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LARGE CARDINALS AS PRINCIPLES OF STRUCTURAL REFLECTION

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Abstract. After discussing the limitations inherent to all set-theoretic reflection principles akin to those studied by A. Lévy et. al. in the 1960s, we introduce new principles of reflection based on the general notion of *Structural Reflection* and argue that they are in strong agreement with the conception of reflection implicit in Cantor's original idea of the unknowability of the *Absolute*, which was subsequently developed in the works of Ackermann, Lévy, Gödel, Reinhardt, and others. We then present a comprehensive survey of results showing that different forms of the new principle of Structural Reflection are equivalent to well-known large cardinal axioms covering all regions of the large-cardinal hierarchy, thereby justifying the naturalness of the latter.

In the framework of Zermelo–Fraenkel (ZF) set theory[1] the universe V of all sets is usually represented as a stratified cumulative hierarchy of sets indexed by the ordinal numbers. Namely, $V = \bigcup_{\alpha \in OR} V_\alpha$, where

$$
\begin{aligned}
&V_0 = \varnothing,\\
&V_{\alpha+1} = \mathcal{P}(V_\alpha), \ \text{ namely the set of all subsets of } V_\alpha, \text{ and}\\
&V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \ \text{if } \lambda \text{ is a limit ordinal.}
\end{aligned}
$$

This view of the set-theoretic universe justifies *a posteriori* the ZF axioms. For not only all ZF axioms are true in V, but they are also necessary to build V. Indeed, the axioms of Extensionality, Pairing, Union, Power-Set, and Separation are used to define the set-theoretic operation given by

$$G(x) = \bigcup \{\mathcal{P}(y) : \exists z(\langle z, y\rangle \in x)\}.$$

The axiom of Replacement is then needed to prove by transfinite recursion that the operation V on the ordinals given by $V(\alpha) = G(V \restriction \alpha)$ is well-

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[1] For all undefined set-theoretic notions see [14] or [15].

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defined and unique. Then $V(\alpha) = V_\alpha$ is as defined above. The two remaining ZF axioms are easily justified: the axiom of Infinity leads the ordinal sequence into the transfinite, and is the essence of set theory, for its negation (namely, ZFC minus Infinity plus the negation of Infinity) yields a theory mutually interpretable with Peano Arithmetic; and the axiom of Regularity simply says, in the presence of the other ZF axioms, that there are no more sets beyond those in V.

In the context of ZFC (ZF plus the Axiom of Choice), the *Axiom of Choice* is clearly true in V, although it is not necessary to build V.

In ZF, or ZFC, other representations of V as a cumulative hierarchy are possible. By "a cumulative hierarchy" we mean a union of transitive sets X_α, defined by transfinite recursion on a *club*, i.e., closed and unbounded, class C of ordinals α, such that:

$$\alpha \leq \beta \text{ implies } X_\alpha \subseteq X_\beta,$$
$$X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha, \text{ if } \lambda \text{ is a limit point of } C, \text{ and}$$
$$V = \bigcup_{\alpha\in C} X_\alpha.$$

For example, let H_κ, for κ an infinite cardinal, be the set of all sets whose transitive closure has cardinality less than κ. Then the H_κ also form a cumulative hierarchy indexed by the club class $CARD$ of infinite cardinal numbers. (Note however that to prove $\bigcup_{\kappa\in CARD} H_\kappa = V$ one needs the Axiom of Choice to guarantee that every (transitive) set has a cardinality, and therefore every set belongs to some H_κ.) Nevertheless, all representations of V as a cumulative hierarchy are essentially the same, in the following sense: Suppose $V = \bigcup_{\alpha\in C} X_\alpha$ and $V = \bigcup_{\alpha\in D} Y_\alpha$ are two cumulative hierarchies, where C and D are club classes of ordinals. Then there is a club class of ordinals $E \subseteq C \cap D$ such that $X_\alpha = Y_\alpha$, for all $\alpha \in E$.

Every representation of V as a cumulative hierarchy is subject to the *reflection* phenomenon, namely the fact that every sentence of the first-order language of set theory that holds in V holds already at some stage V_α of the hierarchy. Indeed, the *Principle of Reflection* of Montague and Lévy [20], provable in ZF, asserts that every formula of the first-order language of set theory true in V holds in some V_α. In fact, for every formula $\varphi(x_1, \dots, x_n)$ of the language of set theory, ZF proves that there exists an ordinal α such that for every $a_1, \dots, a_n \in V_\alpha$,

$$\varphi(a_1, \dots, a_n) \quad \text{if and only if} \quad V_\alpha \models \varphi(a_1, \dots, a_n).$$

Even more is true: for each natural number n, there is a Π_n definable club proper class $C^{(n)}$ of ordinals such that ZF proves that for every $\kappa \in C^{(n)}$,

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every Σ_n formula $\varphi(x_1, \dots, x_n)$, and every $a_1, \dots, a_n \in V_\kappa$,

$$\varphi(a_1, \dots, a_n) \quad \text{if and only if} \quad V_\kappa \models \varphi(a_1, \dots, a_n).$$

That is, $(V_\kappa, \in)$ is a Σ_n-elementary substructure of $(V, \in)$, henceforth written as $(V_\kappa, \in) \preceq_n (V, \in)$, or simply as $V_\kappa \preceq_n V$.

The import of the Principle of Reflection is highlighted by the result of Lévy [20] showing that the ZF axioms of Extensionality, Separation, and Regularity, together with the principle of Complete Reflection (CR), imply ZF. The CR principle is the schema asserting that for every formula $\varphi(x_1, \dots, x_n)$ of the language of set theory there exists a transitive set A closed under subsets (i.e., all subsets of elements of A belong to A) such that for every $a_1, \dots, a_n \in A$,

$$\varphi(a_1, \dots, a_n) \leftrightarrow A \models \varphi(a_1, \dots, a_n).$$

Since the V_α are transitive and closed under subsets, this shows that the reflection phenomenon, as expressed by the CR principle or the Reflection Theorem (given the existence of the V_α's), is not only deeply ingrained in the ZF axioms, but it captures the main content of ZF.

In the series of papers [20–22] Lévy considers stronger principles of reflection, formulated as axiom schemata, and he shows them equivalent to the existence of inaccessible, Mahlo, and Hyper-Mahlo cardinals. The unifying idea behind such principles is clearly stated by Lévy at the beginning of [21]:

> If we start with the idea of the impossibility of distinguishing, by specific means, the universe from partial universes we shall be led to the following axiom schemata, listed according to increasing strength. These axiom schemata will be called *principles of reflection* since they state the existence of standard models (by models we shall mean, for the time being, models whose universes are sets) which reflect in some sense the state of the universe.

This idea of reflection, namely the impossibility of distinguishing the universe from its partial universes (such as the V_α), is also implicit in earlier work of Ackermann [1], but it is Lévy who demonstrates how it can be used to find new natural theories strengthening ZF. Indeed, in his review of Levy's article [22], Feferman [12] writes:

> The author's earlier work demonstrated very well that the diversity of known set-theories could be viewed with more uniformity in the light of various reflection principles, and that these also provided a natural way to "manufacture" new theories. The present paper can only be regarded as a beginning of a systematic attempt to compare the results.

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More recently, it has been argued by many authors that any intrinsic justification of new set-theoretic axioms, beyond ZF, and in particular the axioms of large cardinals, should be based on stronger forms of Lévy's reflection principles. In the next section we shall see some of these arguments, as well as the limitations, clearly exposed by Koellner [16], inherent to all reflection principles akin to those studied by Lévy. In Section 2 we will introduce new principles of reflection based on what we call *Structural Reflection* and will argue that they are in strong agreement with the notion of reflection implicit in the original idea of Cantor's of the unknowability of the *absolute*, which was subsequently developed in the works of Ackermann, Lévy, Gödel, Reinhardt, and others. The rest of the paper, starting with Section 3, will present a series of results showing that different forms of the new principles of Structural Reflection are equivalent to well-known large cardinal axioms covering all regions of the large-cardinal hierarchy, thereby justifying the naturalness of the latter.[2]

§1. Set-Theoretic axioms as reflection principles. Cantor [11, p. 205, note 2] emphasizes the unknowability of the transfinite sequence of all ordinal numbers, which he thinks of as an "appropriate symbol of the absolute":

> The absolute can only be acknowledged, but never known, not even approximately known.

This principle of the unknowability of the absolute, which in Cantor's work seems to have only a metaphysical (non-mathematical) meaning (see [13]), resurfaces again in the 1950s in the work of Ackermann and Lévy, taking the mathematical form of a principle of reflection. Thus, in Ackermann's set theory—in fact, a theory of classes—which is formulated in the first-order language of set theory with an additional constant symbol for the class V of all sets, the idea of reflection is expressed in the form of an axiom schema of comprehension:

> *Ackermann's Reflection:* Let $\varphi(x, z_1, \dots, z_n)$ be a formula which does not contain the constant symbol V.
> Then for every $\vec{a} = a_1, \dots, a_n \in V$,
> $$\forall x(\varphi(x, \vec{a}) \to x \in V) \to \exists y(y \in V \wedge \forall x(x \in y \leftrightarrow \varphi(x, \vec{a}))).$$

A consequence of Ackermann's Reflection is that no formula can define V, or the class OR of all ordinal numbers, and is therefore in agreement

[2]The work presented in the following sections started over 10 years ago. After a talk I gave in Barcelona in 2011 on large cardinals as principles of structural reflection, John Baldwin, who attended the talk and was at the time editor of the BSL, encouraged me to write a survey article on the topic for the Bulletin. Well, here it is. I'm thankful to him for the invitation and I apologise for the long delay.

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with Cantor's principle of the unknowability of the absolute. However, Ackermann's set theory (with Foundation) was shown by Lévy [19] and Reinhardt [30] to be essentially equivalent to ZF, in the sense that both theories are equiconsistent and prove the same theorems about sets. Thus Ackermann's set theory did not provide any real advantage with respect to the simpler and intuitively clearer ZF axioms, and so it was eventually forgotten.

Later on, in the context of the wealth of independence results in set theory that were obtained starting in the mid-1960s thanks to the forcing technique, and as a result of the subsequent need for the identification of new set-theoretic axioms, Gödel (as quoted by Wang [36]), places Ackermann's principle (stated in a Cantorian, non-mathematical form) as the main source for new set-theoretic axioms beyond ZFC:

> All the principles for setting up the axioms of set theory should be reducible to a form of Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases.

Thus, according to Gödel, the fundamental guiding principle in setting up new axioms of set theory is the unknowability of the absolute, and so any new axiom should be based on such principle. Gödel's program consisted, therefore, in formulating stronger and stronger systems of set theory by adding to the base theory, which presumably could be taken as ZFC, new principles akin to Ackermann's.

So the question is how should one understand and formulate the idea of reflection embodied in Ackermann's principle. Some light is provided by Gödel in the following quote from [36, p. 285], where he asserts that the undefinability of V should be the source of all axioms of infinity, i.e., all large-cardinal axioms.

> Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is undefinable, where definability is to be taken in [a] more and more generalized and idealized sense.

One possible interpretation of Gödel's principle of the undefinability of V is as an unrestricted version of the Montague–Lévy Principle of Reflection. Namely: every formula, with parameters, in *any* formal language with the membership relation, that holds in V, must also hold in some V_α. This has been indeed the usual way to interpret Gödel's view of reflection as a justification for the axioms of large cardinals. This is made explicit in

Koellner [16, p. 208], where he identifies reflection principles with generalized forms of the Montague–Lévy Principle of Reflection. Namely,

> Reflection principles aim to articulate the informal idea that the height of the universe is "absolute infinite" and hence cannot be "characterized from below". These principles assert that any statement true in V is true in some smaller V_α.

Moreover, he explicitly interprets Gödel's view of reflection as a source of large cardinals in this way [16, p. 208]:

> Since the most natural way to assert that V is undefinable is via reflection principles and since to assert this in a "more and more generalized and idealized sense" is to move to languages of higher-order with higher-order parameters, Gödel is (arguably) espousing the view that higher-order reflection principles imply all large cardinal axioms.

This kind of reflection, namely generalised forms of the Principle of Reflection of Lévy and Montague allowing for the reflection of second-order formulas, has been used to justify weak large-cardinal axioms, such as the existence of inaccessible, Mahlo, or even weakly compact cardinals. Let us see briefly some examples to illustrate how the arguments work.

The Principle of Reflection also holds with proper classes as additional predicates. Namely, for every (definable, with set parameters) proper class A, and every n, there is an ordinal α such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

Thus, if we interpret second-order quantifiers over V as ranging only over definable classes, then for each n, the following second-order sentence, call it φ_n, is true in V:

$$\forall A \exists \alpha (V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

Now, applying reflection, φ_n must reflect to some V_κ. The second-order universal quantifier in φ_n is now interpreted in V_κ, hence ranging over *all* subsets of V_κ, which are now available in $V_{\kappa+1}$. So we have that for each subset A of V_κ, there is some $\alpha < \kappa$ such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V_\kappa, \in, A).$$

If this is so, and even just for $n = 1$, then κ must be an inaccessible cardinal, and this property actually characterises inaccessible cardinals [20]. Thus the existence of an inaccessible cardinal follows rather easily from the reflection of the single Π^1_1 sentence φ_1 to some V_κ.

Further, the following stronger form of reflection is provable in ZFC (as a schema): if C is a definable club proper class of ordinals, then for every proper class A, and every n, there is a club proper class of α in C such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A).$$

Hence, for each n, the following Π^1_1 sentence, with C as a second-order parameter, holds in V (again, interpreting the universal second-order quantifier as ranging over definable classes):

$$\forall A \forall \beta \exists \alpha > \beta (\alpha \in C \wedge (V_\alpha, \in, A \cap V_\alpha) \preceq_n (V, \in, A)).$$

Applying reflection, there is κ such that for every subset A of V_κ there are unboundedly many α in $C \cap \kappa$ such that

$$(V_\alpha, \in, A \cap V_\alpha) \preceq_n (V_\kappa, \in, A).$$

But since C is a club, $\kappa \in C$, and so (in the case of $n \geq 1$) κ is an inaccessible cardinal in C. Now consider the following Π^1_1 sentence, which we have just shown (using Π^1_1 reflection) to hold in V:

$$\forall C (C \text{ is a club subclass of ordinals} \rightarrow \exists \kappa (\kappa \text{ inaccessible} \wedge \kappa \in C)).$$

Any cardinal that reflects the Π^1_1 sentence consisting of the conjunction of φ_1 above with the last displayed sentence is an inaccessible cardinal λ with the property that every club subset of λ contains an inaccessible cardinal, i.e., λ is a Mahlo cardinal.

Furthermore, if we are willing to assume that all Π^1_1 sentences with second-order parameters reflect, we may as well reflect this property of V. So, consider the following Π^1_2 sentence, which says that V reflects all Π^1_1 sentences with parameters,

$$\forall A \forall \varphi \in \Pi^1_1 \exists \alpha ((V, \in, A) \models \varphi \rightarrow (V_\alpha, \in, A \cap V_\alpha) \models \varphi).$$

If V_κ reflects this sentence, then κ is a Π^1_1-indescribable cardinal, i.e., a weakly compact cardinal (see [14]). Similar arguments, applied to sentences of order n, but only allowing first-order and second-order parameters, yield the existence of Σ^m_n and Π^m_n indescribable cardinals.

While the arguments just given may seem reasonable, or even natural, we do not think they provide a justification for the existence of the large cardinals obtained in that way. The problem is that the relevant second-order statements are true in V only when interpreting second-order variables as ranging over *definable* classes, yet when the statements are reflected to some V_α the second-order variables are reinterpreted as ranging over the full power-set of V_α. It is precisely this transition from definable classes to the full power-set that yields the large-cardinal strength. Thus, the fundamental objection is that second-order reflection from V to some V_α, or to some set, is always problematic because so is unrestricted second-order quantification over V, as the full *power-class* of V is not available.

Nevertheless, even if one is willing to accept the existence of cardinals κ that reflect n-th-order sentences, with parameters of order greater than 2, one does not obtain large cardinals much stronger than the indescribable ones. Indeed, to start with, and as first noted in [29], one cannot even have reflection for Π^1_1 sentences with unrestricted third-order parameters. For suppose, towards a contradiction, that κ is a cardinal that reflects such sentences. Let A be the collection $\{V_\alpha : \alpha < \kappa\}$ taken as a third-order parameter, i.e., as a subset of $\mathcal{P}(V_\kappa)$. Then the Π^1_1 sentence

$$\forall X \exists x (X \in A \to X = x),$$

where X is a second-order variable and x is first-order, asserts that every element of A is a set. The sentence is clearly true in $(V_\kappa, \in, A)$, but false in any $(V_\alpha, \in, A \cap \mathcal{P}(V_\alpha))$ with $\alpha < \kappa$, because V_α belongs to $A \cap \mathcal{P}(V_\alpha)$ but is not an element of V_α.

