# STABILITY THEOREMS FOR WEDGES

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## 1. Introduction

We are concerned with the stability properties of uniformly closed wedges in $C(E)$ (resp. $C^+(E)$), the real-valued (resp. non-negative) continuous functions on a compact space $E$, and solve the following problems in this area:

(*a*) Let $A$ be a closed semi-algebra in $C^+(E)$ such that

$$f, g \in A \Rightarrow fg/(1+g) \in A.$$

Is $A$ an ideal of a type 1 semi-algebra?

(*b*) Let $W$ be a closed wedge in $C^+(E)$ stable under some continuous $F$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $\lim_{x\to\infty} (F(x)-x+1) = 0$. Is $W$ stable under all increasing convex $G$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $G(0) = 0$?

(*c*) Let $n$ be a positive integer and let $A_n$ denote the semi-algebra of vectors in $\boldsymbol{R}^{n+1}$ whose first $n$ differences are non-negative. Is $A_n$ stable under all $F$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ whose first $n$ differences are non-negative?

Here, $W$ is said to be *stable under* $F$ if

$$f \in W \Rightarrow F \circ f \in W.$$

We give affirmative answers to (*a*), (*b*), (*c*) in §§ **2**, **3**, **4** respectively. Question (*c*) has irked us for some time—chiefly because it is tempting to guess that $A_n$ is the restriction to $\{i/n\colon 0 \leqq i \leqq n\}$ of the corresponding semi-algebras on $[0, 1]$ (which motivated early semi-algebra theory). But this is false: in fact, already for $n = 3$ the vector $(0, 0, 1, 3)$ cannot be obtained in this way. A special case of (*b*) under the additional hypothesis that $F$ is increasing and convex was proved by the second author in (**9**), and the " convexity " assumption was removed by F. F. Bonsall in (**5**) where the problem is raised in its present form and a positive answer conjectured. Problem (*a*) is already implicit in (**4**) and was considered in (**1**). Our solution is obtained as a consequence of a general stability theorem which has other interesting implications which we discuss also in §**2** (Theorems 3 and 4).

We wish to thank Professor F. F. Bonsall for a pre-publication copy of (**5**), and Professor E. J. Barbeau for finding a slip in the statement of Theorem 1.

## 2. A general stability theorem

Before stating the main theorem of this section (Theorem 1) we give a general condition for the existence of a suitable " choice " function to be used in the proof.

Let $\mathscr{F}$ denote the class of all continuous functions $F$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ which are increasing, convex, and satisfy $F(0) = 0$.

**Lemma 1.** *Let $F \in \mathscr{F}$ be non-linear and let $W$ be a closed wedge in $C^+(E)$ stable under $F$. Then there exists a sequence $(F_n)$ of functions in $\mathscr{F}$ such that*

(i) *$W$ is stable under $F_n$, $n = 1, 2, \ldots,$*
(ii) *$F_n(x) \to 0$ $(x<1)$, $F_n(x) \to \infty$ $(x>1)$.*

**Proof.** Let $\mathscr{S}(W)$ denote the set of functions in $\mathscr{F}$ under which $W$ is stable. $\mathscr{S}(W)$ is clearly composition closed and stable under multiplication by positive constants.

By the definition of convexity,

$$F(ta) = F(ta+(1-t)0) \leqq tF(a)+(1-t)F(0) = tF(a) \quad (a \in \boldsymbol{R}^+, 0<t<1).$$

Since $F$ is non-linear there exist $a \in \boldsymbol{R}^+$ and $t$ with $0<t<1$ such that $F(ta)<tF(a)$. In particular $F(a) = b>0$ and we define $G \in \mathscr{S}(W)$ by

$$G(x) = b^{-1}F(xa) \quad (x \in \boldsymbol{R}^+).$$

Since $G(1) = 1$, $G(t)<t$ we have

$$G(x)<x \quad (0<x<1), \quad G(x)>x \quad (x>1).$$

In fact, with $\alpha = x/t$, $\beta = (x-t)/(1-t)$, $\gamma = \beta^{-1}$, we have

$$\begin{aligned} G(x) &= G(\alpha t+(1-\alpha)0) \leqq \alpha G(t)+(1-\alpha)G(0)<x && (0<x<t), \\ G(x) &\leqq \beta G(1)+(1-\beta)G(t)<x && (t<x<1), \\ G(x) &\geqq \gamma^{-1}(G(1)-(1-\gamma)G(\mathrm{t}))>x && (x>1). \end{aligned}$$

We define $F_n \in \mathscr{S}(W)$ by

$$F_1 = G,\ F_n = G \circ F_{n-1} \quad (n = 2, 3 \ldots).$$

For $x<1$, the bounded sequence $(F_n(x))$ is monotonic decreasing to a fix-point of $G$ and hence converges to zero. For $x>1$, the monotonic increasing sequence $(F_n(x))$ must be unbounded for otherwise it would converge to a fix-point of $G$ greater than 1. As each $F_n$ belongs to $\mathscr{S}(W)$ this completes the proof.

**Notation.** Let $W$ be a closed wedge in $C^+(E)$. We write

$$[W] = \{f \in C^+(E)\colon fg \in W \text{ whenever } g \in W\}.$$

Note that $[W]$ is a closed semi-algebra which contains the unit function, that $W \subseteq [W]$ if and only if $W$ is a semi-algebra, and that $W$ is then an ideal of $[W]$.

