ON A THEOREM OF OSIMA AND NAGAO

J. S. FRAME AND G. DE B. ROBINSON

1. Introduction. If we define the *weight* b of a Young diagram containing n nodes to be the number of removable p-hooks where $n = a + bp$, then three fundamental theorems stand out in the modular representation theory of the symmetric group S_n.

1.1 *Two irreducible representations of S_n belong to the same block if and only if they have the same p-core.*

This has been proved in various ways (**1**; **5**; **7**).

1.2 *The number l_b of ordinary irreducible representations in a block of weight b is independent of the p-core and is given by*

$$l_b = \sum_{b_1,\ldots b_p} p_{b_1} p_{b_2} \ldots p_{b_p} \qquad \left(\sum_1^p b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

The enumeration here is based on the 1-1 correspondence holding (**5**;**8**) between the representations $[\alpha]$ with a given p-core and the associated star diagrams $[\alpha]_p{}^*$.

1.3 *The number l'_b of modular irreducible representations* (*indecomposables of the regular representation of S_n*)

(i) *is independent of the p-core, and*

(ii) *is given by*

$$l'_b = \sum_{b_1\ldots b_{p-1}} p_{b_1} p_{b_2} \ldots p_{b_{p-1}} \qquad \left(\sum_1^{p-1} b_i = b,\ 0 \leqslant b_i \leqslant b\right).$$

Theorem 1.3 (ii) was recently proven by Osima (**6**) assuming 1.3 (i) (**8**); Nagao (**4**) obtained 1.3 (i) and (ii) directly. We give here another version of Osima's proof which yields, in addition, generating functions for the number of p-cores containing a nodes and the number of blocks (**1**) to which the representations of S_n belong.

2. Proof of 1.3(ii). The partition generating function

$$\mathscr{P}(x) = 1 + p_1x + p_2x^2 + p_3x^3 + \ldots = \{(1-x)(1-x^2)(1-x^3)\ldots\}^{-1} \tag{2.1}$$

is well known (**2**, p. 272). It follows from 1.2 that

$$\mathscr{L}(x) = 1 + l_1x + l_2x^2 + \ldots = [\mathscr{P}(x)]^p. \tag{2.2}$$

Received October 10, 1953.

If we write

(2.3) $$\mathscr{C}(x) = 1 + c_1x + c_2x^2 + \ldots,$$

when c_a is the number of p-cores containing a nodes, then we may enumerate the ordinary representations of S_n lying in all the blocks in the following manner:

(2.4) $$\mathscr{C}(x)\, \mathscr{L}(x^p) = \mathscr{C}(x)\, [\mathscr{P}(x^p)]^p = \mathscr{P}(x),$$

using 2.2 and the fact that $n = a + bp$. On the other hand, assuming 1.3 (i), we may write

(2.5) $$\mathscr{L}'(x) = 1 + l_1'x + l_2'x^2 + \ldots.$$

Since the total number of modular irreducible representations is equal to the number of p-regular classes of S_n, we have

(2.6) $$\mathscr{C}(x)\, \mathscr{L}'(x^p) = \mathscr{P}(x)/\mathscr{P}(x^p).$$

From 2.4 and 2.6 it follows immediately that

(2.7) $$\mathscr{L}'(x^p) = [\mathscr{P}(x^p)]^{p-1},$$

or

(2.8) $$\mathscr{L}'(x) = [\mathscr{P}(x)]^{p-1},$$

which is precisely the relation 1.3 (ii).

3. The number of p-regular classes. We can say a little more, however. Setting

(3.1) $$\mathscr{M}(x) = 1 + m_1x + m_2x^2 + \ldots,$$

where m_n is the number of distinct blocks associated with S_n, we have

(3.2) $$m_i = c_i + c_{i-p} + c_{i-2p} + \ldots,$$

so that

(3.3) $$\mathscr{M}(x) = \mathscr{C}(x)/(1 - x^p) = \mathscr{P}(x)/(1 - x^p)[\mathscr{P}(x^p)]^p,$$

from 2.4.

In this connection it is worth remarking that the generating function on the right hand side of 2.6, namely,

(3.4) $$\frac{\mathscr{P}(x)}{\mathscr{P}(x^p)} = \frac{(1 - x^p)(1 - x^{2p})\ldots}{(1 - x)(1 - x^2)\ldots},$$

can be interpreted in two ways. We may cancel each factor of $\mathscr{P}(x^p)$ with an equal factor of $\mathscr{P}(x)$ and conclude that $\mathscr{P}(x)/\mathscr{P}(x^p)$ generates the number of partitions of n into summands not divisible by p, which is the number of p-regular classes. Or we may divide the kth factor $(1 - x^k)^{-1}$ of $\mathscr{P}(x)$ into the kth factor $(1 - x^{kp})^{-1}$ of $\mathscr{P}(x^p)$ and generate the number of partitions into summands no one of which appears as many as p times. Hence we have:

3.5 *The number of p-regular classes of S_n is equal to the number of partitions of n in which no summand appears as many as p times.*

This result is of interest in the study of the indecomposables of the regular representation of S_n; such partitions may indeed characterize them.

References

1. R. Brauer, *On a conjecture by Nakayama*, Trans. Royal Soc. Canada, III, *41* (1947), 11–19.

2. G. H. Hardy and E. M. Wright, *The theory of numbers* (Oxford, 1945).

3. D. E. Littlewood, *Modular representations of symmetric groups*, Proc. Royal Soc. London (A) *209* (1951), 333–353.

4. H. Nagao, *Note on the modular representations of symmetric groups*, Can. J. Math. *5* (1953), 356–363.

5. T. Nakayama and M. Osima, *Note on blocks of symmetric groups*, Nagoya Math. J. *2* (1951), 111–117.

6. M. Osima, *Some remarks on the characters of the symmetric group*, Can. J. Math. *5* (1953), 336–343.

7. G. de B. Robinson, *On a conjecture by Nakayama*, Trans. Royal Soc. Canada, III, *41* (1947), 20–25.

8. ———, *On a conjecture by J. H. Chung*, Can. J. Math., *4* (1952), 373–380.

Michigan State College