One possible way around the problem of second-order reflection with third-order parameters is to allow such parameters, or even higher-order parameters, but to restrict the kind of sentences to be reflected. This is the approach taken by Tait [35]. He considers the class $\Gamma^{(2)}$ of formulas which, in normal form, have all universal quantifiers restricted to first-order and second-order variables and the only atomic formulas allowed to appear negated are either those of first order or of the form $x \in X$, where x is a variable of first order and X a variable of second order. Tait shows that reflection at some V_κ for the class of $\Gamma^{(2)}$ sentences, allowing parameters of arbitrarily high finite order, implies that κ is an ineffable cardinal (see [16]), and that V_κ reflects all such sentences whenever κ is a measurable cardinal. A sharper upper bound on the consistency strength of this kind of reflection is given by Koellner [16, Theorem 9]. He shows that below the first ω-Erdös cardinal, denoted by $\kappa(\omega)$, there exists a cardinal κ such that V_κ reflects all $\Gamma^{(2)}$ formulas. The existence of $\kappa(\omega)$ is, however, a rather mild large-cardinal assumption, since it is compatible with $V = L$.

At this point, the question is thus whether reflection can consistently hold (modulo large cardinals) for a wider class of sentences. But Koellner [16], building on some results of Tait, shows that no V_κ can reflect the class of formulas of the form $\forall X \exists Y \varphi(X, Y, Z)$, where X is of third-order, Y is of any finite order, Z is of fourth order, and φ has only first-order quantifiers and its only negated atomic subformulas are either of first order or of the form $x \in X$, where x is of first order and X is of second order. Other kinds of restrictions on the class of sentences to be reflected are possible (see [26] for the consistency of some forms of reflection slightly stronger than Tait's $\Gamma^{(2)}$), but Koellner [16] convincingly shows that the existence of a cardinal κ such that V_κ reflects any reasonable expansion of the class of sentences $\Gamma^{(2)}$, with parameters of order greater than 2, either follows from the existence of $\kappa(\omega)$ or is outright inconsistent.

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These results seem to put an end to the program of providing an *intrinsic*[3] justification of large-cardinal axioms, even for axioms as strong as the existence of $\kappa(\omega)$, by showing that their existence follows from strong higher-order reflection properties holding at some V_κ. In particular, the program cannot even provide justification for the existence of measurable cardinals. Thus, the conclusion is that if one understands reflection principles as asserting that some sentences (even of higher order, and with parameters) that hold in V must hold in some V_α, then reflection principles cannot be used to justify the existence of large cardinals up to or beyond $\kappa(\omega)$. Moreover, as we already emphasized, a more fundamental problem with the use of higher-order reflection principles is that either second-order quantification over V is interpreted as ranging over definable classes, in which case second-order reflection does not yield any large cardinals unless one makes the dubious jump from definable classes to the full power-set, or is ill-defined, as the full power-class of V does not exist.

1.1. A remark on the undefinability of *V*. Before we go on to propose a new kind of reflection principle, let us take a pause to consider another possible interpretation of Gödel's principle of the undefinability of V as a justification for large-cardinal axioms.

The statement that a set A is definable is usually understood in two different senses:

(1) There is a formula $\varphi(x)$ that defines A. That is, for every set a, a belongs to A if and only if $\varphi(a)$ holds. The formula φ may have parameters, provided they are *simpler* than A, e.g., their rank is less than the rank of A.

(2) A is the unique solution of a formula $\psi(x)$. Again, ψ may have parameters simpler than A.

There is, however, no essential difference between (1) and (2). For the formula $\varphi(x)$ defines a set A in the sense of (1) if and only if the formula $\forall x(x \in y \leftrightarrow \varphi(x))$ defines A in the sense of (2). But if A is a proper class, then (1) and (2) are very different, even if only because (2) needs to be reformulated to make any sense. If we understand definability as in (1), then there are many formulas that define V, for instance the formula $x = x$. So, the notion of undefinability of V can only be understood in the sense of (2), once properly reformulated. To express that V is not the unique solution of a formula, possibly with some sets as parameters, we need to make sense of the fact that a formula is true *of* V, as opposed to being true *in* V.

Let $\mathcal{L}_V$ be the first-order language of set theory expanded with a constant symbol $\bar{a}$ for every set a, and a new constant symbol v. Define the class $\mathcal{T}$

[3]See [16] for a discussion on *intrinsic* versus *extrinsic* justification of the axioms of set theory.

of sentences of $\mathcal{L}_V$ recursively as follows: φ belongs to $\mathcal{T}$ if and only if:

φ is of the form $\bar{a} \in \bar{b}$ or $\bar{a} \in v$, for some sets a, b such that $a \in b$, or
φ is of the form $\bar{a} = \bar{a}$ for some set a, or $v = v$, or
$\varphi \equiv \neg\psi$, and ψ does not belong to $\mathcal{T}$, or
$\varphi \equiv \psi \wedge \theta$, and both ψ and θ belong to $\mathcal{T}$, or
$\varphi \equiv \exists y \psi(y)$, and there is a set a such that $\psi(\bar{a})$ belongs to $\mathcal{T}$.

The idea is that if φ belongs to $\mathcal{T}$, then φ is true in the structure

$$\bar{V} := \langle V \cup \{V\}, \in, V, \langle \bar{a} \rangle_{a \in V} \rangle,$$

where the constant v is interpreted as V. And conversely, if φ is a sentence in the language $\mathcal{L}_V$ that is true in $\bar{V}$, then $\varphi \in \mathcal{T}$. Thus one may construe the principle of undefinability of V, as expressed in Gödel's quote above, as follows:

Undefinability of V: Every $\varphi \in \mathcal{T}$ is true in some

$$\bar{V}_\alpha := \langle V_\alpha \cup \{V_\alpha\}, \in, V_\alpha, \langle \bar{a} \rangle_{a \in V_\alpha} \rangle.$$

Of course, to express this principle in the first-order language of set theory one needs to do it as a schema. Namely, for each n let $\mathcal{T}_n$ be the class of Σ_n sentences of $\mathcal{T}$, and let

$$\Sigma_n\text{-Undefinability of } V: \text{ Every } \varphi \in \mathcal{T}_n \text{ is true in some } \bar{V}_\alpha.$$

Given a Σ_n sentence φ of the language of set theory, where $n \geq 1$, if it is true in V, i.e., if $\models_n \varphi$ holds, then the sentence φ^v obtained from φ by bounding all quantifiers by v belongs to $\mathcal{T}_n$. Hence, by Σ_n-*Undefinability of* V, φ^v is true in some $\bar{V}_\alpha$, and therefore $V_\alpha \models \varphi$. Thus *Undefinability of* V directly implies the Principle of Reflection of Montague–Lévy (over the theory ZF minus Infinity). Conversely, by induction on the complexity of φ, it is easily shown that ZF proves Σ_n-*Undefinability of* V. Thus, to derive stronger reflection principles based on the undefinability of V one needs to understand "definability," following Gödel's quote above, in a "more and more generalized and idealized sense." One could expand the class $\mathcal{T}$ by adding higher-order sentences that are true of V. However this will not lead us very far. For if $\mathcal{T}'$ is any reasonable class of (higher-order) sentences that are true of V, then *Undefinability of* V for the class $\mathcal{T}'$ will imply the reflection, in the sense of Montague–Lévy, of all sentences in $\mathcal{T}'$. Therefore, the limitations seen above of the extensions of the Montague–Lévy Principle of Reflection to higher-order formulas apply also to these generalized forms of *Undefinability of* V.

§2. Structural Reflection. The main obstacle for the program of finding an intrinsic justification of large-cardinal axioms via strong principles of reflection lies, we believe, on a too restrictive interpretation of the notion of reflection, namely the interpretation investigated by Tait, Koellner, and others, according to which the reflection properties of V are exhausted by generalized forms of the Montague–Lévy Principle of Reflection to higher-order logics.

Let us think again about the notion of reflection as derived from the Cantor–Ackermann principle of the unknowability of the absolute. A different interpretation of this principle may be extracted from another claim made by Gödel, as quoted in [36]:

> The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.

This quote does not immediately suggest that the uncharacterizability of V should be interpreted in the sense of the Montague–Lévy kind of reflection. Rather, what it seems to suggest is some sort of reflection, not (only) of formulas, but of *internal structural properties of the membership relation*. The quote does also state that the properties should be expressible in some logic, and any reasonable logic would do. So maybe Gödel is not saying here anything new, and he is simply advocating for a generalization of the Montague–Lévy type of reflection to formulas belonging to any (reasonable) kind of logic. But whatever the correct interpretation of Gödel's quote above may be, let us consider in some detail the idea of reflection of structural properties of the membership relation. Thus, what one would want to reflect is not the theory of V, but rather the *structural content* of V.

Whatever one might mean by a "structural property of the membership relation," it is clear that such a property should be exemplified in structures of the form $\langle A, \in, \langle R_i \rangle_{i \in I} \rangle$, where A is a set and $\langle R_i \rangle_{i \in I}$ is a family of relations on A, and where I is a set that may be empty. Moreover, any such property should be expressible by some formula of the language of set theory, maybe involving some set parameters. Thus, any internal structural property of the membership relation would be formally given by a formula $\varphi(x)$ of the first-order language of set theory, possibly with parameters, that defines a class of structures $\langle A, \in, \langle R_i \rangle_{i \in I} \rangle$ of the same type. As we shall see later on there is no loss of generality in considering only classes of structures whose members are of the form $\langle V_\alpha, \in, \langle R_i \rangle_{i \in I} \rangle$. Now, Gödel's vague assertion (as quoted above) that V "cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation" can be naturally interpreted in the sense that no formula $\varphi(x)$ characterizes V, meaning that some V_α reflects *the structural property defined by* $\varphi(x)$. Let us emphasize that what is reflected is not the formula $\varphi(x)$, but the structural property defined by $\varphi(x)$, i.e., the class of structures defined by $\varphi(x)$. This is the crucial difference with the Montague–Lévy type of reflection. The most natural way to make this precise is to assert that there exists an ordinal α such that for every structure A in the class (i.e., for every structure A that satisfies $\varphi(x)$) there exists a structure B also in the class which belongs to V_α and is *like* A. Since, in general, A may be much larger than any B in V_α, the closest resemblance of B to A is attained in the case B is isomorphic to an elementary substructure of A, i.e., B can be elementarily embedded into A. We can now formulate (an informal and preliminary version of) the general principle of Structural Reflection as follows:

SR: (*Structural Reflection*) For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\alpha$ and an elementary embedding from B into A.

Observe that when $\mathcal{C}$ is a set the principle becomes trivial.

We do not wish to claim that the SR principle is what Gödel had in mind when talking about reflection of internal structural properties of the membership relation, but we do claim that SR is a form of reflection that derives naturally from the Cantor–Ackermann principle of unknowability of the Absolute and is at least compatible with Gödel's interpretation of this principle.

In the remaining sections we will survey a collection of results showing the equivalence of different forms of SR with the existence of different kinds of large cardinals. Our goal is to illustrate the fact that SR is a general principle underlying a wide variety of large-cardinal principles. Many of the results have already been published [2–4, 7–9, 23] or are forthcoming [6], but some are new (Theorems 4.1, 4.2, 4.5, and 4.7–4.9, Proposition 5.1, Theorem 5.11, Proposition 5.17, and Theorems 5.18, 5.19, 5.21, and 7.6). Each of these results should be regarded as a small step towards the ultimate objective of showing that all large cardinals are in fact different manifestations of a single general reflection principle.

§3. From supercompactness to Vopěnka's Principle. We shall begin with the SR principle, as stated above, which is properly formulated in the first-order language of set theory as an axiom schema. Namely, for each natural number n let

$\mathbf{\Sigma_n}$-SR: ($\mathbf{\Sigma_n}$-*Structural Reflection*) For every $\mathbf{\Sigma}_n$-definable, with parameters, class $\mathcal{C}$ of relational structures of the same type there is an ordinal α that *reflects* $\mathcal{C}$.

$\mathbf{\Pi_n}$-SR may be formulated analogously. We may also define the *lightface*, i.e., parameter-free versions (as customary we use the lightface types Σ_n and Π_n for that). Namely,

Σ_n-SR: (Σ_n-*Structural Reflection*) For every Σ_n-definable, without parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *reflects* $\mathcal{C}$.

Similarly for Π_n-SR.

A standard closing-off argument shows that $\mathbf{\Sigma_n}$-SR is equivalent to the assertion that there exists a proper class of ordinals α such that α reflects *all* classes of structures of the same type that are Σ_n-definable, with parameters in V_α. Also, Σ_n-SR is equivalent to the assertion that there exists an ordinal α that reflects *all* classes of structures of the same type that are Σ_n-definable, without parameters. Similarly for $\mathbf{\Pi_n}$-SR and Π_n-SR. Thus, for Γ a *definability class* (i.e., one of $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, Σ_n, or Π_n), let us say that an ordinal α *witnesses* Γ-SR if α reflects all classes of structures of the same type that are Γ-definable (allowing for parameters in V_α, in the case of boldface classes). Then, in the case of boldface classes Γ, Γ-SR holds if and only if Γ-SR is witnessed by a proper class of ordinals.

The first observation is that, as the next proposition shows, $\mathbf{\Sigma_1}$-SR is provable in ZFC. Recall[4] that, for every $n > 0$, $C^{(n)}$ is the Π_n-definable club proper class of cardinals κ such that $V_\kappa \preceq_{\Sigma_n} V$, i.e., V_κ is a Σ_n-elementary substructure of V. In particular, every element of $C^{(1)}$ is an uncountable cardinal and a fixed point of the $\beth$ function.

PROPOSITION 3.1. *The following are equivalent for every ordinal* α:

(1) α *witnesses* $\mathbf{\Sigma_0}$-SR.
(2) α *witnesses* $\mathbf{\Sigma_1}$-SR.
(3) $\alpha \in C^{(1)}$.

PROOF. The implication (2)$\Rightarrow$(1) is trivial. The implication (3)$\Rightarrow$(2) is proved in [3], using a Löwenheim–Skolem type of argument.

To show that (1) implies (3), let α witness $\mathbf{\Sigma_0}$-SR and suppose $\varphi(a_1, \dots, a_n)$ is a Σ_1 sentence, with parameters $a_1, \dots, a_n$ in V_α, that holds in V. Let $\mathcal{C}$ be the Σ_0-definable, with $a_1, \dots, a_n$ as parameters, class of structures of the form

$$\langle M, \in, \{a_1, \dots, a_n\}\rangle,$$

[4]See [2].

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where M is a transitive set that contains $a_1, \dots, a_n$. Let M be any transitive set such that $M \models \varphi(a_1, \dots, a_n)$. By SR, there is an elementary embedding

$$j : \langle N, \in, \{a_1, \dots, a_n\}\rangle \to \langle M, \in, \{a_1, \dots, a_n\}\rangle,$$

where $\langle N, \in \{a_1, \dots, a_n\}\rangle \in \mathcal{C} \cap V_\alpha$. Since j fixes $a_1, \dots, a_n$, by elementarity $N \models \varphi(a_1, \dots, a_n)$, and by upwards absoluteness for Σ_1 sentences with respect to transitive sets, $V_\alpha \models \varphi(a_1, \dots, a_n)$. This shows $\alpha \in C^{(1)}$. $\dashv$

Thus, $\boldsymbol{\Sigma_1}$-SR is provable in ZFC, and therefore does not yield any large cardinals. But Π_1-SR does, and is indeed very strong. The following theorem hinges on Magidor's characterization of the first supercompact cardinal as the first cardinal that reflects the Π_1-definable class of structures of the form $\langle V_\alpha, \in\rangle$, α an ordinal [24].

THEOREM 3.2 [2, 3]. *The following are equivalent*:

(1) Π_1-SR.
(2) Σ_2-SR.
(3) *There exists a supercompact cardinal.*

The proof of the theorem shows in fact that the following are equivalent for an ordinal κ:

(1) κ is the least ordinal that witnesses SR for the Π_1-definable class of structures $\langle V_\alpha, \in\rangle$, α an ordinal.
(2) κ is the least cardinal that witnesses Π_1-SR.
(3) κ is the least cardinal that witnesses $\boldsymbol{\Sigma_2}$-SR.
(4) κ is the least supercompact cardinal.

The following global parametrized version then follows. Namely,

THEOREM 3.3 [2, 3]. *The following are equivalent*:

(1) $\boldsymbol{\Pi_1}$-SR.
(2) $\boldsymbol{\Sigma_2}$-SR.
(3) *There exists a proper class of supercompact cardinals.*

The proof of the theorem also shows that if κ witnesses $\boldsymbol{\Pi_1}$-SR, then κ is either supercompact or a limit of supercompact cardinals.