**Theorem 1.** *Let $W$ be a closed wedge in $C^+(E)$ stable under a continuous function $G$: $(\boldsymbol{R}^+)^2 \to \boldsymbol{R}^+$ such that, for each $y$, $G(0, y) = 0$ and*

$$\lim_{x \to \infty,\, y' \to y} G(x, y') = y.$$

*Then $[W]$ is stable under all continuous increasing $H$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$. In particular $[W]$ is a type 1 semi-algebra.*

*If, in addition, $W$ is stable under some non-linear $F \in \mathscr{F}$ then $W$ is an ideal of the semi-algebra $[W]$ and, in particular, $W$ is stable under all elements of $\mathscr{F}$.*

**Corollary.** *The closed ideals of the type 1 semi-algebras in $C^+(E)$ are precisely those closed semi-algebras $A$ in $C^+(E)$ which have the following property:*

$$f, g \in A \Rightarrow fg(1+g)^{-1} \in A.$$

**Remark.** E. J. Barbeau showed in (**1**) that a closed semi-algebra in $C^+(E)$ which is an inf-wedge is an ideal of a type 1 semi-algebra—a fact we can deduce from Theorem 1 by taking $G(x, y) = x \wedge y$ and $F(x) = x^2$ (since a semi-algebra admits squaring). On the other hand there exist ideals of type 1 semi-algebras which are not inf-wedges—see (**4**, p. 113). Thus, of the functions $x \wedge y$ and $xy/(1+y)$ on $(\boldsymbol{R}^+)^2$, which both satisfy the conditions of Theorem 1, stability under the first implies stability under the second but not vice-versa. There is scope here for study of two-dimensional stability properties in more generality.

**Proof of theorem.** Since $1 \in [W]$ we may as well consider continuous increasing $H$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $H(0) = 0$. Suppose that $f \in [W]$ but $H \circ f \notin [W]$ for some such $H$. Then there exists $g \in W$ such that $(H \circ f)g \notin W$. By the version of the Hahn-Banach theorem appropriate to ordered spaces there is a continuous linear functional $\mu$ on $C(E)$ such that

$$\mu((H \circ f)g) = 1, \quad \mu(h) \leqq 0 \quad (h \in W). \tag{1}$$

For $\alpha > 0$, $n$ a positive integer we define $K_{\alpha, n}$ and $u_\alpha$ by

$$K_{\alpha, n} = G \circ ((\alpha^{-1} f)^n g, g)$$

and

$$u_\alpha(x) = \begin{cases} 1 & f(x) > \alpha, \\ 0 & f(x) \leqq \alpha. \end{cases}$$

Because $|\mu|$ is a countably-additive measure there exists a countable set $\Gamma \subseteq \boldsymbol{R}^+$ such that $E_\alpha = \{x \in E : f(x) = \alpha\}$ is $|\mu|$-null for $\alpha \in \boldsymbol{R}^+ \sim \Gamma$, and it is easy to check that, outside $E_\alpha$, $K_{\alpha, n}$ converges pointwise to $gu_\alpha$ as $n \to \infty$. Since $G$ is bounded on $\boldsymbol{R}^+ \times \text{range } g$ it follows that $K_{\alpha, n}$ is bounded uniformly with respect to $n$, and an application of Lebesgue's dominated convergence theorem shows that

$$\lim_{n \to \infty} \int K_{\alpha, n} d\mu = \int g u_\alpha d\mu \quad (\alpha \in \boldsymbol{R}^+ \sim \Gamma). \tag{2}$$

But $[W]$ is a semi-algebra so that $(\alpha^{-1} f)^n g \in W$ and hence $K_{\alpha, n} \in W$. In conjunction with (1) and (2) this proves that

$$\int g u_\alpha d\mu \leqq 0 \quad (\alpha \in \boldsymbol{R}^+ \sim \Gamma). \tag{3}$$

Now fix $\varepsilon > 0$ and choose $\alpha_1, \alpha_2, \ldots \alpha_n \in \boldsymbol{R}^+ \sim \Gamma$, $\alpha_0 = 0$, such that

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n, \quad \max_E f \leqq \alpha_n, \quad H(\alpha_k) - H(\alpha_{k-1}) < \varepsilon, \quad k = 1, \ldots, n.$$

Defining $h = \sum_{k=1}^{n} (H(\alpha_k) - H(\alpha_{k-1}))u_{\alpha_k}g$, we deduce from (3) that $\int hd\mu \leqq 0$, and a little calculation shows that $\left| \int ((H \circ f)g - h)d\mu \right| \leqq \varepsilon \int gd \left| \mu \right|$. For small $\varepsilon$ these inequalities contradict (1) and establish the first part of the statement of the theorem. The particular choice $H(x) = x(1+x)^{-1}$ shows that $[W]$ is a type 1 semi-algebra.

