}(W) \le \mathcal{CD}(B)$. That then implies that $\mathfrak{M}_W = \mathfrak{M}_B$, since $C_W(U) \le C_B(U)$ for any $U \le B$. Hence if $U \in \mathcal{CD}(B)$ then $U \in \mathcal{CD}(W)$, too. Thus $\mathcal{CD}(W) = \mathcal{CD}(U)$.

Let $U \in \mathcal{CL}(W)$. Then $U \in \mathcal{CD}(W)$ and $C_W(U) = \mathrm{Z}(U)$. By the arguments earlier, $U \in \mathcal{CD}(B)$. Additionally $C_B(U) \le C_W(U)$, so we conclude that $U \in \mathrm{CLB}$, giving $\mathcal{CL}(W) \subseteq \mathcal{CL}(B)$. Suppose that $U \in \mathcal{CL}(B)$; hence $U \in \mathcal{CD}(B)$ and $C_B(U) = \mathrm{Z}(U)$. Then we can conclude that $U \in \mathcal{CD}(W)$, but then the preceding paragraph gives $C_W(U) \le C_B(U)$. Therefore $U \in \mathcal{CL}(W)$ and $\mathcal{CL}(B) = \mathcal{CL}(W)$. □

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