Some remarks are in order. First, the equivalence of Π_1-SR and Σ_2-SR, and also of their boldface forms, is due to the following general fact. Given a Σ_{n+1} definable (possibly with parameters, and with $n > 0$) class $\mathcal{C}$ of relational structures of the same type, let $\mathcal{C}^*$ be the class of structures of the form $\langle V_\alpha, \in, A\rangle$, where α is the least cardinal in $C^{(n)}$ such that $A \in V_\alpha$ and $V_\alpha \models \varphi(A)$, where $\varphi(x)$ is a fixed Σ_{n+1} formula that defines $\mathcal{C}$. Then,

$$A \in \mathcal{C} \text{ if and only if } \langle V_\alpha, \in, A\rangle \in \mathcal{C}^*.$$

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Now notice that $\mathcal{C}^*$ is Π_n definable, with the same parameters as $\mathcal{C}$ (see [2]). Moreover, if a cardinal κ reflects the class $\mathcal{C}^*$, then it also reflects $\mathcal{C}$: for if $A \in \mathcal{C}$, let α be the least cardinal in $C^{(n)}$ such that $\langle V_\alpha, \in, A\rangle \models \varphi(A)$, where $\varphi(x)$ is a fixed Σ_{n+1} formula that defines $\mathcal{C}$. Let $j : \langle V_\beta, \in, B\rangle \to \langle V_\alpha, \in, A\rangle$ be elementary with $\langle V_\beta, \in, B\rangle \in \mathcal{C}^* \cap V_\kappa$. Then, since $\beta \in C^{(n)}$ and $V_\beta \models \varphi(B)$, we have that $B \in \mathcal{C}$ and the restriction map $j \restriction A : A \to B$ is an elementary embedding.

For P a set or a proper class and Γ a definability class, we shall write $\Gamma(P)$-SR for the assertion that SR holds for all Γ-definable, with parameters in P, classes of structures of the same type. Thus, e.g., $\boldsymbol{\Sigma}_{\mathbf{n}}$-SR is $\Sigma_n(V)$-SR, and Σ_n-SR is $\Sigma_n(\varnothing)$-SR. Our remarks above yield now the following:

PROPOSITION 3.4. *For P any set or proper class, the assertions* $\Pi_n(P)$-SR *and* $\Sigma_{n+1}(P)$-SR *are equivalent. In particular* Π_n-SR *and* Σ_{n+1}-SR *are equivalent, and so are* $\boldsymbol{\Pi}_{\mathbf{n}}$-SR *and* $\boldsymbol{\Sigma}_{\mathbf{n+1}}$-SR.

Second, the remarks above also show that for principles of Structural Reflection of the form $\Gamma(P)$-SR the relevant structures to consider are those of the form $\langle V_\alpha, \in, A\rangle$, where $A \in V_\alpha$. Let us say that a structure is *natural* if it is of this form. Therefore, we may reformulate Γ-SR, for Γ a lightface definability class, as follows:

Γ-SR: (Γ-*Structural Reflection. Second version*) There exists a cardinal κ that reflects all Γ-definable classes $\mathcal{C}$ of natural structures, i.e., for every $A \in \mathcal{C}$ there exist $B \in \mathcal{C} \cap V_\kappa$ and an elementary embedding $j : B \to A$.

The version for Γ a boldface definability class being as follows:

> There exists a proper class of cardinals κ that reflect all Γ-definable, with parameters in V_κ, classes $\mathcal{C}$ of natural structures.

At the next level of definitional complexity, i.e., $n = 2$, we have the following:

THEOREM 3.5 [2, 3]. *The following are equivalent*:

(1) Π_2-SR.
(2) *There exists an extendible cardinal.*

The proof of the theorem shows that the first extendible cardinal is precisely the first cardinal that witnesses SR for one particular Π_2-definable class of natural structures. The parameterized version also holds:

THEOREM 3.6 [2, 3]. *The following are equivalent*:

(1) $\boldsymbol{\Pi}_{\mathbf{2}}$-SR.
(2) *There exists a proper class of extendible cardinals.*

Moreover, if κ witnesses $\mathbf{\Pi_2}$-SR, then κ is either extendible or a limit of extendible cardinals.

For the higher levels of definitional complexity we need the notion of $C^{(n)}$-extendible cardinal from [2, 3]: κ is $C^{(n)}$-*extendible* if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$. Note that the only difference with the notion of extendibility is that we require the image of the critical point to be in $C^{(n)}$. Also note that every extendible cardinal is $C^{(1)}$-extendible. We then have the following level-by-level characterizations of SR in terms of the existence of large cardinals:

THEOREM 3.7 [2, 3]. *The following are equivalent for $n \geq 1$:*

(1) Π_{n+1}-SR.
(2) *There exists a $C^{(n)}$-extendible cardinal.*

THEOREM 3.8 [2, 3]. *The following are equivalent for $n \geq 1$:*

(1) $\mathbf{\Pi_{n+1}}$-SR.
(2) *There exists a proper class of $C^{(n)}$-extendible cardinals.*

Similarly as in the case of supercompact and extendible cardinals, the proofs of the theorems above actually show that the first $C^{(n)}$-extendible cardinal is the first cardinal that witnesses SR for one single Π_{n+1}-definable class of natural structures. Also, if κ witnesses $\mathbf{\Pi_{n+1}}$-SR, then κ is either a $C^{(n)}$-extendible cardinal or a limit of $C^{(n)}$-extendible cardinals.

Recall that *Vopěnka's Principle* (VP) is the assertion that for every proper class $\mathcal{C}$ of relational structures of the same type there exist $A \neq B$ in $\mathcal{C}$ such that A is elementarily embeddable into B. In the first-order language of set theory VP can be formulated as a schema. The following corollary to the theorems stated above yields a characterization of VP in terms of SR. Moreover, it shows that, globally, the lightface and boldface forms of SR are equivalent.

THEOREM 3.9 [2, 3]. *The following schemata are equivalent*:

(1) SR, *i.e.*, $\mathbf{\Pi_n}$-SR *for all n.*
(2) Π_n-SR *for all n.*
(3) *There exists a $C^{(n)}$-extendible cardinal, for every n.*
(4) *There is a proper class of $C^{(n)}$-extendible cardinals, for every n.*
(5) VP.

§4. Structural Reflection below supercompactness. We have just seen that a natural hierarchy of large cardinals in the region between the first supercompact cardinal and VP can be characterized in terms of SR. Now the question is if the same is true for other well-known regions of the large cardinal hierarchy. Since $\mathbf{\Sigma_1}$-SR is provable in ZFC and Π_1-SR implies already the existence of a supercompact cardinal, if large cardinals weaker than supercompact admit a characterization as principles of structural reflection, then we need to look either for SR restricted to particular (collections of) Π_1-definable classes of structures, or for classes of structures whose definitional complexity is between Σ_1 and Π_1 (e.g., Σ_1-definability with additional Π_1 predicates), or for weaker forms of structural reflection. Let us consider first the SR principle restricted to particular definable classes of structures contained in canonical inner models.

4.1. Structural Reflection relative to canonical inner models. There is one single class $\mathcal{C}$ of structures in L that is Π_1-definable in V, without parameters, and such that $\mathrm{SR}(\mathcal{C})$ is equivalent to the existence of $0^\sharp$. Namely, let $\mathcal{C}$ be the class of structures of the form $\langle L_\beta, \in, \gamma\rangle$, with $\gamma < \beta$ uncountable cardinals (in V).

THEOREM 4.1. *The following are equivalent*:

(1) $\mathrm{SR}(\mathcal{C})$.
(2) $0^\sharp$ *exists.*

PROOF. (1) implies (2): Suppose that α reflects $\mathcal{C}$. Pick V-cardinals $\gamma < \beta$ with $\alpha \leq \gamma$. By reflection, there are V-cardinals $\gamma' < \beta' < \alpha$ and an elementary embedding

$$j : \langle L_{\beta'}, \in, \gamma'\rangle \to \langle L_\beta, \in, \gamma\rangle.$$

Since $j(\gamma') = \gamma$, j is not the identity. Let κ be the critical point of j. Thus, $\kappa \leq \gamma' < \beta'$. Hence by Kunen's Theorem [18] $0^\sharp$ exists.

(2) implies (1): Assume $0^\sharp$ exists. Let α be an uncountable limit cardinal in V. We claim that α reflects $\mathcal{C}$. For suppose $\langle L_\beta, \in, \gamma\rangle \in \mathcal{C}$ with $\alpha \leq \beta$. Let $\gamma' < \beta' < \alpha$ be uncountable cardinals in V such that $\gamma' \leq \gamma$. Let I denote the class of Silver indiscernibles. Let $j : I \cap [\gamma', \beta'] \to I \cap [\gamma, \beta]$ be order-preserving and such that $j(\gamma') = \gamma$ and $j(\beta') = \beta$. Then j generates an elementary embedding

$$j : \langle L_{\beta'}, \in, \gamma'\rangle \to \langle L_\beta, \in, \gamma\rangle,$$

as required. ⊣

The existence of $0^\sharp$ yields also the SR principle restricted to classes of structures that are definable in L.

THEOREM 4.2. *If* $0^\sharp$ *exists, then* $\mathrm{SR}(\mathcal{C})$ *holds for every class* $\mathcal{C}$ *that is definable in* L, *with parameters.*

PROOF. Fix $\mathcal{C}$ and a formula $\varphi(x)$, possibly with ordinals $\alpha_0 < \cdots < \alpha_m$ as parameters, that defines it in L. Let κ be a limit of Silver indiscernibles greater than α_m. We claim that κ reflects $\mathcal{C}$. For suppose $B \in \mathcal{C}$. Without loss of generality, $B \notin L_\kappa$. Since $0^\sharp$ exists, there is an increasing sequence of Silver indiscernibles $i_0, \dots, i_n, i_{n+1}$ and a formula $\psi(y, z_0, \dots, z_n)$, without parameters, such that

$$B = \{y : L_{i_{n+1}} \models \psi(y, i_0, \dots, i_n)\}.$$

Choose indiscernibles $j_0 < \cdots < j_n < j_{n+1} < \kappa$, with $\alpha_m < j_0$, and let

$$A = \{y : L_{j_{n+1}} \models \psi(y, j_0, \dots, j_n)\}.$$

Thus $A \in L_\kappa$. Since $L \models \varphi(B)$, we have that

$$L \models \forall x(\forall y(y \in x \leftrightarrow L_{i_{n+1}} \models \psi(y, i_0, \dots, i_n)) \to \varphi(x)).$$

By indiscernibility,

$$L \models \forall x(\forall y(y \in x \leftrightarrow L_{j_{n+1}} \models \psi(y, j_0, \dots, j_n)) \to \varphi(x)),$$

which implies $L \models \varphi(A)$, i.e., $A \in \mathcal{C}$.

Let $j : L \to L$ be an elementary embedding that sends i_k to j_k, all $k \leq n+1$. Then by indiscernibility, the map $j \restriction A : A \to B$ is an elementary embedding. ⊣

However, the SR principle restricted to classes of structures that are definable in L falls very short of yielding $0^\sharp$, as we shall next show. Let us recall the following definition:

Definition 4.3 [4]. A cardinal κ is *n-remarkable*, for $n > 0$, if for all $\lambda > \kappa$ in $C^{(n)}$ and every $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega, <\kappa)}$ there exists an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$ and $a \in \mathrm{range}(j)$.

A cardinal κ is 1-remarkable if and only if it is remarkable, in the sense of Schindler (see [4] and Definition 7.2).

If $0^\sharp$ exists, then every Silver indiscernible is completely remarkable in L (i.e., n-remarkable for every $n > 0$). Moreover, the consistency strength of the existence of a 1-remarkable cardinal is strictly weaker than the existence of a 2-iterable cardinal, which in turn is weaker than the existence of an ω-Erdös cardinal (see [4]).

A weaker notion than n-remarkability is obtained by eliminating from its definition the requirement that $j(\mathrm{crit}(j)) = \kappa$. So, let's define:

Definition 4.4. A cardinal κ is *almost n-remarkable*, for $n > 0$, if for all $\lambda > \kappa$ in $C^{(n)}$ and every $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega, <\kappa)}$ there exists an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $a \in \mathrm{range}(j)$.

We say that κ is *almost completely remarkable* if it is almost-n-remarkable for every n.

THEOREM 4.5. *A cardinal κ is almost n-remarkable if and only if in $V^{\mathrm{Coll}(\omega,<\kappa)}$ κ witnesses* $\mathrm{SR}(\mathcal{C})$ *for every class $\mathcal{C}$ that is Π_n-definable in V with parameters in V_κ.*

PROOF. Assume κ is almost n-remarkable. Fix $\mathcal{C}$ and a Π_n formula $\varphi(x)$ that defines it in V, possibly with parameters in V_κ. Suppose $B \in \mathcal{C}$. In V, let $\lambda \in C^{(n)}$ be greater than the rank of B. Thus, $V_\lambda \models \varphi(B)$. Since κ is almost n-remarkable, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there exists an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $B \in \mathrm{range}(j)$. Let A be the preimage of B under j. So $A \in V_\kappa$. By elementarity of j, $V_{\bar{\lambda}} \models \varphi(A)$. Hence, since $\bar{\lambda} \in C^{(n)}$, $A \in \mathcal{C}$. Moreover, $j \restriction A : A \to B$ is an elementary embedding.

Conversely, assume that in $V^{\mathrm{Coll}(\omega,<\kappa)}$, κ witnesses $\mathrm{SR}(\mathcal{C})$ for every class $\mathcal{C}$ that is Π_n-definable in V with parameters in V_κ. Let $\mathcal{C}$ be the Π_n-definable class of structures of the form $\langle V_\alpha, \in, a\rangle$ where $\alpha \in C^{(n)}$ and $a \in V_\alpha$. Given $\lambda \in C^{(n)}$ and $a \in V_\lambda$, in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there exists some $\langle V_{\bar{\lambda}}, \in, b\rangle \in \mathcal{C}$ together with an elementary embedding

$$j : \langle V_{\bar{\lambda}}, \in, b\rangle \to \langle V_\lambda, \in, a\rangle.$$

Since $j(b) = a$, $a \in \mathrm{range}(j)$. This shows κ is almost n-remarkable. $\dashv$

COROLLARY 4.6. *If κ is an almost completely remarkable cardinal in L, then in $L^{\mathrm{Coll}(\omega,<\kappa)}$ κ witnesses* $\mathrm{SR}(\mathcal{C})$ *for all classes $\mathcal{C}$ of structures that are definable in L with parameters in L_κ.*

Similar arguments yield analogous results for $X^\sharp$, for every set of ordinals X. Given a set of ordinals X, let $\mathcal{C}_X$ be the class of structures of the form $\langle L_\beta[X], \in, \gamma\rangle$, where γ and β are cardinals (in V) and $sup(X) < \gamma < \beta$. Clearly, $\mathcal{C}_X$ is Π_1 definable with X as a parameter.

THEOREM 4.7.

(1) $\mathrm{SR}(\mathcal{C}_X)$ *holds if and only if $X^\sharp$ exists.*
(2) *If $X^\sharp$ exists, then* $\mathrm{SR}(\mathcal{C})$ *holds for all classes $\mathcal{C}$ that are definable in $L[X]$, with parameters.*

These results suggest the following forms of SR restricted to inner models. Let M be an inner model. Writing M_α for $(V_\alpha)^M$, consider the following principle for Γ a lightface definability class:

Γ-$\mathrm{SR}(M)$: (Γ-*Structural Reflection for M*) There exists an ordinal α that reflects every Γ-definable class $\mathcal{C}$ of relational structures of the same type such that $\mathcal{C} \subseteq M$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap M_\alpha$ and an elementary embedding j from B into A. (*Warning: j may not be in M.*)

The corresponding version for a boldface Γ being as follows:

> There exists a proper class of ordinals α that reflect every Γ-definable, with parameters in M_α, class $\mathcal{C}$ of relational structures of the same type such that $\mathcal{C} \subseteq M$.

The last theorem shows that, for every set of ordinals X, the existence of $X^\sharp$ implies $\mathrm{SR}(L[X])$, i.e., $\mathbf{\Sigma_n}$-$\mathrm{SR}(L[X])$, for every n.

Similar results may be obtained for other canonical inner models, e.g., $L[U]$, the canonical inner model for one measurable cardinal κ, where U is the (unique) normal measure on κ in $L[U]$. Let $\mathcal{C}^U$ be the class of structures of the form $\langle L_\beta[U], \in, \gamma\rangle$, with $\gamma < \beta$ uncountable cardinals (in V). The class $\mathcal{C}^U$ is Π_1-definable in V, with U as a parameter. By using well-known facts about $0^\dagger$ due to Solovay (see [15] 21), and arguing similarly as in Theorems 4.1 and 4.2, respectively, one obtains the following:

THEOREM 4.8. *The following are equivalent*:

(1) $\mathrm{SR}(\mathcal{C}^U)$ *holds if and only if* $0^\dagger$ *exists.*
(2) *If* $0^\dagger$ *exists, then* $\mathrm{SR}(\mathcal{C})$ *holds for every class* $\mathcal{C}$ *that is definable in* $L[U]$, *with parameters.*

Also, similarly as in Theorem 4.7, one can obtain the analogous result for $X^\dagger$, for every set of ordinals X. Namely, given a set of ordinals X, let $\mathcal{C}^U_X$ be the class of structures of the form $\langle L_\beta[U, X], \in, \gamma\rangle$, where γ and β are cardinals (in V) and $sup(X) < \gamma < \beta$. Then $\mathcal{C}^U_X$ is Π_1-definable with U and X as parameters.

THEOREM 4.9.