Now suppose that $W$ is $F$-stable for some non-linear $F$ in $\mathscr{F}$. Let $(F_n)$ be a sequence of the kind guaranteed by Lemma 1, and redefine $K_{\alpha,n}$ by

$$K_{\alpha,n} = G \circ (F_n \circ (\alpha^{-1}f), g),$$

where $g$ *and* $f$ are fixed elements of $W$. Repetition of the previous argument shows that $(H \circ f)g \in W$. Taking $H(x) = x$ we see $W$ is a semi-algebra and hence an ideal of $[W]$. The final assertion follows from the fact that every $F \in \mathscr{F}$ can be written as $xH(x)$ where $H$ is increasing on $\boldsymbol{R}^+$. Thus

$$f \in W \Rightarrow F \circ f = (H \circ f)f \in W.$$

**Remark.** Our method of proof is based on the technique employed by F. F. Bonsall in obtaining Theorems 1, 2 of (5). His results are stated for general topological spaces $E$ when $C(E)$ has the topology of uniform convergence on compacta—the extension of the preceding theorem to that case is straightforward. Theorem 2 of (5) is an easy corollary of our result and a similar extension of Theorem 1 of that paper is given below.

**Theorem 2.** *Let $W$ be a closed wedge in $C^+(E)$ stable under a continuous function $G$: $(\boldsymbol{R}^+)^2 \to \boldsymbol{R}^+$ such that, for each $y$, $G(0, y) = y$ and*

$$\lim_{x \to \infty,\, y' \to y} G(x, y') = 0.$$

*Then $[W]$ is stable under all continuous $H$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$. In particular $[W]$ is a type 0 semi-algebra.*

*If, in addition, $W$ is stable under some non-linear $F \in \mathscr{F}$ then $W$ is an ideal of the semi-algebra $[W]$ and is itself stable under all continuous $H$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $H(0) = 0$. Moreover $W - W$ is a closed subalgebra of $C(E)$ and*

$$W = (W - W) \cap C^+(E).$$

**Corollary.** *The closed ideals of the type 0 semi-algebras in $C^+(E)$ coincide with the positive parts of the closed subalgebras of $C(E)$ and are precisely those closed semi-algebras in $C^+(E)$ which have the following property:*

$$f, g \in A \Rightarrow f(1+g)^{-1} \in A.$$

**Proof of theorem.** A method similar to the previous proof shows first that $[W]$ is stable under *decreasing* continuous $H$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ and hence $\left( \text{taking } H(x) = \dfrac{1}{1+x} \right)$ is a type 0 semi-algebra. That the closed type 0 semi-algebras are the positive parts of the closed unital subalgebras of $C(E)$ is the first result of semi-algebra theory (F. F. Bonsall (3)) and it follows that the word " decreasing " can be omitted. The proof that $W$ is an ideal if $F$-stable for non-linear $F$ in $\mathscr{F}$ is also similar to the previous proof and the remaining assertions follow easily from the fact that a closed ideal of a type 0 semi-algebra is the positive part of a closed subalgebra. This is a simple consequence of results in (4) but does not appear to have been explicitly stated. We give the following argument for completeness. Let $W$ be a closed ideal in a closed type 0 semi-algebra $A$. $W$ is obviously a type 1 semi-algebra and is thus of the form

$$W = \{f \in C^+(E)\colon f(x) \leqq f(y) \text{ whenever } x \leqq_W y \text{ and } f(x) = 0 \quad (x \in N)\},$$

where $\leqq_W$ is the partial order defined by

$$x \leqq_W y \Leftrightarrow f(x) \leqq f(y) \quad (f \in W)$$

and $N$ is the null set of $W$,

$$N = \{x \in E\colon f(x) = 0 \quad (f \in W)\}.$$

We note also that the positive part of a closed subalgebra can be characterized in a similar fashion (via the Stone-Weierstrass theorem) where the partial order is, in fact, an equivalence relation. Accordingly it will suffice to prove that

$$f \in W,\ 0 < f(x) < f(y) \Rightarrow x \nleqq_W y.$$

Given $f \in W$, $y \in E$, it follows that $f(y)1 - f \in A - A$, hence that

$$(f(y)1 - f) \vee 0 \in A,$$

and consequently $g = (f(y)f - f^2) \vee 0 \in W$. But if $0 < f(x) < f(y)$, then

$$g(x) > 0 = g(y).$$

This proves (1) and yields the required result.

**Remarks.** Theorem 2 is of less interest than Theorem 1, in the sense that whereas a two-dimensional stability function seems appropriate to characterize type 1 ideals among the wedges in $C^+(E)$, it is over-powerful for the type 0 ideals, where in fact a pair of one-dimensional functions will suffice. We recall from (6), Theorem 3 that a closed wedge $W$ in $C^+(E)$ is a type 1 semi-algebra if and only if

$$f \in W \Rightarrow f^2 \in W \text{ and } f - \tfrac{1}{2}f^2 \in W \text{ whenever } \| f \| \leqq 1.$$

This together with a slight extension of the argument at the end of the preceding proof shows that $W$ is an ideal of a type 0 semi-algebra if and only if

$$f \in W \Rightarrow f^2 \in W \text{ and } (f - \tfrac{1}{2}f^2)^+ \in W.$$

(Of course the latter condition implies $(\lambda f - f^2)^+ \in W$ for all $f \in W$, $\lambda \geqq 0$.) It is natural to ask whether the pair of functions $x^2$, $(x - \frac{1}{2}x^2)^+$ can be replaced by any non-linear $F \in \mathscr{F}$ together with any non-zero continuous $G\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that $G(0) = \lim_{x \to \infty} G(x) = 0$.