(1) $\mathrm{SR}(\mathcal{C}^U_X)$ *holds if and only if* $X^\dagger$ *exists.*
(2) *If* $X^\dagger$ *exists, then* $\mathrm{SR}(\mathcal{C})$ *holds for all classes* $\mathcal{C}$ *that are definable in* $L[U, X]$, *with parameters.*

Analogous results should also hold for canonical inner models for stronger large-cardinal notions. For example, for the canonical inner model $L[\mathcal{E}]$ for a strong cardinal, as in [17] or [27], and its sharp, *zero pistol* $0^\P$. Also for a canonical inner model for a proper class of strong cardinals, as in [32], and its sharp, *zero hand grenade*. For inner models for stronger large cardinal notions, e.g., one Woodin cardinal, the situation is less clear, although analogous results should hold given the appropriate canonical inner model and its corresponding sharp.

§5. Product Structural Reflection. Recall that for any set S of relational structures $\mathcal{A} = \langle A, \dots\rangle$ of the same type, the set-theoretic product $\prod S$ is the structure whose universe is the set of all functions f with domain S such that $f(\mathcal{A}) \in A$, for every $\mathcal{A} \in S$, and whose relations are defined point-wise.

In this section we shall consider the following general product form of structural reflection, which is a variation of the Product Reflection Principle (PRP) introduced in [9]:

PSR: (*Product Structural Reflection*) For every definable class of relational structures $\mathcal{C}$ of the same type, τ, there exists an ordinal α that *product-reflects* $\mathcal{C}$, i.e., for every $\mathcal{A}$ in $\mathcal{C}$ there exists a set S of structures of type τ (although not necessarily in $\mathcal{C}$) with $\mathcal{A} \in S$ and an elementary embedding $j : \prod(\mathcal{C} \cap V_\alpha) \to \prod S$.

Similarly as in the case of SR (see Section 2), we may formally define PSR as a schema. Thus, we say that an ordinal α witnesses $\Gamma(P)$-PSR (where Γ is a definability class and P a set or a proper class) if α product-reflects all classes $\mathcal{C}$ that are Γ-definable with parameters in P. Our remarks in Section 2 also apply here. In particular, an ordinal α witnesses $\mathbf{\Pi_n}$-PSR if and only if it witnesses $\mathbf{\Sigma_{n+1}}$-PSR. Moreover, we obtain equivalent principles by restricting to classes of natural structures. Thus, for Γ a lightface definability class, we define:

Γ-PSR: (Γ-*Product Structural Reflection*) There exists an ordinal α that product-reflects all Γ-definable classes $\mathcal{C}$ of natural structures.

The corresponding version for Γ boldface being as follows:

There exists a proper class of ordinals α that product-reflect all Γ-definable, with parameters in V_α, classes $\mathcal{C}$ of natural structures.

As in [9, Proposition 3.2] one can show that every cardinal κ in $C^{(1)}$ witnesses $\mathbf{\Sigma_1}$-PSR. The converse also holds, and in fact we have the following:

Proposition 5.1. *For every n, if κ witnesses* $\mathbf{\Pi_n}$-PSR, *then* $\kappa \in C^{(n+1)}$.

Proof. We shall prove the case $n = 1$. The general case follows by induction, using a similar argument. The case $n = 0$ is similar to the case $n = 1$, but simpler, as it suffices to consider a class of structures with domain a transitive set (see the proof of Proposition 3.1). So, suppose $\varphi(x, y)$ is a Π_1 formula with x, y as the only free variables, $a \in V_\kappa$, and $V \models \exists x \varphi(x, a)$. Let $\mathcal{C}$ be the Π_1-definable, with a as a parameter, class of structures of the form $\mathcal{A}_\alpha = \langle V_\alpha, \in, a, \{R^\alpha_\varphi\}_{\varphi \in \Pi_1}\rangle$ with $\alpha \in C^{(1)}$, and where $\{R^\alpha_\varphi\}_{\varphi\in\Pi_1}$ is the Π_1 relational diagram for $\langle V_\alpha, \in, a\rangle$, i.e., if $\varphi(x_1, \dots, x_n, \overline{a})$, $\overline{a}$ a constant symbol, is a Π_1 formula in the language of $\langle V_\alpha, \in, a\rangle$, then

$$R^\alpha_\varphi = \{\langle a_1, \dots, a_n\rangle : \langle V_\alpha, \in, a\rangle \models \text{“}\varphi[a_1, \dots, a_n, a]\text{”}\}.$$

Let $\lambda > \kappa$ be in $C^{(2)}$, so that $V_\lambda \models \exists x \varphi(x, a)$. By PSR there exists a set S that contains $\mathcal{A}_\lambda$ and an elementary embedding

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$$j : \prod_{\alpha<\kappa} \mathcal{A}_\alpha \to \prod S.$$

Since $\mathcal{A}_\lambda \models \exists x \varphi(x, a)$, we have $R^\lambda_\varphi \neq \varnothing$. Hence, since $\mathcal{A}_\lambda \in S$,

$$\prod S \not\models \langle R^\alpha_\varphi \rangle_{\mathcal{A}_\alpha \in S} = \varnothing,$$

and therefore, by elementarity of j,

$$\prod_{\alpha<\kappa} \mathcal{A}_\alpha \not\models \langle R^\alpha_\varphi \rangle_{\alpha<\kappa} = \varnothing,$$

which implies that $\mathcal{A}_\alpha \models R^\alpha_\varphi \neq \varnothing$, for some $\alpha < \kappa$. Hence, $\mathcal{A}_\alpha \models \exists x \varphi(x, a)$.

Note that, if $\varphi(x, y)$ had been a bounded formula, instead of Π_1, then we would have, by upward absoluteness, that $\mathcal{A}_\kappa \models \exists x \varphi(x, y)$, thus showing that $\kappa \in C^{(1)}$. Thus, since $\alpha, \kappa \in C^{(1)}$, we have that $V_\alpha \preceq_{\Sigma_1} V_\kappa$, and therefore $V_\kappa \models \exists x \varphi(x, a)$.

Now suppose $V_\kappa \models \exists x \varphi(x, a)$. Since $\kappa \in C^{(1)}$, by upward absoluteness, $V \models \exists x \varphi(x, a)$. ⊣

Recall that a cardinal κ is *λ-strong*, where $\lambda > \kappa$, if there exists an elementary embedding $j : V \to M$, with M transitive, $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and with V_λ contained in M. A cardinal κ is *strong* if it is λ-strong for every cardinal $\lambda > \kappa$.

The following proposition is proved similarly as in [9, Proposition 3.3].

Proposition 5.2. *If κ is a strong cardinal, then κ witnesses* $\mathbf{\Pi_1}$-PSR.

Proof. Let κ be a strong cardinal and let $\mathcal{C}$ be a Π_1-definable, with parameters in V_κ, proper class of structures in a fixed relational language $\tau \in V_\kappa$. Let $\varphi(x)$ be a Π_1 formula defining it.

Given any $\mathcal{A} \in \mathcal{C}$, let $\lambda \in C^{(1)}$ be greater than or equal to κ and with $\mathcal{A} \in V_\lambda$.

Let $j : V \to M$ be an elementary embedding, with $crit(j) = \kappa$, $V_\lambda \subseteq M$, and $j(\kappa) > \lambda$.

By elementarity, the restriction of j to $\mathcal{C} \cap V_\kappa$ yields an elementary embedding

$$h : \prod (\mathcal{C} \cap V_\kappa) \to \prod (\{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}).$$

Let $S := \{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}$. Since $\mathcal{A} \in V_\lambda$ and $\varphi(x)$ is Π_1, by downward absoluteness $V_\lambda \models \varphi(\mathcal{A})$. Hence, since the fact that $\lambda \in C^{(1)}$ is Π_1-expressible and therefore downwards absolute for transitive classes, and since $V_\lambda \subseteq M$, it follows that $V_\lambda \preceq_{\Sigma_1} M$ and therefore $M \models \varphi(\mathcal{A})$. Moreover $\mathcal{A} \in V_\lambda \subseteq V^M_{j(\kappa)}$. Thus, $\mathcal{A} \in S$. ⊣

Since the Product Reflection Principle (PRP) introduced in [9] when restricted to Π_1-definable classes is an easy consequence of Π_1-PSR, from [9] we obtain the following:

THEOREM 5.3. *The following are equivalent*:

(1) Π_1-PSR.
(2) *There exists a strong cardinal.*

as well as its boldface version:

THEOREM 5.4. *The following are equivalent*:

(1) $\mathbf{\Pi_1}$-PSR.
(2) *There exists a proper class of strong cardinals.*

This shows that strong cardinals are related to PSR as supercompact cardinals are to SR (Theorems 3.2 and 3.3). For the higher levels of definability, i.e., $n > 1$, the large cardinal notion that corresponds to Π_n-PSR, analogous to the notion of $C^{(n)}$-extendible cardinal in the case of SR, is the following:

DEFINITION 5.5 [9]. For Γ a definability class and an ordinal λ, a cardinal κ is λ-Γ-*strong* if for every Γ-definable (without parameters) class A there is an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

A cardinal κ is Γ-*strong* if it is λ-Γ-strong for every ordinal λ.

As with the case of strong cardinals, standard arguments show (cf. [15] 26.7(b)) that κ is λ-Γ-*strong* if and only if for every Γ-definable (without parameters) class A there is an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $j(\kappa) > \lambda$, and $A \cap V_\lambda \subseteq j(A)$. As shown in [9], every strong cardinal is Σ_2-strong. Also, a cardinal is Π_n-strong if and only if is Σ_{n+1}-strong. Moreover, if $n \geq 1$ and $\lambda \in C^{(n+1)}$, then the following are equivalent for a cardinal $\kappa < \lambda$:

(1) κ is λ-Π_n-strong.
(2) There is an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $M \models$ "$\lambda \in C^{(n)}$."

Similarly as in Proposition 5.2, one can prove the following (see [9] for details):

PROPOSITION 5.6. *If κ is a Π_n-strong cardinal, then κ witnesses* $\mathbf{\Pi_n}$-PSR.

The following theorem then follows from the main result in [9]:

THEOREM 5.7. *The following are equivalent for $n \geq 1$*:

(1) Π_n-PSR.
(2) *There exists a Π_n-strong cardinal.*

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The corresponding boldface version also holds. Namely,

THEOREM 5.8. *The following are equivalent for* $n \geq 1$:

(1) $\mathbf{\Pi_n}$-PSR.
(2) *There exists a proper class of* Π_n*-strong cardinals.*

Finally, for the PSR principle, the statement analogous to Vopěnka's Principle in the case of SR is the following:

DEFINITION 5.9 [9]. Ord *is Woodin* if for every definable $A \subseteq V$ there exists some α which is A-strong, i.e., for every γ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Note that if δ is a Woodin cardinal (see [15] for the definition of Woodin cardinal and its equivalent formulation in terms of A-strength), then V_δ satisfies Ord is Woodin. The following equivalences then follow (cf. Theorem 3.9):

THEOREM 5.10 [9]. *The following schemata are equivalent*:

(1) PSR, *i.e.*, $\mathbf{\Pi_n}$-PSR *for all* n.
(2) Π_n-PSR *for all* n.
(3) *There exists a* Π_n*-strong cardinal, for every* n.
(4) *There is a proper class of* Π_n*-strong cardinals, for every* n.
(5) Ord *is Woodin*.

A close inspection of the proofs of Propositions 5.2 and 5.6 reveals that, for $n > 0$, if κ is a Π_n-strong cardinal, then for every Π_n-definable (with parameters in V_κ) class $\mathcal{C}$ of relational structures of the same type τ, and for every $\beta \geq \kappa$, there exists a set S of structures of type τ (although possibly not in $\mathcal{C}$) that contains $\mathcal{C} \cap V_\beta$ and there exists an elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$ with the following properties:

(1) *Faithful*: For every $f \in \prod(\mathcal{C} \cap V_\kappa)$, $h(f) \restriction (\mathcal{C} \cap V_\kappa) = f$.
(2) $\subseteq$*-chain-preserving*: If $f \in \prod(\mathcal{C} \cap V_\kappa)$ is so that $f(\mathcal{A}) \subseteq f(\mathcal{A}')$ whenever $A \subseteq A'$, then so is $h(f)$.

Moreover, if κ witnesses Π_n-PSR, then some cardinal less than or equal to κ is Π_n-strong. Thus, the following is an equivalent reformulation of Γ-PSR, for $\Gamma = \Gamma_n$ a lightface definability class with $n > 0$:

Γ-PSR: (Γ-*Product Structural Reflection. Second version*) There exists a cardinal κ that *product-reflects* all Γ-definable proper classes $\mathcal{C}$ of relational structures of the same type τ, i.e., for every β there exists a set S of structures of type τ that contains $\mathcal{C} \cap V_\beta$, and there exists a faithful and $\subseteq$-chain-preserving elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$.

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The corresponding version for a boldface Γ being as follows:

> There exists a proper class of cardinals κ that *product-reflect* all Γ-definable, with parameters in V_κ, proper class $\mathcal{C}$ of relational structures of the same type.

The next theorem implies that strong cardinals can be characterized in terms of Π_1-PSR. The proof follows closely [9, Theorem 5.1], with the properties of faithfulness and $\subseteq$-chain preservation ((1) and (2) above) playing now a key role.

THEOREM 5.11. *There is a* Π_1*-definable, without parameters, class* $\mathcal{C}$ *of natural structures such that if a cardinal* κ *product-reflects* $\mathcal{C}$ *(second version), then* κ *is a strong cardinal.*

PROOF. Let C be the class of all ordinals $\alpha < \kappa$ of uncountable cofinality such that α is the α-th element of $C^{(1)}$. Let $\mathcal{C}$ be the Π_1-definable class of all structures

$$\mathcal{A}_\alpha := \langle V_{\lambda_\alpha}, \in, \alpha \rangle,$$

where $\alpha \in C$, and λ_α is the least cardinal in $C^{(1)}$ greater than α

Note that, since by Proposition 5.1, $\kappa \in C^{(2)}$, κ is a limit point of C.

Pick any γ in C greater than κ. We will show that κ is γ-strong. By PSR there is a faithful $\subseteq$-chain-preserving elementary embedding $j : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$, where S is some set with $(\mathcal{C} \cap V_\kappa) \cup \{\mathcal{A}_\gamma\} \subseteq S$.

Now pick any $\mathcal{A}_\beta \in \mathcal{C} \cap S$ and let

$$h_\beta : \prod S \to \mathcal{A}_\beta$$

be the projection map. Let $I := C \cap \kappa$ and define

$$k_\beta : V_{\kappa+1} \to V_{\beta+1}$$

by

$$k_\beta(X) = h_\beta(j(\{X \cap V_\alpha\}_{\alpha \in I})).$$

Since j is elementary, for all formulas $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in \prod(\mathcal{C} \cap V_\kappa)$, if $\prod(\mathcal{C} \cap V_\kappa) \models \varphi[a_1, \dots, a_n]$, then $\prod S \models \varphi[j(a_1) \dots, j(a_n)]$. It easily follows that k_β preserves Boolean operations, the subset relation, and is the identity on $\omega + 1$.

Note that $k_\beta(\kappa) = h_\beta(j(\{\alpha\}_{\alpha \in I})) = \beta$.

For each $a \in [\beta]^{<\omega}$, define E^β_a by

$$X \in E^\beta_a \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k_\beta(X)\,.$$

Since $k_\beta(\kappa) = \beta$ and $k_\beta(|a|) = |a|$, we also have $k_\beta([\kappa]^{|a|}) = [\beta]^{|a|}$, hence $[\kappa]^{|a|} \in E^\beta_a$. Since k_β preserves Boolean operations and the $\subseteq$ relation, E^β_a is a proper ultrafilter over $[\kappa]^{|a|}$. Moreover, since $k_\beta(\omega) = \omega$, a simple argument shows that E^β_a is ω_1-complete, hence the ultrapower $\mathrm{Ult}(V, E^\beta_a)$ is well-founded. Furthermore, since j is faithful, if $\beta < \kappa$, then E^β_a is the principal ultrafilter generated by $\{a\}$. Let

$$j^\beta_a : V \to M^\beta_a \cong \mathrm{Ult}(V, E^\beta_a),$$

with M^β_a transitive, be the corresponding ultrapower embedding. Note that if $\beta < \kappa$, then $M^\beta_a = V$ and j^β_a is the identity.

Let $\mathcal{E}_\beta := \{E^\beta_a : a \in [\beta]^{<\omega}\}$. As in [9], one can show that $\mathcal{E}_\beta$ is normal and coherent. Thus, for each $a \subseteq b$ in $[\beta]^{<\omega}$ the maps $i^\beta_{ab} : M^\beta_a \to M^\beta_b$ given by

$$i^\beta_{ab}([f]_{E^\beta_a}) = [f \circ \pi_{ba}]_{E^\beta_b}$$

for all $f : [\kappa]^{|a|} \to V$, are well-defined and commute with the ultrapower embeddings j^β_a (see [15, p. 26]).