A summary of our knowledge of type 1 ideals in $C^+(E)$ may be in order. We have given an algebraic description of those wedges which can be ideals of a suitably chosen type 1 semi-algebra. E. J. Barbeau (**1**, p. 212) has characterized the class $\mathscr{I}_A$ of ideals of a given type 1 semi-algebra $A$ in $C^+(E)$ in terms of an elementary subclass $\mathscr{E}$ of $\mathscr{I}_A$: a result "discovered" recently by the second author and at least one other worker in the field. Barbeau's result is as follows:

Let $\mathscr{U}$ denote the class of all closed upper sets in $E$ with respect to the order $\leqq_A$. For any $\mu \in C(E)^*$ define

$$I_\mu = \left\{ f \in A : \int_U f d\mu \geqq 0 \text{ for all } U \in \mathscr{U} \right\}.$$

Then $I_\mu$ is an ideal of $A$ and the set $\mathscr{E}$ of all such $I_\mu$ is an elementary class for $\mathscr{I}_A$.

In the special case where $E$ is a compact subset of $\boldsymbol{R}$ with the usual ordering, A. K. Roy (**10**) has recently characterized those $\mu$ for which $I_\mu$ admits inf, sup, or both, and has shown that each ideal of one of these types is an intersection of $I_\mu$ of the same type; while B. A. Barnes (**2**) has found a simple description of those $\mu$ for which $J_\mu = \{f \in A : \mu(f) \geqq 0\}$ is an ideal of $A$ (which occurs if and only if $J_\mu$ and $I_\mu$ coincide).

**Theorem 3.** *If $W$ is a closed inf-wedge in $C^+(E)$ then the following are equivalent:*

(i) *$W$ is stable under some non-linear $F \in \mathscr{F}$.*
(ii) *$W$ is stable under all $F \in \mathscr{F}$.*
(iii) *$W$ is a type 1 semi-algebra ideal.*

*The closed inf-wedges in $C^+(E)$ satisfying the equivalent conditions* (i)-(iii) *are the subsets $W$ of $C^+(E)$ of the form $W = \bigcap_{i \in I} A_{x_i, \mu_i}$ where for $x_i \in E$, $\mu_i$ a non-negative measure on $E$,*

$$A_{x_i, \mu_i} = \left\{ f \in C^+(E) : \int f d\mu_i \leqq f(x_i) \text{ and } f(x) \leqq f(x_i) \quad (x \in \operatorname{supp} \mu_i) \right\}.$$

**Proof.** A wedge $W$ is an inf-wedge if it is stable under $G(x, y) = x \wedge y$ so that the equivalence of (i), (ii), (iii) follows from Theorem 1. It is elementary to verify that $\cap A_{x_i, \mu_i}$ is an inf-wedge satisfying (i). Let us suppose then that $W$ is a closed inf-wedge satisfying (ii). By the celebrated Choquet-Deny theorem (7),

$$W = \bigcap_{i \in I} W_{x_i, \mu_i} \text{ where } W_{x_i, \mu_i} = \left\{ f \in C^+(E) : \int f d\mu_i \leqq f(x_i) \right\}.$$

If $\mu_i = 0$ there is no difficulty. Suppose that for some $i$, we have

$$f(x) > f(x_i) \text{ for some } x \in \operatorname{supp}(\mu_i).$$

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Then $\int (f-f(x_i))_+ d\mu_i > 0 = (f-f(x_i))_+(x_i)$, and this leads to a contradiction since $W$ is stable under the function $F(x) = (x-\lambda)_+$, for constant $\lambda \geqq 0$.

**Remark.** The implication (i) $\Rightarrow$ (ii) under the above conditions with " concave " replacing " convex " is of long standing. (See (**8**), Theorem 1.)

In general the stability properties of wedges in $C(E)$ are more complicated than those of wedges of non-negative functions. It is of interest therefore to reduce the structure theory of inf-wedges in $C(E)$ stable under squaring to the case covered by Theorem 3. The obvious way to obtain a wedge of this kind is to take all elements of a closed subalgebra of $C(E)$ whose restrictions to some closed subset $F$ of $E$ form a square-stable inf-wedge in $C^+(F)$. In fact this simple process exhausts the possibilities.

**Theorem 4.** *Let $W$ be a closed inf-wedge in $C(E)$ which is stable under squaring. Then there exists a closed subalgebra $A$ of $C(E)$, a closed subset $F$ of $E$, and a closed inf-wedge $W'$ in $C^+(F)$ stable under squaring, such that*

$$W = \{f \in A : f|_F \in W'\},$$

*where $f|_F$ is the restriction of $f$ to $F$. In particular $W$ is a semi-algebra.*

**Proof.** Let $U = (-W) \cap C^+(E)$. Then from Theorem 4 of (**5**) it follows that $U \subseteq W \cap (-W)$ and $U$ is the positive part of a closed subalgebra of $C(E)$. We show first that $W$ is a semi-algebra. Since $W$ is an inf-wedge,

$$f \in W \Rightarrow -f_- = f \wedge 0 \in W.$$

Hence for each $f$ in $W$, $f_- \in U \subseteq W \cap (-W)$. Also $f \vee 0 = f_+ = f + f_- \in W$. By Theorem 1, $W \cap C^+(E)$ is a semi-algebra, so that

$$f, g \in W \Rightarrow f_+ g_+, f_- g_- \in W.$$

Now fix $f, g \in W$. By square-stability, for each $\lambda > 0$,

$$(g_- - \lambda f_+)^2 = g_-^2 - 2\lambda f_+ g_- + \lambda^2 f_+^2 \in W.$$

Since $g_- \in U$, it follows that $g_-^2 \in (U-U) \cap C^+(E) = U \subseteq -W$, so that

$$-f_+ g_- + \tfrac{1}{2}\lambda f_+^2 \in W, \quad \lambda > 0.$$

Letting $\lambda \to 0$, we deduce that $-f_+ g_- \in W$, and similarly we obtain that $-g_+ f_- \in W$.

It is now clear that $fg = (f_+ - f_-)(g_+ - g_-) \in W$, and we have proved that $W$ is a semi-algebra.

Let $F = \{x \in E : f(x) = 0 \ (f \in U-U)\}$ and define the relation $\sim$ on $E \sim F$ by

$$x \sim y \Leftrightarrow f(x) = f(y) \quad (f \in U-U).$$

Let $A$ be the closed subalgebra defined by

$$A = \{f \in C(E) : f(x) = f(y) \text{ whenever } x \sim y\}.$$

We now show that $W$ respects $\sim$. In fact, given $x, y \in E \sim F$ with $x \sim y$ there exists $h$ in $U-U$ such that $h \geqq 0$ and $h(x) = h(y) = 1$. But

$$U = (U-U) \cap C^+(E)$$

so that $h \in U \subseteq -W$. Then, for each $f$ in $W$, $hf_+$ and $hf_-$ belong to

$$-W \cap C^+(E) = U,$$

so that

$$hf = hf_+ - hf_- \in U-U,$$

and hence

$$f(x) = h(x)f(x) = h(y)f(y) = f(y).$$

It is clear that the restriction to $F$ of each $f$ in $W$ is non-negative (since $f_- \in U$ whenever $f \in W$) and it follows immediately that these restrictions form a closed inf-wedge $W'$ which is stable under squaring. Finally suppose that $g \in A$, $g \equiv f$ on $F$ for some $f$ in $W$. Then $g-f$ belongs to closed subalgebra $U-U$ (by the Stone-Weierstrass theorem) and, since $U-U \subseteq W$,

$$g = f + (g-f) \in W.$$

This completes the proof.

## 3. Type 2 wedges

In this section we answer question (*b*) of the Introduction.

**Lemma 2.** *Let $F$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ be continuous with $\lim_{x\to\infty} (F(x)-x+1) = 0$. Let $F_1 = F$, and $F_n = F \circ F_{n-1}$ ($n = 2, 3, \ldots$). Then*

$$\frac{1}{n} F_n(nx) \to (x-1)^+$$

*uniformly on $\boldsymbol{R}^+$ as $n \to \infty$.*

**Remark.** This result under the additional assumption that $F$ be increasing is proved by Bonsall (**5**, p. 140), and we make use of this result in the present proof.

**Proof.** Define the functions $U$ and $L$ by

$$\left.\begin{aligned} U(x) &= \sup\{F(t)\colon 0 \leqq t \leqq x\} \\ L(x) &= \inf\{F(t)\colon x \leqq t < \infty\} \end{aligned}\right\} \quad (x \in \boldsymbol{R}^+).$$

Since $F$ is non-negative, and bounded on bounded intervals, $U$ and $L$ are finite, non-negative, continuous and increasing, and satisfy

$$L(x) \leqq F(x) \leqq U(x) \quad (x \in \boldsymbol{R}^+).$$

It follows at once from the definition of $L$ that if for any $\varepsilon > 0$ we have

$$|F(t) - t + 1| \leqq \varepsilon$$

for all $t \geqq x_0$, then $|L(x_0) - x_0 + 1| \leqq \varepsilon$; replacing $x_0$ by any $x \geqq x_0$ we have $|L(x) - x + 1| \leqq \varepsilon$ ($x \geqq x_0$) and so

$$\lim (L(x) - x + 1) = 0.$$

We now show that $\lim (U(x)-x+1) = 0$. In fact, choose any $\varepsilon>0$. There is $\xi_0 \in \boldsymbol{R}^+$ such that $|F(x)-x+1|<\varepsilon$ $(x \geqq \xi_0)$. Let

$$\xi = \max(\xi_0, U(\xi_0)+1+2\varepsilon).$$

For any $x \geqq \xi$ we have $U(x) \geqq F(x)>x-1-\varepsilon \geqq \xi-1-\varepsilon$ so that

$$U(x)>U(\xi_0)+\varepsilon.$$

There is $t \leqq x$ such that $F(t)>U(x)-\varepsilon>U(\xi_0)$. Then $t>\xi_0$ by the definition of $U$, so that

$$|F(t)-t+1|<\varepsilon.$$

We then deduce

$$x-1-\varepsilon<F(x)<U(x)<F(t)+\varepsilon<(t-1+\varepsilon)+\varepsilon \leqq x-1+2\varepsilon.$$

Hence $|U(x)-x+1|<2\varepsilon$ $(x \geqq \xi)$ and it follows that $\lim (U(x)-x+1) = 0$.