Let $M_{\mathcal{E}_\beta}$ be the direct limit of the directed system

$$\langle\langle M^\beta_a : a \in [\beta]^{<\omega}\rangle, \langle i^\beta_{ab} : a \subseteq b\rangle\rangle,$$

and let $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ be the corresponding direct limit elementary embedding, i.e.,

$$j_{\mathcal{E}_\beta}(x) = [a, [c^a_x]_{E^\beta_a}]_{\mathcal{E}_\beta}$$

for some (any) $a \in [\beta]^{<\omega}$, and where $c^a_x : [\kappa]^{|a|} \to \{x\}$.

As in [9] one can also show that $M_{\mathcal{E}_\beta}$ is well-founded. So, let $\pi_\beta : M_{\mathcal{E}_\beta} \to N_\beta$ be the transitive collapse, and let $j_{N_\beta} : V \to N_\beta$ be the corresponding elementary embedding, i.e., $j_{N_\beta} = \pi \circ j_{\mathcal{E}_\beta}$. Then, as in [9] we can show that $V_\beta \subseteq N_\beta$ and $j_{N_\beta}(\kappa) \geq \beta$. If $\beta > \kappa$, this implies that $\mathrm{crit}(j_{N_\beta}) \leq \kappa$. (If $\beta < \kappa$, then $j_{N_\beta} : V \to V$ is the identity.)

Let $I_S := \{\beta : \mathcal{A}_\beta \in \mathcal{C} \cap S\}$.

Claim 5.12. If $\beta \leq \beta'$ are in I_S, then $E^\beta_a = E^{\beta'}_a$, for every $a \in [\beta]^{<\omega}$. ⊣

Proof of claim. Since $E^\beta_a, E^{\beta'}_a$ are proper ultrafilters over $[\kappa]^{|a|}$, it is sufficient to see that $E^\beta_a \subseteq E^{\beta'}_a$. So, suppose $X \in E^\beta_a$. Then $a \in k_\beta(X) = h_\beta(j(\{X \cap V_\alpha\}_{\alpha \in I}))$. Since $\{X \cap V_\alpha\}_{\alpha \in I}$ forms a $\subseteq$-chain and j is $\subseteq$-chain-preserving, so does $j(\{X \cap V_\alpha\}_{\alpha \in I})$. Hence, $k_\beta(X) \subseteq k_{\beta'}(X)$, and therefore $a \in k_{\beta'}(X)$, which yields $X \in E^{\beta'}_a$. ⊣

By the claim above, for every $\beta < \beta'$ in I_S the map

$$k_{\beta,\beta'} : M_{\mathcal{E}_\beta} \to M_{\mathcal{E}_{\beta'}}$$

given by

$$k_{\beta,\beta'}([a, [f]_{E^\beta_a}]_{\mathcal{E}_\beta}) = [a, [f]_{E^{\beta'}_a}]_{\mathcal{E}_{\beta'}}$$

is well-defined and elementary. Moreover, it commutes with the embeddings $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ and $j_{\mathcal{E}_{\beta'}} : V \to M_{\mathcal{E}_{\beta'}}$. Let M be the direct limit of

$$\langle\langle M_{\mathcal{E}_\beta} : \beta \in I' \rangle, \langle k_{\beta,\beta'} : \beta < \beta' \text{ in } I_S \rangle\rangle,$$

and let $j_M : V \to M$ be the corresponding direct limit elementary embedding, which is given by

$$j_M(x) = [\beta, [a, [c_x^a]_{E_a^\beta}]_{\mathcal{E}_\beta}]$$

for some (any) $a \in [\beta]^{<\omega}$. Let $\pi^M : M \to N$ be the transitive collapse, and let $j_N = \pi^M \circ j_M : V \to N$.

Let $\xi = \sup(I_S)$. Note that, as $\gamma \in I_S$, $\xi > \kappa$.

Claim 5.13. $j_N(\kappa) = \xi$.

Proof of claim. As in [9], we can show that $j_{N_\beta}(\kappa) \geq \beta$, for every $\beta \in I_S \setminus \kappa$. So, for such a β, letting $\ell_{\beta,N}$ be the unique elementary embedding such that $j_N = \ell_{\beta,N} \circ j_{N_\beta}$, we have

$$j_N(\kappa) = \ell_{\beta,N}(j_{N_\beta}(\kappa)) \geq \ell_{\beta,N}(\beta) \geq \beta.$$

Hence, $j_N(\kappa) \geq \xi$. Also, $j_{E_a^\beta}(\kappa)$ can be computed in V_β, for all $a \in [\beta]^{<\omega}$, and therefore $j_{N_\beta}(\kappa) \leq \beta$. Hence, $j_N(\kappa) \leq \xi$. ⊣

Since $\kappa < \xi$, it follows from the claim above that $\mathrm{crit}(j_N) \leq \kappa$. But since for $\beta < \kappa$ the map j_{N_β} is the identity, we must have $\mathrm{crit}(j_N) = \kappa$. Also, since $\gamma \in I_S$, $V_\gamma \subseteq N_\gamma$, hence $V_\gamma \subseteq N$. This shows that κ is γ-strong, as wanted.

From Proposition 5.2 and Theorem 5.11 we obtain now the following characterization of strong cardinals.

Corollary 5.14. *A cardinal κ is strong if and only if it witnesses* Π_1-PSR (*second version*).

Similar results can be proven for Γ-strong cardinals (Definition 5.5). On the one hand, Proposition 5.6 shows that if κ is Π_n-strong, then κ witnesses $\mathbf{\Pi_{n+1}}$-PSR. On the other hand, similarly as in Theorem 5.11, we can prove that if κ witnesses Π_n-PSR, then κ is Π_n-strong. This yields the following characterization of Π_n-strong cardinals:

Theorem 5.15. *For every $n > 0$, a cardinal κ is Π_n-strong if and only if it witnesses* Π_n-PSR (*second version*).

5.1. Strong Product Structural Reflection. Let us consider next the following, arguably more natural, strengthening of PSR:

SPSR: (*Strong Product Structural Reflection*) For every definable class of relational structures $\mathcal{C}$ of the same type, τ, there exists an ordinal α that *strongly product-reflects* $\mathcal{C}$, i.e., for every $\mathcal{A}$ in $\mathcal{C}$ there

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exist an ordinal β with $\mathcal{A} \in V_\beta$ and an elementary embedding $j : \prod(\mathcal{C} \cap V_\alpha) \to \prod(\mathcal{C} \cap V_\beta)$.

Similarly as in the case of PSR, let us say that a cardinal κ witnesses $\Gamma(P)$-SPSR if κ strongly product-reflects all classes $\mathcal{C}$ that are Γ-definable (with parameters in P). Also, a cardinal κ witnesses $\mathbf{\Pi_n}$-SPSR if and only if it witnesses $\mathbf{\Sigma_{n+1}}$-SPSR, and similarly for the lightface definability classes. Moreover, we obtain equivalent principles by restricting to classes of natural structures. Thus, we may formally define Γ-SPSR, for Γ a lightface definability class, as follows:

Γ-SPSR: (Γ-*Strong Product Structural Reflection*) There exists a cardinal κ that strongly product-reflects all Γ-definable classes $\mathcal{C}$ of natural structures.

The boldface version is as follows:

There exists a proper class of cardinals κ that strongly product-reflect all Γ-definable, with parameters in V_κ, class $\mathcal{C}$ of natural structures.

Note that Γ-SPSR implies Γ-PSR, for any definability class Γ.

We shall see next that the large cardinal notions that correspond to the SPSR principle are those of superstrong, globally superstrong, and $C^{(n)}$-globally superstrong cardinals.

Definition 5.16 [10]. A cardinal κ is *superstrong above* λ, for some $\lambda \geq \kappa$, if there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_{j(\kappa)} \subseteq M$.

A cardinal κ is *globally superstrong* if it is superstrong above λ, for every $\lambda \geq \kappa$.

More generally, a cardinal κ is $C^{(n)}$-*superstrong above* λ, for some $\lambda \geq \kappa$, if there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_{j(\kappa)} \subseteq M$, and $j(\kappa) \in C^{(n)}$.

A cardinal κ is $C^{(n)}$-*globally superstrong* if it is $C^{(n)}$-superstrong above λ, for every $\lambda \geq \kappa$.

Note that every globally superstrong cardinal is $C^{(1)}$-globally superstrong. Also, every globally superstrong cardinal is superstrong, and every $C^{(n)}$-globally superstrong cardinal belongs to $C^{(n+2)}$ [10]. As shown in [10], on the one hand, if κ is $C^{(n)}$-globally superstrong, then there are many $C^{(n)}$-superstrong cardinals below κ. On the other hand, if κ is $(\kappa+1)$-extendible, then V_κ satisfies that there is a proper class of $C^{(n)}$-globally superstrong cardinals, for every n. Moreover, if κ is $C^{(n)}$-extendible, then there are many $C^{(n)}$-globally superstrong cardinals below κ.

Similarly as in Proposition 5.2 we can prove the following:

PROPOSITION 5.17. *If κ is $C^{(n)}$-globally superstrong, then it witnesses* $\mathbf{\Pi_n}$-SPSR.

PROOF. Let κ be $C^{(n)}$-globally superstrong and let $\mathcal{C}$ be a class of relational structures of the same type that is definable by a Π_n formula $\varphi(x)$, with parameters in V_κ.

Given any $\mathcal{A} \in \mathcal{C}$, let $\lambda \in C^{(n)}$ be greater than or equal to κ and with $\mathcal{A} \in V_\lambda$.

Let $j : V \to M$ be an elementary embedding witnessing that κ is $C^{(n)}$-superstrong above λ. Then the restriction of j to $\mathcal{C} \cap V_\kappa$ yields an elementary embedding

$$h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}).$$

Let $S := \{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}$. Since $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$, we have $V_{j(\kappa)} \models \varphi(\mathcal{A})$. Hence, since $\kappa \in C^{(n)}$ (in fact $\kappa \in C^{(n+2)}$) [10], by elementarity $V_{j(\kappa)} = V^M_{j(\kappa)} \preceq_{\Sigma_n} M$, and thus $M \models \varphi(\mathcal{A})$. It follows that $\mathcal{A} \in S$. Moreover, if $\mathcal{B} \in S$, then $M \models \varphi(\mathcal{B})$. Hence, since $V^M_{j(\kappa)} \preceq_{\Sigma_n} M$, we have that $V^M_{j(\kappa)} = V_{j(\kappa)} \models \varphi(\mathcal{B})$. Since $j(\kappa) \in C^{(n)}$, $\varphi(\mathcal{B})$ holds in V, and therefore $\mathcal{B} \in \mathcal{C}$. This shows that $S = \mathcal{C} \cap V_{j(\kappa)}$, and so

$$h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\mathcal{C} \cap V_{j(\kappa)}).$$

Hence, h witnesses SPSR for $\mathcal{A}$. ⊣

Observe that since the function h in the proof of the last proposition is the restriction of j to $\prod(\mathcal{C} \cap V_\kappa)$, and j is elementary, it preserves all first-order properties. In particular, it is $\subseteq$-chain-preserving, and since $\kappa = \mathrm{crit}(j)$, h is faithful. Thus, taking into consideration our remarks from Section 2, as well as those made in the previous section before we stated the second version of the PSR schema, we may reformulate Γ-SPSR for Γ a lightface definability class as follows:

Γ-SPSR: (Γ-*Strong Product Structural Reflection. Second version*) There exists a cardinal κ that *strongly product-reflects* all Γ-definable classes $\mathcal{C}$ of natural structures, i.e., for every $\mathcal{A} \in \mathcal{C}$ there exist an ordinal β with $\mathcal{A} \in V_\beta$ and a faithful and $\subseteq$-chain-preserving elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\mathcal{C} \cap V_\beta)$.

The corresponding version for boldface Γ being:

> There exists a proper class of cardinals κ that *strongly product-reflect* all Γ-definable, with parameters in V_κ, proper classes $\mathcal{C}$ of natural structures.

Arguing similarly as in the proof of Theorem 5.11 (and [9]), we can now prove the following:

THEOREM 5.18. *For every $n > 0$, there is a Π_n-definable, without parameters, class $\mathcal{C}$ of natural structures such that if a cardinal κ strongly product-reflects $\mathcal{C}$, then κ is a $C^{(n)}$-globally superstrong cardinal.*

PROOF. Let $\mathcal{C}$ be the Π_n-definable class of all structures

$$\mathcal{A}_\alpha := \langle V_{\lambda_\alpha}, \in, \alpha \rangle,$$

where α has uncountable cofinality and is the α-th element of $C^{(n)}$, and λ_α is the least cardinal in $C^{(n)}$ greater than α.

Let κ witness SPSR for $\mathcal{C}$. Let $I := \{\alpha : \mathcal{A}_\alpha \in V_\kappa\}$. Since $\kappa \in C^{(n+1)}$ (Proposition 5.1), $\sup(I) = \kappa$.

Pick any ordinal $\lambda \geq \kappa$ and let us show that κ is λ-$C^{(n)}$-superstrong. Let $\mathcal{A}_\beta$ in $\mathcal{C}$ with $\lambda < \beta$. Let κ' be such that $\mathcal{A}_\beta \in V_{\kappa'}$ and there is a faithful $\subseteq$-chain-preserving elementary embedding

$$j : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\mathcal{C} \cap V_{\kappa'}).$$

Let

$$h_\beta : \prod(\mathcal{C} \cap V_{\kappa'}) \to \mathcal{A}_\beta$$

be the projection map and define $k_\beta : V_{\kappa+1} \to V_{\beta+1}$ by

$$k_\beta(X) = h_\beta(j(\{X \cap V_\alpha\}_{\alpha \in I})).$$

As in Theorem 5.11, for each $a \in [\beta]^{<\omega}$, define E_a^β by

$$X \in E_a^\beta \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k_\beta(X).$$

Then E_a^β is an ω_1-complete proper ultrafilter over $[\kappa]^{|a|}$, and so the ultrapower $\mathrm{Ult}(V, E_a^\beta)$ is well-founded. Furthermore, since j is faithful, if $\beta \in I$, then E_a^β is the principal ultrafilter generated by $\{a\}$. Let

$$j_a^\beta : V \to M_a^\beta \cong \mathrm{Ult}(V, E_a^\beta),$$

and let $\mathcal{E}_\beta := \{E_a^\beta : a \in [\beta]^{<\omega}\}$. As in [9], $\mathcal{E}_\beta$ is normal and coherent. Let $M_{\mathcal{E}_\beta}$ be the direct limit of

$$\langle \langle M_a^\beta : a \in [\beta]^{<\omega} \rangle, \langle i_{ab}^\beta : a \subseteq b \rangle \rangle,$$

where the i_{ab}^β are the standard projection maps, and let $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ be the corresponding limit elementary embedding. As in [9], $M_{\mathcal{E}_\beta}$ is well-founded. So, let $\pi_\beta : M_{\mathcal{E}_\beta} \to N_\beta$ be the transitive collapse, and let $j_{N_\beta} = \pi \circ j_{\mathcal{E}_\beta} : V \to N_\beta$. We have that $V_\beta \subseteq N_\beta$ and $j_{N_\beta}(\kappa) \geq \beta$ (see [9]).

By Claim 5.12, if $\beta \leq \beta'$ are in $I' := \{\alpha : \mathcal{A}_\alpha \in V_{\kappa'}\}$, then $E^\beta_a = E^{\beta'}_a$, for every $a \in [\beta]^{<\omega}$. Hence, for every $\beta < \beta'$ in I', the map

$$k_{\beta,\beta'} : M_{\mathcal{E}_\beta} \to M_{\mathcal{E}_{\beta'}}$$

given by

$$k_{\beta,\beta'}([a,[f]_{E_a}]_{\mathcal{E}_\beta}) = [a,[f]_{E_a}]_{\mathcal{E}_{\beta'}}$$

is well-defined, elementary, and commutes with the embeddings $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ and $j_{\mathcal{E}_{\beta'}} : V \to M_{\mathcal{E}_{\beta'}}$. Let M be the direct limit of

$$\langle\langle M_{\mathcal{E}_\beta} : \beta \in I'\rangle, \langle k_{\beta,\beta'} : \beta < \beta' \text{ in } I'\rangle\rangle,$$

and let $j_M : V \to M$ be the corresponding limit elementary embedding. Let $\pi^M : M \to N$ be the transitive collapse, and let $j_N = \pi^M \circ j_M : V \to N$.

Let $\xi = \sup(I')$. Note that $\xi \in C^{(n)}$ and $\xi > \kappa$. As in Claim 5.13, $j_N(\kappa) = \xi$, hence $\mathrm{crit}(j_N) \leq \kappa$. But since for $\beta \in I$ the map j_{N_β} is the identity, $\mathrm{crit}(j_N) = \kappa$. Also, since $V_\xi = \bigcup_{\beta\in I} V_\beta$, and $V_\beta \subseteq N_\beta$ for all $\beta \in I$, it follows that $V_\xi \subseteq N$. This shows that κ is ξ-superstrong, hence also λ-$C^{(n)}$-superstrong, as wanted. ⊣

We have thus proved the following:

Theorem 5.19. *For every $n \geq 1$, the following are equivalent for any cardinal κ*:

(1) *κ witnesses* Π_n-SPSR.
(2) *κ is a $C^{(n)}$-globally superstrong cardinal.*

Corollary 5.20. *The following are equivalent*:

(1) SPSR, *i.e.*, $\mathbf{\Pi_n}$-SPSR *for every n.*
(2) Π_n-SPSR *for every n.*
(3) *There exists a $C^{(n)}$-globally superstrong cardinal, for every n.*
(4) *There exists a proper class of $C^{(n)}$-globally superstrong cardinals.*

5.2. Bounded Product Structural Reflection. Let us consider next some *bounded* forms of PSR. Namely, for Γ a lightface definability class and any ordinal β let:

Γ-PSR$_\beta$: There exists a cardinal κ that *β-product-reflects* every Γ-definable proper class $\mathcal{C}$ of natural structures, i.e., for every $\mathcal{A}$ in $\mathcal{C}$ of rank $\leq \kappa + \beta$ there exist a set S with $\mathcal{A}$ in S and an elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$.