Now let $U_n$, $L_n$ be the $n$-th iterates of $U$ and $L$ defined in the same way as the $F_n$. An inductive argument using the monotonicity of $U$ and $L$ shows

$$L_n(x) \leqq F_n(x) \leqq U_n(x) \quad (x \in \boldsymbol{R}^+,\ n = 1, 2, \ldots). \tag{3.1}$$

Now $U$ and $L$ satisfy the conditions of Bonsall's lemma, so that $n^{-1}L_n(nx)$, $n^{-1}U_n(nx) \to (x-1)^+$ uniformly on $\boldsymbol{R}^+$ as $n\to\infty$, and the desired result follows at once from (3.1).

**Remark.** Bonsall's proof makes no use of the continuity of $F$, so that ours uses it only in order to ensure that $F$ is bounded on bounded intervals.

**Theorem 5.** *Let $F$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ be continuous with* $\lim_{x\to\infty} (F(x)-x+1) = 0$, *and let $W$ be a closed wedge in $C^+(E)$ stable under $F$. Then $W$ is reductive, type* 2, *and stable under all $G \in \mathscr{F}$.*

**Remark.** *Reductive* and *type* 2 mean stable under the functions $(x-1)^+$, $x^2/(1+x)$ respectively.

**Proof.** Let $\{F_n\}$ be the sequence of the previous lemma. For any $f \in W$,

$$\frac{1}{n} F_n \circ (nf) \in W \quad (n = 1, 2, \ldots).$$

As $n\to\infty$ this sequence converges uniformly to $(f-1)^+$ and hence $W$ is reductive.

That a closed reductive wedge in $C^+(E)$ is stable under all elements of $\mathscr{F}$ is an early result of stability theory (see (9), (5)) and the type 2 property is a particular case of this.

## 4. Semi-algebras of completely monotonic vectors

**Notation.** Given a continuous function $F$: $I \to \boldsymbol{R}$ where $I$ is an interval (possibly infinite) in $\boldsymbol{R}$, and given $h \in \boldsymbol{R}^+$, we define $\Delta_h F$ by

$$\Delta_h F(t) = F(t+h)-F(t)$$

whenever $t$, $t+h \in I$. We write $\Delta_h^2 F$ for $\Delta_h(\Delta_h F)$, etc., and we have the general formula for the *n-th difference of* $F$ (of length $h$):

$$\Delta_h^n F(t) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} F(t+rh)$$

defined whenever $t$, $(t+h)$, ..., $(t+nh) \in I$.

**Definition.** $F$ is called *n-fold monotonic* if $F \geqq 0$ and for $k = 1, 2, \ldots, n$ and each $h>0$ we have $\Delta_h^k F(t) \geqq 0$ whenever it is defined.

**Notation.** Let $J$ be an interval $\{i: p \leqq i \leqq q\}$ in $Z$ and let $x: J \to R$, so that $x$ may be thought of as a vector $(x_p, \ldots, x_q)$ in space of $q-p+1$ dimensions. Similarly to above we define the *n-th difference of* $x$ by

$$(\Delta^n x)_i = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} x_{i+r}$$

whenever $p \leqq i \leqq i+n \leqq q$. Clearly $\Delta^n x$ is a vector with $q-p-n+1$ components $(n \leqq q-p)$ but fails to be defined if $n>q-p$. We shorten $(\Delta^n x)_i$ to $\Delta^n x_i$.

**Definition.** $x$ is called *completely monotonic* (c.m.) if $(\Delta^n x)_i \geqq 0$ whenever it is defined.

**Notation.** We denote by $A_n$ $(n = 0, 1, 2, \ldots)$ the set of all $x$: $\{0, 1, \ldots, n\} \to R$ which are c.m.; which we can regard as a subset of $R^{n+1}$.

Clearly $A_n$ is a cone in $R^{n+1}$. With multiplication defined pointwise it is well known that

$A_n$ is a semi-algebra.

Given continuous $F$: $R^+ \to R^+$ and $x \in A_n$ we denote by $F \circ x$ the function composition of $F$ and $x$, so that in the vector notation $F \circ x = (F(x_0), \ldots, F(x_n))$.

**Theorem 6.** *For* $n = 0, 1, 2, \ldots$ *the semi-algebra* $A_n$ *is stable under all* $F$: $R^+ \to R^+$ *which are continuous and n-fold monotonic.*

**Corollary.** *Since the function* $t \to t^n/(1+t)$ *is n-fold monotonic it follows that* $A_n$ *is a type n semi-algebra.*

**Remark.** The continuous analogue of this, namely that the semi-algebra $B_n$ of $n$-fold monotonic functions in $C[0, 1]$ is stable under all continuous, $n$-fold monotonic $F$: $R^+ \to R^+$, is of long standing and easily proved by differentiation, using induction and Lemma 3.

**Preliminaries to the proof.** We shall proceed by induction on $n$. The next lemma, a well-known result in the theory of convex functions, is used in reducing the problem to one with a smaller value of $n$.