Thus Γ-PSR holds if and only if there exists a cardinal κ that witnesses Γ-PSR$_\beta$ for all (equivalently, a proper class of) ordinals β.

The following theorem shows that measurable cardinals can be characterized in terms of bounded PSR.

THEOREM 5.21. *The following are equivalent*:

(1) Π_1-PSR_1.
(2) *There exists a measurable cardinal.*

PROOF. (1) implies (2): Let $\mathcal{C}$ be the Π_1-definable class of $\langle V_\gamma, \in\rangle$, $\gamma \geq \omega$. Let κ witness PSR_1 for $\mathcal{C}$, and let S be a set that contains $V_{\kappa+1}$ such that there exists an elementary embedding

$$h : \prod(\mathcal{C} \cap V_\kappa) \to \prod S.$$

Define $k : V_{\kappa+1} \to V_{\kappa+2}$ by

$$k(X) = h_{\kappa+1}(h(\{X \cap V_\gamma\}_{\gamma<\kappa})),$$

where $h_{\kappa+1}$ is the projection on $V_{\kappa+1}$. Then k preserves Boolean operations and the subset relation, and is the identity on $\omega + 1$. Moreover, $k(\kappa) = \kappa + 1$. Now for each $a \in V_{\kappa+1}$, define $\mathcal{U}_a$ by

$$X \in \mathcal{U}_a \quad \text{iff} \quad X \subseteq \kappa \text{ and } a \in k(X).$$

Clearly, $\kappa \in \mathcal{U}_a$. Also, since k preserves Boolean operations and the $\subseteq$ relation, $\mathcal{U}_a$ is a proper ultrafilter over κ. Moreover, since $k(\omega) = \omega$, $\mathcal{U}_a$ is ω_1-complete. Furthermore, since $|V_{\kappa+1}| = 2^{|V_\kappa|} > |\kappa|$, some $\mathcal{U}_a$ is non-principal. So, some cardinal less than or equal to κ is measurable.

(2) implies (1): Let κ be a measurable cardinal, and let $\varphi(x)$ be a Π_1 formula (we may allow parameters in V_κ) that defines a proper class $\mathcal{C}$ of natural structures. We claim that κ 1-product-reflects $\mathcal{C}$.

Let $j : V \to M$ be an ultrapower elementary embedding, given by some κ-complete normal measure over κ. Thus, $\mathrm{crit}(j) = \kappa$ and $V^M_{\kappa+1} = V_{\kappa+1}$. By elementarity, the restriction of j to $\mathcal{C} \cap V_\kappa$ yields an elementary embedding

$$h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}).$$

Let $S := \{X : M \models \varphi(X)\} \cap V^M_{j(\kappa)}$. If $\mathcal{A} = \langle V_\gamma, \in\rangle \in \mathcal{C}$, with $\gamma \leq \kappa + 1$, then $\mathcal{A} \in M$, and since $\varphi(\mathcal{A})$ holds in V, by Π_1 downward absoluteness for transitive classes it also holds in M. Moreover, since $j(\kappa) > \kappa + 1$, $\mathcal{A} \in V^M_{j(\kappa)}$, hence $\mathcal{A} \in S$. ⊣

Let us note that the proof of the theorem above also shows that the least measurable cardinal is precisely the least cardinal κ that witnesses PSR_1 for all classes $\mathcal{C}$ that are Π_1-definable with parameters in V_κ.

§6. Large cardinals below measurability. We shall next consider Structural Reflection for classes of relational structures that are Σ_1-definable in the language of set theory extended with additional Π_1 predicates. That is, classes of structures of complexity between Σ_1 and Σ_2.

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Let $\mathcal{R}$ be a set of Π_1 predicates or relations. A class $\mathcal{C}$ of structures in a fixed countable relational type is said to be $\Sigma_1(\mathcal{R})$-definable if it is definable by means of a Σ_1 formula of the first-order language of set theory with additional predicate symbols for the predicates in $\mathcal{R}$, without parameters. We define the following form of SR:

$\Sigma_1(\mathcal{R})$-SR: For every $\Sigma_1(\mathcal{R})$-definable class $\mathcal{C}$ of structures of the same type there exists a cardinal κ that *reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\kappa$ and an elementary embedding from B into A.

For the rest of this section we shall write $\mathrm{SR}_{\mathcal{R}}$ instead of the more cumbersome $\Sigma_1(\mathcal{R})$-SR. Also, if $\mathcal{R} = \{R_1, \dots, R_n\}$, then we may write $\mathrm{SR}_{R_1,\dots,R_n}$ for $\mathrm{SR}_{\mathcal{R}}$.

We have that $\mathrm{SR}_{\varnothing}$, i.e., Σ_1-SR, is provable in ZFC (Proposition 3.1). However, if R is the Π_1 relation "x is an ordinal and $y = V_x$," then SR_R holds if and only if there exists a supercompact cardinal ([2, 3]; see also Theorem 3.2). Moreover, if κ is supercompact, then $\mathrm{SR}_{\mathcal{R}}$ holds for κ, for any set $\mathcal{R}$ of Π_1 predicates (cf. Theorem 3.2).

6.1. The principle $\mathrm{SR}^-_{\mathcal{R}}$**.** For Γ any lightface definability class, the following is a natural restricted form of Γ-SR:

Γ-SR^-: There exists a cardinal κ such that for every Γ-definable class $\mathcal{C}$ of structures of the same type and every $A \in \mathcal{C}$ of cardinality κ there exist $B \in \mathcal{C} \cap H_\kappa$ and an elementary embedding from B into A. We say that the cardinal κ κ-*reflects* $\mathcal{C}$.

The restriction of SR^- to $\Sigma_1(\mathcal{R})$-definable classes of structures was first introduced in [8]. Namely, for $\mathcal{R}$ a finite set of Π_1 predicates or relations, let

$\mathrm{SR}^-_{\mathcal{R}}$: There exists a cardinal κ that κ-reflects every $\Sigma_1(\mathcal{R})$-definable (with parameters in H_κ) class $\mathcal{C}$ of structures of the same type.

6.1.1. The Cardinality predicate. Let Cd be the Π_1 predicate "x is a cardinal." Magidor and Väänänen [25] show that the principle SR_{Cd} implies $0^\sharp$, and much more, e.g., there are no good scales. The principle SR^-_{Cd} is much weaker, but it does have some large-cardinal strength, as the next theorem shows.

THEOREM 6.1 [8]. SR^-_{Cd} *holds, witnessed by* κ, *then there exists a weakly inaccessible cardinal* $\lambda \leq \kappa$.

It is shown in [25] that, starting form a supercompact cardinal, one can produce a model of ZFC in which SR_{Cd} holds for the first weakly inaccessible cardinal. Thus, no large-cardinal properties beyond weak inaccessibility may be proved in ZFC to hold for the least cardinal witnessing SR_{Cd}.

6.1.2. The Regularity predicate. Let Rg be the Π_1 predicate "x is a regular ordinal."

THEOREM 6.2 [8]. *If* SR^-_{Rg} *holds, witnessed by* κ, *then there exists a weakly Mahlo cardinal* $\lambda \leq \kappa$.

It follows from [25] that one cannot hope to get from SR_{Rg} more than a weakly Mahlo cardinal $\leq \kappa$, for starting from a weakly Mahlo cardinal one can obtain a model in which SR_{Rg} is witnessed by the least weakly Mahlo cardinal. One cannot hope either to show that the least κ witnessing SR_{Rg} is strongly inaccessible, for in [34] it is shown that one can have SR_{Rg} witnessed by $\kappa = 2^{\aleph_0}$.

Let us note that, since the predicate Cd is $\Sigma_1(Rg)$-definable (see [8]), the principle $\mathrm{SR}_{Cd,Rg}$ is equivalent to SR_{Rg}.

6.1.3. The Weakly Inaccessible predicate. There is a principle between SR^-_{Cd} and SR^-_{Rg}, namely $\mathrm{SR}^-_{Cd,WI}$, where WI is the Π_1 predicate "x is weakly inaccessible."

PROPOSITION 6.3 [8]. *If* $\mathrm{SR}^-_{Cd,WI}$ *holds, witnessed by* κ, *then there exists a* 2*-weakly inaccessible cardinal* $\lambda \leq \kappa$.

We may also consider predicates α-WI, for α an ordinal. That is, the predicate "x is α-weakly inaccessible." Then, similar arguments as in [8] would show that the principle $\mathrm{SR}^-_{Cd,\alpha\text{-}WI}$ holding for κ implies that there is an $(\alpha+1)$-weakly inaccessible cardinal $\lambda \leq \kappa$.

6.1.4. Weak compactness. Let $WC(x,\alpha)$ be the Π_1 relation "α is a limit ordinal and x is a partial ordering with no chain of order-type α."

THEOREM 6.4 [8]. *If* $\mathrm{SR}^-_{Cd,WC}$ *holds, witnessed by some* κ *such that if* $\gamma \leq \kappa$ *is weakly inaccessible, then* $2^\delta \leq \gamma$ *for all cardinals* $\delta < \gamma$, *then there exists a weakly compact cardinal* $\lambda \leq \kappa$.[5]

Since the first weakly Mahlo cardinal may satisfy SR_{Rg} [25], we cannot prove the existence of a weakly compact cardinal $\leq \kappa$ just from SR_{Rg}. Hence, $\mathrm{SR}_{Cd,WC}$ is stronger than SR_{Rg}.

Let $PwSet$ be the Π_1 relation $\{(x,y) : y = \mathcal{P}(x)\}$. Then we have the following:

[5]In [8] it is only assumed that κ witnesses $\mathrm{SR}^-_{Cd,WC}$. However, Lücke [23] has shown that some additional assumption on κ is needed. Note that our assumption on κ in the current statement of the theorem follows form the GCH.

THEOREM 6.5 [24]. *κ is the least cardinal witnessing* SR_{PwSet} *if and only if κ is the first supercompact cardinal.*

It follows from Theorem 3.2 that SR_{PwSet} is in fact equivalent to Π_1-SR, and also equivalent to Σ_2-SR. However, Lücke [23] has established that SR^-_{PwSet} is much weaker than SR_{PwSet}. Indeed, he shows that SR^-_{PwSet} is equivalent to the existence of a weakly shrewd cardinal, a large cardinal notion obtained by weakening the definition of shrewd cardinal studied by Rathjen in [28] and whose consistency strength is strictly between the large cardinal notions of total indescribability and subtleness. However, as shown in [23], shrewd and weakly shrewd cardinal are equiconsistent.

DEFINITION 6.6 [23]. A cardinal κ is *weakly shrewd* if for every formula $\varphi(x, y)$ of the language of set theory, every cardinal $\theta > \kappa$, and every $A \subseteq \kappa$ such that $\varphi(A, \kappa)$ holds in H_θ, there exist cardinals $\bar{\kappa} < \bar{\theta}$ such that $\bar{\kappa} < \kappa$ and $\varphi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $H_{\bar{\theta}}$.

THEOREM 6.7 [23]. *The following are equivalent*:

(1) *κ is the least weakly shrewd cardinal.*
(2) *κ is the least cardinal witnessing* SR^-_{PwSet}.
(3) *κ is the least cardinal witnessing* Σ_2-SR^-.

Since, as shown in [23], weakly shrewd cardinals may be smaller than $2^{\aleph_0}$, the principle Σ_2-SR^-, and therefore also $\mathrm{SR}^-_{\mathcal{R}}$, for any set $\mathcal{R}$ of Π_1 predicates, does not imply the existence of a strongly inaccessible cardinal. Moreover, [23] shows that it is consistent, modulo the existence of a weakly shrewd cardinal that is not shrewd (a large cardinal notion consistency-wise weaker than subtleness), that there exists a cardinal less than $2^{\aleph_0}$ witnessing the principle SR^- for all definable classes of structures of the same type, taken as a schema, i.e., Σ_n-SR^-, for all $n < \omega$. Thus, even SR^- cannot imply the existence of a strongly inaccessible cardinal.

6.2. Strong $\Sigma_1(\mathcal{R})$-definability. Notice that a class $\mathcal{C}$ is Σ_1-definable iff there is a Σ_1 formula φ such that for every A, $A \in \mathcal{C}$ if and only if some transitive structure $\langle M, \in\rangle$ that contains A satisfies $\varphi(A)$. Now let $\mathcal{L}_{\dot{R}}$ be the language of set theory expanded with an additional predicate symbol $\dot{R}$, and suppose R is a predicate. Naturally, one may define a class $\mathcal{C}$ to be $\Sigma_1(R)$ if it is Σ_1-definable in the language $\mathcal{L}_{\dot{R}}$ with $\dot{R}$ being interpreted as R. However, unlike the case of Σ_1-definability, this is not equivalent to saying that there is a Σ_1 formula φ of the language $\mathcal{L}_{\dot{R}}$ such that for every A, $A \in \mathcal{C}$ if and only if some transitive structure $\langle M, \in, R^M\rangle$ that contains A satisfies $\varphi(A)$. For the equivalence to hold we need to require that R^M is precisely $R \cap M$. Namely,

PROPOSITION 6.8. *The following are equivalent for all classes $\mathcal{C}$ and predicates R*:

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(1) *$\mathcal{C}$ is $\Sigma_1(R)$, i.e., there exists a Σ_1 formula $\varphi(x)$ of $\mathcal{L}_{\dot{R}}$ such that*

$$\mathcal{C} = \{A : \varphi(A),\ \textit{with } \dot{R} \textit{ interpreted as } R\}.$$

(2) *There is a Σ_1 formula $\varphi(x)$ of the language $\mathcal{L}_{\dot{R}}$ such that for every A, $A \in \mathcal{C}$ if and only if*

$$\langle M, \in, R \cap M \rangle \models \varphi(A)$$

for some transitive structure $\langle M, \in \rangle$ that contains A.

Notice also that if $\mathcal{C}$ is a Σ_1-definable class of structures of the same type, then the closure of $\mathcal{C}$ under isomorphisms is also Σ_1-definable, and we have the following equivalences:

Proposition 6.9. *The following are equivalent for any class $\mathcal{C}$ of structures of the same type that is closed under isomorphisms*:

(1) *$\mathcal{C}$ is Σ_1.*
(2) *There is a Σ_1 formula $\varphi(x)$ of the language of set theory such that for every A, $A \in \mathcal{C}$ if and only if*

$$\langle M, \in \rangle \models \varphi(B)$$

for some transitive structure $\langle M, \in \rangle$ of size $|A|$ that contains B, where B is isomorphic to A.

Based on the considerations above, the following is therefore a natural definition for a class of structures closed under isomorphisms to be Σ_1-definable with an additional predicate R. This is a reformulation, for the case $n = 1$, of Lücke's [23] definition of *local $\Sigma_n(R)$-class*:

Definition 6.10. A class $\mathcal{C}$ of structures of the same type and closed under isomorphisms is $\Sigma_1(R)^*$ if there is a Σ_1 formula $\varphi(x)$ of the language $\mathcal{L}_{\dot{R}}$ such that for every A, $A \in \mathcal{C}$ if and only if

$$\langle M, \in, R \cap M \rangle \models \varphi(B)$$

for some transitive structure $\langle M, \in \rangle$ of size $|A|$ that contains B, where B is isomorphic to A.

Observe that although every $\Sigma_1(R)^*$ class $\mathcal{C}$ is $\Sigma_1(R)$, the converse is not true, even assuming closure under isomorphisms. An example is the class $\mathcal{C}$ of all structures isomorphic to some transitive $\langle M, \in, \mathrm{Cd} \cap M \rangle$, where Cd is the class of cardinals.

The closure under isomorphisms of the $\Sigma_1(\mathcal{R})$-definable classes of structures that are used in the proofs of Theorems 6.1 and 6.2, Proposition 6.3, and Theorem 6.4 (as given in [8]) are easily seen to be $\Sigma_1(\mathcal{R})^*$, for the corresponding $\mathcal{R}$. Thus, the results follow from the weaker $\Sigma_1(\mathcal{R})^*$-SR^- corresponding assumptions. Also, the argument in the proof of Theorem 5.5

from [8] can be adapted to show that if $\mathcal{L}^*$ and $\mathcal{R}$ are symbiotic, then the $SLST(\mathcal{L}^*)$ property implies $\Sigma_1(\mathcal{R})^*$-SR$^-$ (see [8]).[6] Thus, from the results in Section 8 of [8] one may obtain the following equivalences:

THEOREM 6.11.

(1) [23] *κ is the least weakly inaccessible cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd)^*$-SR$^-$.
(2) [23] *κ is the least weakly Mahlo cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Rg)^*$-SR$^-$.
(3) *κ is the least α-weakly inaccessible cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd, \alpha\text{-}WI)^*$-SR$^-$.
(4) *Suppose κ is such that if $\gamma \leq \kappa$ is weakly inaccessible, then $2^\delta \leq \gamma$ for all cardinals $\delta < \gamma$. Then κ is the least weakly compact cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd, WC)^*$-SR$^-$.

Under the assumption of GCH, or just assuming that every weakly inaccessible cardinal is inaccessible, the theorem above yields exact characterizations in terms of SR for the first inaccessible, Mahlo, α-inaccessible, and weakly compact cardinals.

COROLLARY 6.12 (GCH).

(1) *κ is the least inaccessible cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd)^*$-SR$^-$.
(2) *κ is the least Mahlo cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Rg)^*$-SR$^-$.
(3) *κ is the least α-inaccessible cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd, \alpha\text{-}WI)^*$-SR$^-$.
(4) *κ is the least weakly compact cardinal iff κ is the least cardinal witnessing* $\Sigma_1(Cd, WC)^*$-SR$^-$.

In items (1)–(4) above one may, equivalently, strengthen $\Sigma_1(\mathcal{R})^*$-SR$^-$ by allowing κ-reflection for classes of structures $\mathcal{C}$ that are $\Sigma_1(\mathcal{R})^*$-definable with parameters in H_κ.

§7. Generic Structural Reflection.

If A and B are structures of the same type, we say that an elementary embedding $j : A \to B$ is *generic* if it exists in some forcing extension of V. We shall next consider the following *generic* version of SR:

GSR: (*Generic Structural Reflection*) For every definable (with parameters) class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *generically reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\alpha$ and a generic elementary embedding from B into A.

[6] However, as shown in [23], it does not imply $(\mathrm{SR})^-_{\mathcal{R}}$, as claimed in [8].

Thus, GSR is just like SR, but the elementary embeddings may not exist in V but in some forcing extension of V. The next proposition shows that this is equivalent to requiring that the elementary embedding exists in any forcing extension resulting from collapsing the structure B to make it countable.

PROPOSITION 7.1 [4]. *The following are equivalent for structures B and A of the same type.*

(1) $V^{\mathrm{Coll}(\omega,B)} \models$ *"There is an elementary embedding $j : B \to A$."*
(2) *For some forcing notion $\mathbb{P}$,*
$V^{\mathbb{P}} \models$ *"There is an elementary embedding $j : B \to A$."*

Taking into account similar considerations as in the case of SR and PSR, we may properly formulate GSR as a schema. Namely, for Γ a lightface definability class, let:

Γ-GSR: (Γ-*Generic Structural Reflection*) There exists a cardinal κ that *generically reflects* all Γ-definable, with parameters in V_κ, classes $\mathcal{C}$ of natural structures, i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ such that in $V^{\mathrm{Coll}(\omega,B)}$ there is an elementary embedding from B into A.

The boldface version being:

There exists a proper class of cardinals κ that *generically reflect* all Γ-definable, with parameters in V_κ, classes $\mathcal{C}$ of natural structures.

The assertion that κ witnesses Γ-GSR, for Γ a boldface definability class, is equivalent to the *Generic Vopěnka Principle* gVP(κ, Γ) introduced in [4].

Similar considerations as in the case of SR (see the remarks before and after Proposition 3.4) show that $\mathbf{\Pi_n}$-GSR and $\mathbf{\Sigma_{n+1}}$-GSR are equivalent, and also Π_n-GSR and Σ_{n+1}-GSR are equivalent.

We shall see next that some large cardinals, such as Schindler's *remarkable* cardinals, can be characterized in terms of GSR.

DEFINITION 7.2 [31, 33]. A cardinal κ is *remarkable* if for every regular cardinal $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H^V_{\bar{\lambda}} \to H^V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.

A cardinal is remarkable if and only if it is 1-remarkable (Definition 4.3). Remarkable cardinals are downward absolute to L and their consistency strength is strictly below a 2-iterable cardinal. Remarkable cardinals are in $C^{(2)}$, and they are totally indescribable and ineffable, hence limits of totally indescribable cardinals (see [4]).

THEOREM 7.3 [4]. *The following are equiconsistent*:

(1) Π_1-GSR.
(2) *There exists a cardinal* κ *that witnesses* $\mathbf{\Pi_1}$-GSR.
(3) *There exists a remarkable cardinal.*

Let us say that a cardinal κ is *almost remarkable* if it is almost-1-remarkable (Definition 4.4), namely: for all $\lambda > \kappa$ in $C^{(1)}$ and every $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(1)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there exists an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $a \in \mathrm{range}(j)$. Then Theorem 4.5 yields the following:

THEOREM 7.4. *A cardinal* κ *witnesses* $\mathbf{\Pi_1}$-GSR *if and only if* κ *is almost remarkable.*

It follows that the notions of remarkable cardinal and of almost-remarkable cardinal are equiconsistent.

Magidor [24] shows that a cardinal κ is supercompact if and only if for every regular cardinal $\lambda > \kappa$ there is a regular cardinal $\bar{\lambda} < \kappa$ and an elementary embedding $j : H_{\bar{\lambda}} \to H_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$. One can thus view a remarkable cardinal as a *virtually supercompact*[7] cardinal. In analogy with Theorem 3.2 one might therefore expect Π_1-GSR to be not just equiconsistent, but actually equivalent with the existence of a remarkable cardinal. Even more, since the first supercompact cardinal is precisely the first cardinal that witnesses $\mathbf{\Pi_1}$-SR, one might conjecture that the first remarkable cardinal is the first cardinal that witnesses $\mathbf{\Pi_1}$-GSR. This is almost true, but not exactly. On the one hand, if κ is a remarkable cardinal, then κ witnesses $\mathbf{\Pi_1}$-GSR [4]. On the other hand, as shown in Theorem 7.6, if there is no ω-Erdös cardinal in L, then the least cardinal witnessing $\mathbf{\Pi_1}$-GSR is also the first remarkable cardinal.

The following equivalent definition of remarkability was given in [4]: a cardinal κ is remarkable if and only if for every $\lambda > \kappa$ there exist some $\bar{\lambda} < \kappa$ and a generic elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.

Wilson [37] defines the notion of weakly remarkable cardinal by not requiring that $\bar{\lambda}$ is strictly below κ. Namely,

DEFINITION 7.5 [37]. κ is *weakly remarkable* if and only if for every $\lambda > \kappa$ there exist some $\bar{\lambda}$ and a generic elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.

Wilson shows that if there exists a weakly remarkable non-remarkable, cardinal κ, then some ordinal greater than κ is an ω-Erdös cardinal in L. Moreover, the statements "There exists an ω-Erdös cardinal" and "There

[7] We choose to call it *virtually supercompact*, as in [4], instead of the perhaps more natural *generic supercompact*, for the latter notion already exists in the literature with a different meaning.

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exists a weakly remarkable non-remarkable cardinal" are equiconsistent modulo ZFC, and equivalent assuming $V = L$.

Observe that if κ is cardinal witnessing $\mathbf{\Pi_1}$-GSR, then κ also witnesses $\mathbf{\Pi_1}$-GSR in L. Thus, if κ witnesses $\mathbf{\Pi_1}$-GSR and in L λ is the least inaccessible cardinal above κ, then L_λ is a model of ZFC in which κ satisfies Π_1-GSR and there is no ω-Erdös cardinal above κ. By combining arguments from [4, 37] we have the following:

THEOREM 7.6. *Assume there is no ω-Erdös cardinal in L. Then, the least cardinal that satisfies* Π_1-GSR, *if it exists, is remarkable.*

PROOF. Let κ be the least cardinal witnessing Π_1-GSR. Let $\mathcal{C}$ be the Π_1-definable class of structures of the form $\langle V_{\lambda+1}, \in\rangle$ with $\lambda \in C^{(1)}$. Pick a singular cardinal $\lambda \in C^{(2)}$ greater than κ. By Π_1-GSR, let $j : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ be a generic elementary embedding with $\bar{\lambda} < \kappa$. Let $\bar{\alpha} = \mathrm{crit}(j)$. Note that $\bar{\alpha} < \bar{\lambda}$, because $\bar{\alpha}$ is regular and $\bar{\lambda}$ is not.

We claim that $\bar{\alpha}$ is weakly remarkable up to $\bar{\lambda}$. So, fix some $\delta > \bar{\alpha}$ smaller than $\bar{\lambda}$. Consider the restriction $j : V_\delta \to V_{j(\delta)}$, which has $j(\mathrm{crit}(j)) = j(\bar{\alpha})$. Then $V_{\lambda+1}$ satisfies that for some $\bar{\delta}$ there exists a generic elementary embedding $j^* : V_{\bar{\delta}} \to V_{j(\delta)}$ such that $j^*(\mathrm{crit}(j^*)) = j(\bar{\alpha})$. Hence, by elementarity $V_{\bar{\lambda}+1}$ satisfies that for some $\bar{\delta}$ there exists a generic elementary embedding $j^* : V_{\bar{\delta}} \to V_\delta$ with $j^*(\mathrm{crit}(j^*)) = \bar{\alpha}$.

By elementarity, $\alpha := j(\bar{\alpha})$ is weakly remarkable up to λ, and since $\lambda \in C^{(2)}$, α is weakly remarkable. Since the existence of a weakly remarkable non-remarkable cardinal implies the existence of an ω-Erdös cardinal in L [37], by our assumption we have that α is in fact remarkable. Hence, since every remarkable cardinal witnesses Π_1-GSR [4], we have that $\kappa \leq \alpha$.

The theorem will be proved by showing that $\alpha = \kappa$. For suppose, aiming for a contradiction, that $\kappa < \alpha$. Since α is remarkable and therefore belongs to $C^{(2)}$, we have

$$V_\alpha \models \text{“}\kappa \text{ witnesses } \Pi_1\text{-GSR.”}$$

By elementarity, there is some $\gamma < \bar{\alpha}$ such that

$$V_{\bar{\alpha}} \models \text{“}\gamma \text{ witnesses } \Pi_1\text{-GSR.”}$$

Hence, since $j(\gamma) = \gamma$, again by elementarity,

$$V_\alpha \models \text{“}\gamma \text{ witnesses } \Pi_1\text{-GSR”}$$

and therefore γ witnesses Π_1-GSR, thus contradicting the minimality of κ. ⊣

COROLLARY 7.7. *Assume there is no ω-Erdös cardinal in L. Then, the following are equivalent for a cardinal* κ:

(1) *κ is the least cardinal witnessing* Π_1-GSR.

(2) *κ is the least cardinal witnessing* $\mathbf{\Pi_1}$-GSR.
(3) *κ is the least almost remarkable cardinal.*
(4) *κ is the least weakly remarkable cardinal.*
(5) *κ is the least remarkable cardinal.*

We don't know if the assumption that there is no ω-Erdös cardinal in L is necessary for the equivalence above to hold. However, we have the following:

PROPOSITION 7.8. *For every $n > 0$, if κ witnesses* $\mathbf{\Pi_n}$-GSR, *then $\kappa \in C^{(n+1)}$. In particular, if κ witnesses* $\mathbf{\Pi_1}$-GSR, *then $\kappa \in C^{(2)}$.*

PROOF. Let us prove the case $n = 1$. The general case follows by induction, using a similar argument. So, suppose $a \in V_\kappa$, $\varphi(x, y)$ is a Π_1 formula with x, y as the only free variables, and $V \models \exists x\varphi(x, a)$. Pick $\lambda \in C^{(2)}$ greater than κ, so that $V_\lambda \models \exists x\varphi(x, a)$. Since the class of structures of the form $\langle V_\alpha, \in, a\rangle$ is Π_1-definable with a as a parameter, there exists a generic elementary embedding $j : \langle V_{\bar\lambda}, \in, a\rangle \to \langle V_\lambda, \in, a\rangle$ with $\bar\lambda < \kappa$. Note that, on the one hand, since λ belongs to $C^{(1)}$ so does $\bar\lambda$, hence by downward absoluteness for Π_1 sentences, $V_\kappa \models$ "$\bar\lambda \in C^{(1)}$." On the other hand, by elementarity of j, $V_{\bar\lambda} \models \exists x\varphi(x, a)$. Hence by upwards absoluteness, $V_\kappa \models \exists x\varphi(x, a)$.

A simpler similar argument, using the fact that Σ_1 sentences are absolute for transitive sets, shows that $\kappa \in C^{(1)}$. Hence, if $\varphi(x, y)$ and a are as above, and $V_\kappa \models \exists x\varphi(x, a)$, then by upwards absoluteness, $V \models \exists x\varphi(x, a)$.

For the general case $n > 1$, assume $\kappa \in C^{(n)}$, and consider the Π_n-definable (with a as a parameter) class of structures of the form $\langle V_\alpha, \in, a\rangle$ with $\alpha \in C^{(n)}$. ⊣

As Wilson [37] shows that a cardinal is remarkable if and only if it is weakly remarkable and belongs to $C^{(2)}$, if the least cardinal κ that witnesses $\mathbf{\Pi_1}$-GSR is not remarkable, then is not weakly remarkable. Thus, the question is if it is provable in ZFC that the least cardinal κ witnessing $\mathbf{\Pi_1}$-GSR, if it exists, is weakly remarkable. Notice, however, that the proof of Theorem 7.6 does show that if κ is the least cardinal witnessing Π_1-GSR, then either κ is remarkable or there is a weakly remarkable cardinal below κ. Also, if κ is the least cardinal witnessing $\mathbf{\Pi_1}$-GSR, then either κ is remarkable or there are unboundedly many weakly remarkable cardinals below κ.

More generally, recall (Definition 4.3) that a cardinal κ is *n-remarkable*, for $n > 0$, if for every $\lambda > \kappa$ in $C^{(n)}$, there is $\bar\lambda < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$, there is an elementary embedding $j : V_{\bar\lambda} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$. Equivalently, we may additionally require that for any given $a \in V_\lambda$, a is in the range of j. A cardinal κ is *completely remarkable* if it is n-remarkable for every $n > 0$. Remarkable cardinals are precisely the 1-remarkable cardinals.

As shown in [4], if $0^\sharp$ exists, then every Silver indiscernible is completely remarkable in L. Moreover, if κ is 2-iterable, then V_κ is a model of ZFC in which there exists a proper class of completely remarkable cardinals.

Theorem 7.3 also holds for n-remarkable cardinals. Namely,

THEOREM 7.9. *The following are equiconsistent for $n > 0$:*

(1) Π_n-GSR.
(2) *There exists a cardinal κ that witnesses $\mathbf{\Pi_n}$-GSR.*
(3) *There exists an n-remarkable cardinal.*

As it turns out, $(n+1)$-remarkable cardinals correspond precisely to the virtual form of $C^{(n)}$-extendible cardinals. Namely,

DEFINITION 7.10 [4]. A cardinal κ is *virtually extendible* if for every $\alpha > \kappa$ there is a generic elementary embedding $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

A cardinal κ is *virtually $C^{(n)}$-extendible* if additionally $j(\kappa) \in C^{(n)}$.

Note that virtually extendible cardinals are virtually $C^{(1)}$-extendible because $j(\kappa)$ must be inaccessible in V.

In contrast with the definition of extendible cardinal, in which the requirement that $j(\kappa) > \alpha$ is superfluous, in the definition of virtually extendible cardinal it is necessary. The reason is that while there is no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, such an embedding may exist generically (see [4]).

THEOREM 7.11 [4]. *A cardinal κ is virtually extendible if and only if it is 2-remarkable. More generally, κ is virtually $C^{(n)}$-extendible if and only if it is $(n+1)$-remarkable.*

The requirement that $j(\kappa) > \alpha$ in the definition of virtually extendible cardinals suggests the following strengthening of GSR. Let us say that an elementary embedding $j : V_\alpha \to V_\beta$ is *overspilling* if j has a critical point and $j(\mathrm{crit}(j)) > \alpha$. For Γ a lightface definability class, let:

Γ-SGSR: (Γ-*Strong Generic Structural Reflection*) There exists a cardinal κ that *strongly generically reflects* all Γ-definable classes $\mathcal{C}$ of natural structures, i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ such that in $V^{\mathrm{Coll}(\omega,B)}$ there is an overspilling elementary embedding from B into A.

With the boldface version being:

There exists a proper class of cardinals κ that *strongly generically reflect* all Γ-definable, with parameters in V_κ, classes $\mathcal{C}$ of natural structures.