**Lemma 3.** *Let* $F$ *be as in the statement of the theorem. Then for* $n \geqq 2$, $F$ *is absolutely continuous and* $F'$ *(suitable values being inserted at the points of a* *null set) is $(n-1)$-fold monotonic. For $n \geqq 3$ we have the stronger result that $F$ is continuously differentiable on $\boldsymbol{R}^+$.*

For a fixed $n$, let $u_k$ $(0 \leqq k \leqq n)$ denote the vector

$$\left(\binom{0}{k}, \binom{1}{k}, \ldots, \binom{n}{k}\right),$$

**Lemma 4.** *Each $x \in \boldsymbol{R}^{n+1}$ is uniquely expressible as $\Sigma \lambda_k u_k$ where $\lambda_k = \Delta_k x_0$. In particular $A_n$ is precisely the conical hull of the $u_k$.*

**Proof.** Let $S$: $\boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$, $R$: $\boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$ be the operators defined by

$$S(x_0, x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_n), \quad R(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_{n-1}).$$

Since $R$, $S$ commute and $\Delta = S - R$ we have

$$S^r = (R+\Delta)^r = \sum_{k=0}^{r} \binom{r}{k} \Delta^k R^{r-k}.$$

But for, $0 \leqq k \leqq r$,

$$(R^{r-k}\Delta^k x)_0 = (\Delta^k x)_0,$$

so that

$$x_r = (S^r x)_0 = \sum_{k=0}^{r} \binom{r}{k} \lambda_k \text{ where } \lambda_k = \Delta^k x_0.$$

Since $\binom{r}{k} = 0$ for $r+1 \leqq k \leqq n$ we may replace $\sum_0^r$ by $\sum_0^n$, thus

$$x_r = \sum_{k=0}^{n} \lambda_k \binom{r}{k} \quad (0 \leqq r \leqq n),$$

so that

$$x = \sum_0^n \lambda_k u_k.$$

The number of $u_k$ is equal to the dimension of the space, so they form a basis and the expansion is unique. Hence, if $y = \Sigma u_k \mu_k$ then $\mu_k = \Delta^k y_0$.

It follows that each $y \in A_n$ is a conical combination of the $u_k$. Conversely, let $x = \Sigma \mu_k u_k$ with $\mu_k \geqq 0$. We need to show that $\Delta^q x_r \geqq 0$ $(0 \leqq q \leqq q+r \leqq n)$, and this is true, since

$$\begin{aligned}\Delta^q x_r = \Delta^q S^r x_0 &= \Delta^q (R+\Delta)^r x_0 \\ &= \sum_{k=0}^{r} \binom{r}{k} \Delta^{q+k} x_0 = \sum_{k=0}^{r} \binom{r}{k} \mu_{q+k} \geqq 0.\end{aligned}$$

This completes the proof.

**Proof of Theorem 6.** Let $P(n)$ denote the assertion of the theorem. Assume inductively that $P(m)$ holds for $0 \leqq m < n$, where $n \geqq 2$. (It is easy to verify that $P(0)$ and $P(1)$ are satisfied.)

Let a fixed $n$-fold monotonic function $F$ be chosen.

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**Notation.** By a *part* of a vector $(x_p, \ldots, x_q)$ we shall mean a " subvector " of the form $(x_{p'}, \ldots, x_{q'})$ where $p \leqq p' \leqq q' \leqq q$. It is clear from the definition that any part of a c.m. vector is c.m. From this fact and the fact that $F$ is certainly $k$-fold monotonic if $0 \leqq k < n$, we deduce by the inductive hypothesis that for

$$x \in A_n, \quad \Delta^k(F \circ x)_i \geqq 0 \quad (0 \leqq k < n, 0 \leqq i \leqq i+k \leqq n)$$

so that it only remains to prove that for any $x \in A_n$.

$$\Delta^n(F \circ x)_0 \geqq 0,$$

this being the only $n$-th difference that exists.

By Lemma 4, this is equivalent to proving the non-negativity on the positive cone in $R^{n+1}$ of the function $\hat{F}(\lambda) = \hat{F}(\lambda_0, \lambda_1, \ldots, \lambda_n) = \Delta^n(F \circ x)_0$ where $x = \Sigma \lambda_k u_k$. From the definitions,

$$\hat{F}(\lambda) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} F\left(\sum_{k=0}^{r} \lambda_k \binom{r}{k}\right).$$

By Lemma 3, $F$ is absolutely continuous, so that $\hat{F}$ is also absolutely continuous with respect to each $\lambda_k$. Now,

$$\frac{\partial \hat{F}}{\partial \lambda_k} = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \binom{r}{k} F'\left(\sum_{k=0}^{r} \lambda_k \binom{r}{k}\right)$$

$$= \binom{n}{k} \sum_{r=0}^{n} (-1)^{n-r} \binom{n-k}{r-k} (F' \circ x)_r, \quad \left(\text{since} \binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}\right).$$

Setting $s = r-k$ and discarding terms for which the inner binomial coefficient vanishes we obtain

$$\frac{\partial \hat{F}}{\partial \lambda_k} = \binom{n}{k} \sum_{s=0}^{n-k} (-1)^{n-k-s} \binom{n-k}{s} y_s,$$

$$\frac{\partial \hat{F}}{\partial y_k} = \binom{n}{k} \Delta^{n-k} y_0,$$

where $y = F' \circ (x_k, x_{k+1}, \ldots, x_n)$. Now $(x_k, \ldots, x_n)$ is a part of the c.m. vector $x$, and hence c.m., and provided $k \geqq 1$ it follows by the inductive hypothesis and Lemma 3 that $y$ is c.m., whence

$$\frac{\partial \hat{F}}{\partial \lambda_k} \geqq 0 \text{ on } (R^{n+1})^+ \text{ for } k = 1, \ldots, n.$$

By absolute continuity we thus have for any $(\lambda_0, \ldots, \lambda_n)$,

$$\begin{aligned} \hat{F}(\lambda_0, \ldots, \lambda_n) &\geqq \hat{F}(\lambda_0, 0, \ldots, 0) \\ &= \Delta^n(F \circ \lambda_0 u_0)_0 \\ &= \Delta^n(F(\lambda_0) . (1, 1, \ldots, 1))_0 = 0. \end{aligned}$$

This establishes the induction step and the theorem is proved.

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**Remarks.** (i) It is easy to extend this result to deal with the semi-algebra of " $n$-fold monotonic " vectors in $\boldsymbol{R}^{m+1}$ with $m>n$, that is, the set

$$A_{m,n}=\{x=(x_0,\ldots,x_m)\colon \Delta^k x_i \geqq 0 \text{ for } 0 \leqq k \leqq n, \text{ and all } i \text{ such that this is defined}\}.$$

For clearly $x \in A_{m,n}$ if and only if each of its parts, of " length " $n+1$, is c.m. We thus obtain

$$F \text{ } n\text{-fold monotonic}, \ x \in A_{m,n} \Rightarrow F \circ x \in A_{m,n}.$$

(ii) The case where $F$ is defined on an open or closed bounded subinterval of $\boldsymbol{R}^+$ can be dealt with in much the same way but with more technical detail since $F$ will generally not be extendable to an $n$-fold monotonic function on $\boldsymbol{R}^+$, and $F'$ may be unbounded even when $F$ is bounded.

**5. A problem on the representation of certain finite measures**

Let $X$ denote the space of continuous real functions on $\boldsymbol{R}^+$ with the topology of pointwise convergence. Then $X^*$, which we endow with the $w^*$-topology, is the space of all measures of finite support on $\boldsymbol{R}^+$. The set

$$\mathscr{F}_n=\{f \in X\colon f \text{ is } n\text{-fold monotonic}\}$$

is a closed wedge in $X$, being determined by the inequalities

$$\mu_{h,t,k}(f)=\left(\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\varepsilon_{t+rh}\right)(f) \geqq 0 \quad (0 \leqq k \leqq n;\ h, t \in \boldsymbol{R}^+).$$

It is well known that the dual cone

$$\mathscr{F}_n^d=\{\mu \in X^*\colon \mu(f) \geqq 0 \text{ for all } f \in \mathscr{F}_n\}$$

is therefore the closed conical hull of the measures $\mu_{h,t,k}$. Now Theorem 6 states that if $0 \leqq k \leqq n$, $x \in A_k$, and $\nu_x$ denotes the measure

$$\binom{k}{0}\varepsilon_{x_k}-\binom{k}{1}\varepsilon_{x_{k-1}}+\ldots+(-1)^k\binom{k}{k}\varepsilon_{x_0}$$

then $\nu_x(f)=\Delta^k(f \circ x)_0 \geqq 0$ for all $f \in \mathscr{F}_n$; that is

$$\nu_x \in \mathscr{F}_n^d \quad (x \in A_k).$$

Hence each such $\nu_x$ is the limit of finite positive combinations of the $\mu_{h,t,k}$. Note that $\mu_{h,t,k}$ is a particular case of a $\nu_x$, in which

$$x=(t, t+h, \ldots, t+kh) \in A_k.$$

**Question.** Is it true that $\nu_x$ is in fact a finite positive combination of measures $\mu_{h,t,k}$ without the intervention of a limit process? If not, is there a somewhat extended representation possible, say one corresponding not to powers $\Delta_h^k$ but to products of different $\Delta_h$ leading to measures of the form

$$\mu(f)=\left(\prod_{i=1}^{k}\Delta_{h_i}\right)f(t).$$

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It is easy to verify a formula of the above form in case $n = 2$, and possible but tedious for $n = 3$. The representation is not unique in either case, and it would be of interest to find a canonical form and an algorithm for obtaining this.

**Example.** In the case $n = 2$, let us represent an arbitrary

$$x = (x_0, x_1, x_2) \in A_2$$

in the form $\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2$ according to Lemma 4, where

$$u_0 = (1, 1, 1), \quad u_1 = (0, 1, 2), \quad u_2 = (0, 0, 1).$$

Then

$$\begin{aligned} v_x(f) &= f(x_2) - 2f(x_1) + f(x_0) \\ &= f(\lambda_2 + 2\lambda_1 + \lambda_0) - 2f(\lambda_1 + \lambda_0) + f(\lambda_0) \\ &= \Delta_{\lambda_2} f(2\lambda_1 + \lambda_0) + \Delta^2_{\lambda_1} f(\lambda_0) \end{aligned}$$

so that

$$v_x = \mu_{\lambda_2,\, 2\lambda_1 + \lambda_0,\, 1} + \mu_{\lambda_1,\, \lambda_0,\, 2}.$$

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