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Then we have the following:

THEOREM 7.12. *The following are equivalent for every* $n \geq 1$:

(1) Π_n-SGSR.
(2) *There exists an* $(n+1)$*-remarkable cardinal.*
(3) *There exists a virtually* $C^{(n)}$*-extendible cardinal.*

§8. Beyond VP. We have seen that a variety of large cardinal notions, ranging from weakly inaccessible to Vopěnka's Principle, can be characterised as some form of Structural Reflection for classes of relational structures of some degree of complexity. The question is now if the same is true for large-cardinal notions stronger than VP, up to rank-into-rank embeddings, or even for large cardinals that contradict the Axiom of Choice (see [5]). This is largely a yet unexplored realm, although there are some very recent results showing that this is indeed the case. In [7], we introduce a simple form of SR, which we call *Exact Structural Reflection* (ESR), and show that some natural large-cardinal notions in the region between almost-huge and superhuge cardinals can be characterised in terms of ESR. Also, sequential forms of ESR akin to generalised versions of Chang's Conjecture yield large-cardinal principles at the highest reaches of the known large-cardinal hierarchy, and beyond. We give next a brief summary of the results.

Given infinite cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, let

$\mathrm{ESR}_{\mathcal{C}}(\kappa,\lambda)$: (*Exact Structural Reflection*) For every $A \in \mathcal{C}$ of rank λ, there exist some $B \in \mathcal{C}$ of rank κ and an elementary embedding from B into A.

We let $\Gamma(P)$-$\mathrm{ESR}(\kappa,\lambda)$ denote the statement that $\mathrm{ESR}_{\mathcal{C}}(\kappa,\lambda)$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters from P.

The general ESR principle restricted to classes of structures that are closed under isomorphic images is just equivalent to VP:

THEOREM 8.1 [7]. *Over the theory ZFC, the following schemata of sentences are equivalent*:

(1) *For every class* $\mathcal{C}$ *of structures of the same type that is closed under isomorphic images, there is a cardinal* κ *with the property that* $\mathrm{ESR}_{\mathcal{C}}(\kappa,\lambda)$ *holds for all* $\lambda > \kappa$.
(2) VP.

However, even the principle Π_1-$\mathrm{ESR}(\kappa,\lambda)$ holding for some $\kappa < \lambda$ already implies the existence of large cardinals, the *weakly exact cardinals*, whose consistency strength is beyond that of VP.

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DEFINITION 8.2 [7]. Given a natural number $n > 0$, an infinite cardinal κ is *weakly n-exact for a cardinal* $\lambda > \kappa$ if for every $A \in V_{\lambda+1}$, there exist a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$, and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in \mathrm{range}(j)$.

If we further require that $j(\mathrm{crit}\, j) = \kappa$, then we say that κ is *weakly parametrically n-exact for* λ.

We have the following equivalence:

THEOREM 8.3 [7]. *The following statements are equivalent for all cardinals* κ *and all natural numbers* $n > 0$:

(1) κ *is the least cardinal such that* $\Pi_n(V_\kappa)$-ESR(κ, λ) *holds for some* λ.
(2) κ *is the least cardinal that is weakly n-exact for some* λ.
(3) κ *is the least cardinal that is weakly parametrically n-exact for some* λ.

In contrast with the SR principles considered in previous sections, the ESR principles for Π_n-definable and Σ_{n+1}-definable classes of structures are not equivalent. Indeed, for Σ_n-definable classes, the relevant large cardinals are the *exact cardinals*:

DEFINITION 8.4 [7]. Given a natural number n, an infinite cardinal κ is *n-exact for some cardinal* $\lambda > \kappa$ if for every $A \in V_{\lambda+1}$, there exist a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an $X \preceq H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in \mathrm{range}(j)$.

If we further require that $j(\mathrm{crit}(j)) = \kappa$ holds, then we say that κ is *parametrically n-exact for* λ.

The characterization of ESR for Σ_n-definable classes of structures in terms of exact cardinals is now given by the following:

THEOREM 8.5 [7]. *The following statements are equivalent for all cardinals* κ *and all natural numbers* $n > 0$:

(1) κ *is the least cardinal such that* $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) *holds for some* λ.
(2) κ *is the least cardinal that is n-exact for some* λ.
(3) κ *is the least cardinal that is parametrically n-exact for some* λ.

The strength of weakly n-exact and n-exact cardinals, and therefore also of their corresponding equivalent forms of ESR, goes beyond VP, for as shown in [7] they imply the existence of almost huge cardinals:[8] If $\kappa < \lambda$ are cardinals such that κ is either parametrically 0-exact for λ or weakly

[8]Recall that a cardinal κ is *almost huge* if there exist a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\mathrm{crit}\, j = \kappa$ and ${}^{<j(\kappa)}M \subseteq M$. We then say that a cardinal κ is almost huge with *target* λ if there exists such a j with $j(\kappa) = \lambda$.

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parametrically 1-exact for λ, then the set of cardinals that are almost huge with target κ is stationary in κ. Also, if κ is parametrically 0-exact for some cardinal $\lambda > \kappa$, then it is almost huge with target λ.

As for upper bounds, if κ is huge with target λ, then it is weakly parametrically 1-exact for λ. Hence, $\Pi_1(V_\mu)$-ESR(μ, ν) holds for some $\mu \leq \kappa$ and $\nu > \mu$. Moreover, $\Pi_1(V_\kappa)$-ESR(κ, λ') holds in V_λ, for some λ'. However, if κ is the least huge cardinal, then κ is not 1-exact for any cardinal $\lambda > \kappa$. The best upper bound for the consistency strength of exact cardinals is given by the following:

Proposition 8.6 [7]. *If κ is a 2-huge cardinal,*[9] *then there exist an inaccessible cardinal $\lambda > \kappa$ and a cardinal $\rho > \lambda$ such that V_ρ is a model of* ZFC *and, in V_ρ, the cardinal κ is weakly parametrically n-exact for λ, for all $n > 0$.*

As for direct implication, the best known upper bound for the existence of exact cardinals is given by the following:

Proposition 8.7 [7]. *Let κ be an $I3$-cardinal,*[10] *witnessed by $j : V_\delta \to V_\delta$. If $l, m, n < \omega$, then, in V_δ, the cardinal $j^l(\kappa)$ is parametrically n-exact for $j^{l+m+1}(\kappa)$.*

The existence of an $I3$-cardinal is a very strong principle which implies the consistency of n-huge cardinals, for every n, and much more (see [15, p. 24], also [2, Theorem 7.1]). Yet even stronger large-cardinal principles bordering the inconsistency with ZFC are implied by the following sequential forms of ESR, also introduced in [7].

8.1. Sequential ESR. Let $0 < \eta \leq \omega$ and let $\mathcal{L}$ be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_i \mid i < \eta \rangle$.

Given a sequence $\vec{\mu} = \langle \mu_i \mid i < \eta \rangle$ of cardinals with supremum μ, an $\mathcal{L}$-structure A has *type* $\vec{\mu}$ (*with respect to* $\vec{P}$) if the universe of A has rank μ and $\mathrm{rank}(\dot{P}_i^A) = \mu_i$ for all $i < \eta$.

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures and a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta \rangle$ of cardinals, let

ESR$_{\mathcal{C}}(\vec{\lambda})$: (*Sequential* ESR) For every structure B in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < 1 + \eta \rangle$, there exists an elementary embedding of a structure A in $\mathcal{C}$ of type $\langle \lambda_i \mid i < \eta \rangle$ into B.

[9] That is, there is an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, and ${}^{j^2(\kappa)}M \subseteq M$.

[10] That is, the critical point of a non-trivial elementary embedding $j : V_\delta \to V_\delta$, for some limit ordinal δ. Then V_δ is a model of ZFC and the sequence $\langle j^m(\kappa) \mid m < \omega \rangle$ is cofinal in δ.

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The large-cardinal notions corresponding to sequential ESR are the sequential analogs of weakly exact and exact cardinals (given in Definitions 8.2 and 8.4). Namely,

DEFINITION 8.8 [7]. Let $0 < \eta \leq \omega$ and let $\vec{\lambda} = \langle \lambda_m \mid m < \eta \rangle$ be a strictly increasing sequence of cardinals with supremum λ.

(1) Given $0 < n < \omega$, a cardinal $\kappa < \lambda_0$ is *weakly n-exact for* $\vec{\lambda}$ if for every $A \in V_{\lambda+1}$, there are a cardinal ρ, a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\rho \cup \{\rho\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$, and an e. e. $j : M \to H_{\lambda'}$ with $A \in \operatorname{range} j$, $j(\rho) = \lambda$, $j(\kappa) = \lambda_0$, and $j(\lambda_{m-1}) = \lambda_m$, all m.

If we further require that $j(\operatorname{crit} j) = \kappa$, then we say that κ is *parametrically weakly n-exact for* $\vec{\lambda}$.

(2) Given $n < \omega$, a cardinal $\kappa < \lambda_0$ is *n-exact for* $\vec{\lambda}$ if for every $A \in V_{\lambda+1}$, there are a cardinal ρ, a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\rho$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an $X \preceq H_{\kappa'}$ with $V_\rho \cup \{\rho\} \subseteq X$, and an e. e. $j : X \to H_{\lambda'}$ with $A \in \operatorname{range} j$, $j(\rho) = \lambda$, $j(\kappa) = \lambda_0$, and $j(\lambda_{m-1}) = \lambda_m$, all m.

If we further require that $j(\operatorname{crit} j) = \kappa$, then we say that κ *is parametrically n-exact for* $\vec{\lambda}$.

Then we have the following equivalences:

THEOREM 8.9 [7]. *Let $0 < n < \omega$, let $0 < \eta \leq \omega$, and let $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta \rangle$ be a strictly increasing sequence of cardinals.*

(1) *The cardinal λ_0 is weakly n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$ if and only if Π_n-ESR$(\vec{\lambda})$ holds.*
(2) *If λ_0 is weakly parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$, then $\Pi_n(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.*

Also:

(1) *The cardinal λ_0 is n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$ if and only if Σ_{n+1}-ESR$(\vec{\lambda})$ holds.*
(2) *If λ_0 is parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$, then $\Sigma_{n+1}(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.*

In the case of finite sequences $\vec{\lambda}$ of length n, the sequentially 1-exact cardinals correspond roughly to n-huge cardinals. More precisely [7]: If κ is an n-huge cardinal, witnessed by an elementary embedding $j : V \to M$, then κ is weakly parametrically 1-exact for the sequence $\langle j^{m+1}(\kappa) \mid m < n \rangle$. Also, if κ is a cardinal and $\vec{\lambda} = \langle \lambda_m \mid m \leq n \rangle$ is a sequence of cardinals such that κ is either weakly 1-exact for $\vec{\lambda}$ or 0-exact for $\vec{\lambda}$, then some cardinal less than κ is n-huge.

As for infinite sequences $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$, there is a dramatic increase in consistency strength, as shown by the following facts proved in [7]: Let λ be the supremum of $\vec{\lambda}$ and let $\kappa < \lambda_0$ be a cardinal. If κ is either weakly 1-exact for $\vec{\lambda}$ or 0-exact for $\vec{\lambda}$, then there exists an $I3$-embedding $j : V_\lambda \to V_\lambda$. Also, if κ is either parametrically weakly 1-exact for $\vec{\lambda}$ or parametrically 0-exact for $\vec{\lambda}$, then the set $I3$-cardinals is stationary in κ.

To prove the existence of a weakly parametrically 1-exact cardinal, for some infinite sequence $\vec{\lambda}$, the best known upper bound is an $I1$-cardinal[11] [7]: If κ is an $I1$-cardinal and $k > 0$ is a natural number, then κ is weakly parametrically 1-exact for the sequence $\langle j^{k(m+1)}(\kappa) \mid m < \omega \rangle$. In particular, for $k = 1$, κ is weakly parametrically 1-exact for $\langle j^{(m+1)}(\kappa) \mid m < \omega \rangle$, hence $\Pi_1(V_\kappa)$-ESR$(\vec{\lambda})$ holds.

Many open questions remain (see [7]), the most pressing one being the consistency with ZFC of the principle Σ_2-ESR$(\vec{\lambda})$ for some sequence $\vec{\lambda}$ of length ω.

§9. Summary. The following tables summarize the results exposed in previous sections. Presented in this form, the equivalences between the various kinds of SR and (mostly already well-known) different large cardinal notions illustrate the fact that SR is a general reflection principle that underlies (many stretches of) the large cardinal hierarchy, thus unveiling its concealed uniformity. Table 1 encompasses the region of the large-cardinal hierarchy comprised between supercompact and VP, Table 2 the region between strong and "Ord is Woodin," and Table 3 the region between globally superstrong and $C^{(n)}$-globally superstrong, all n. The tables are

Table 1. Between supercompact and VP.

Complexity	SR
Σ_1	ZFC
Π_1, Σ_2	Supercompact
Π_2, Σ_3	Extendible
Π_3, Σ_4	$C^{(2)}$-Extendible
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	$C^{(n-1)}$-Extendible
$\vdots$	$\vdots$
Π_n, all n	VP

[11] That is, the critical point of a non-trivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ for some limit ordinal λ.

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Table 2. Between strong and "Ord is Woodin".

Complexity	PSR
Σ_1	ZFC
Π_1, Σ_2	Strong
Π_2, Σ_3	Π_2-Strong
Π_3, Σ_4	Π_3-Strong
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	Π_n-Strong
$\vdots$	$\vdots$
Π_n, all n	Ord is Woodin

Table 3. Between globally superstrong and $C^{(n)}$-globally superstrong, all n.

Complexity	SPSR
Σ_1	ZFC
Π_1, Σ_2	Globally Superstrong
Π_2, Σ_3	$C^{(2)}$-Globally Superstrong
Π_3, Σ_4	$C^{(3)}$-Globally Superstrong
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	$C^{(n)}$-Globally Superstrong
$\vdots$	$\vdots$
Π_n, all n	$C^{(n)}$-Globally Superstrong, all n

Table 4. SR relative to inner models.

Class	Inner Model M	SR(M)
Σ_1	Any	ZFC
$\mathcal{C}$	L	$0^\sharp$ exists
$\mathcal{C}_X$	$L[X]$	$X^\sharp$ exists
$\mathcal{C}^U$	$L[U]$	$0^\dagger$ exists
$\mathcal{C}_X^U$	$L[U, X]$	$X^\dagger$ exists
$\vdots$	$\vdots$	$\vdots$

read as, e.g., Γ-SR for Γ being Π_1 or Σ_2 is equivalent to the existence of a supercompact cardinal (Table 1). For the boldface definability classes we have the equivalence of Γ-SR with a proper class of the corresponding large cardinals. In the limit cases, i.e., VP in Table 1, "Ord is Woodin" in Table 2, and $C^{(n)}$-globally superstrong, all n, in Table 3, the lightface and the boldface versions are equivalent.

Table 5. Small large cardinals.

Complexity	SR^-	$\mathrm{SR}^- + \mathrm{GCH}$
$\Sigma_1(PwSet), \Sigma_2$	Weakly shrewd	
$\Sigma_1(Cd)^*$	Weakly inaccessible	Inaccessible
$\Sigma_1(Cd, \alpha\text{-}WI)^*$	α-Weakly inaccessible	α-Inaccessible
$\Sigma_1(Rg)^*$	Weakly Mahlo	Mahlo
$\Sigma_1(Cd, WC)^*$		Weakly compact

Table 6. From almost remarkable to virtually $C^{(n)}$-extendible.

Complexity	GSR	GSR+No ω-Erdös in L	SGSR
Π_1, Σ_2	Almost-Remarkable	Remarkable Almost-remarkable Weakly remarkable	
Π_n, Σ_{n+1}	Almost-n-remarkable		$(n+1)$-Remarkable Virt. $C^{(n)}$-Extend.

Table 7. Beyond Vopěnka's Principle.

Complexity	ESR
Π_1	Weakly 1-exact (Between almost huge and huge)
Π_n	Weakly n-exact (Consistency-wise below 2-huge)
Σ_{n+1}	n-Exact (I3-embedding is an upper bound)

Table 8. Beyond I3-cardinals.

Complexity	$\mathrm{ESR}(\vec{\lambda})$, $\mathrm{lh}(\vec{\lambda}) = \eta + 1$	$\mathrm{ESR}(\vec{\lambda})$, $\mathrm{lh}(\vec{\lambda}) = \omega$
Π_1	Weak. 1-exact for $\vec{\lambda}$ of lh. η (Implied by η-huge)	Weak. 1-exact for $\vec{\lambda}$ of lh. ω (Implies I3-cardinals. I1-cardinal an upper bound)
Π_n	Weak. n-exact for $\vec{\lambda}$ of lh. η	Weak. n-exact for $\vec{\lambda}$ of lh. ω
Σ_{n+1}	n-Exact for $\vec{\lambda}$ of length η	n-Exact for $\vec{\lambda}$ of length ω

We also have the equivalence between Π_1-PSR_1 and the existence of a measurable cardinal (Theorem 5.21).

Table 4 summarizes the results on SR relative to inner models. The classes $\mathcal{C}, \mathcal{C}_X, \mathcal{C}^U$, and $\mathcal{C}_X^U$ are defined in Section 4.1. Similar results should hold for

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canonical inner models for stronger large cardinals, and their corresponding *sharps*.

Tables 5 and 6 summarize the results characterizing large cardinals of consistency strength below $0^\sharp$ in terms of restricted SR and generic SR, respectively.

Finally, Tables 7 and 8 cover some equivalences in the region above Vopěnka's Principle.

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