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BI-POSITIVE SEQUENCES THE BILATERAL MOMENT PROBLEM

BY

JAIME VINUESA & RAFAEL GUADALUPE

ABSTRACT. We pose a "moment problem" in a more general setting than the classical one. Then we find a necessary and sufficient condition for a sequence $\{s_k\}_{-\infty}^{+\infty}$ to have a solution of the "problem"

$$s_k = \int_{-\infty}^{+\infty} x^k \, d\sigma(x) \qquad (k = 0, \pm 1, \pm 2, \dots)$$

where σ is a "distribution function".

1. **Preliminaries**. The Classical Moment Problem is well known: Given a sequence $\{s_k\}_0^\infty$, of real numbers, the necessary and sufficient condition for the existence of a distribution function σ in R (i.e. a function which is non-decreasing, bounded and with an infinite set of points of effective growth) such that

$$s_k = \int_{-\infty}^{+\infty} x^k \, d\sigma(x) \qquad (k = 0, 1, 2, \dots)$$

is that the given sequence should be *positive* in the sense that the determinants

$$\Delta_k = \begin{vmatrix} s_0 & s_1 & \cdots\cdots & s_k \\ \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{vmatrix}$$

should be positive for all k. (see [1], [6])

This problem is very closely related to the theories of Orthogonal Polynomials and Continued Fractions. As a matter of fact, the answer can be obtained as a limit of step functions with jumps at the zeros of the Orthogonal Polynomials corresponding to $\{s_n\}_0^\infty$.

In this note we pose a generalization of that problem to the case of sequences of the type $\{s_k\}_{-\infty}^{+\infty}$ (of moments with either positive or negative exponents). It may be worth to remark that the truncated sequences $\{s_{2k+n}\}_{n=0}^\infty$ of a bi-positive sequence such as the one in Definition 2 below, generate, for each integer value of k, a family of Orthogonal Polynomials $\{P_n^{(k)}(x)\}_{n=0}^\infty$. The idea of the answer we give to our problem is a peculiar "passing to the limit" of those families as $k \to -\infty$ (see for instance that the polynomial $P(x)$ in the proof of Theorem 3 is the nth orthogonal polynomial of one of them).

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A partial result was found by Jones and Thron in connection with their development of the theory of T-fractions (see [4], Chapter 9 and [5]). Their result answers what they call the "strong Stieltjes moment problem", which corresponds, in our setting, to the classical Stieltjes problem (i.e., for distribution functions supported in $(0, +\infty)$). The condition that they find for their case is obviously equivalent to ours, taking into account the sign change for the odd terms in the sequence $\{s_k\}_{-\infty}^{+\infty}$, due to the fact that they pose the problem with powers of $-x$ instead of powers of x (see our Corollary 4 and their Theorem 6.3 in page 527 of [5], where a characterization of the uniqueness of the solution can also be found; we will not study uniqueness here).

2. **Definition (Bi-positive sequences)**. Given a sequence of real numbers of the type $\{s_k\}_{-\infty}^{+\infty}$, we say that it is *bi-positive* if each of the determinants

$$\Delta_n^{(k)} = \begin{vmatrix} s_{2k} & s_{2k+1} & \cdots & s_{2k+n} \\ s_{2k+1} & s_{2k+2} & \cdots & s_{2k+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_{2k+n} & s_{2k+n+1} & \cdots & s_{2k+2n} \end{vmatrix}$$

$$(k = 0, \pm 1, \pm 2, \ldots \quad n = 0, 1, 2, \ldots)$$

is positive.

3. THEOREM (Bilateral Moment Problem). *"Let $\{s_k\}_{-\infty}^{+\infty}$ be a sequence of real numbers. The necessary and sufficient condition for the existence of a distribution function $\sigma : R \to R$ such that*

$$s_k = \int_{-\infty}^{+\infty} x^k \, d\sigma(x) \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

is that the given sequence should be bi-positive".

PROOF.

i) The condition is necessary.

Indeed, let us choose a fixed $k \in Z$ and prove by induction on n that $\Delta_n^{(k)} > 0$. It is of course evident that

$$\Delta_0^{(k)} = s_{2k} = \int_{-\infty}^{+\infty} x^{2k} \, d\sigma(x) > 0.$$

Let us suppose then that $\Delta_{n-1}^{(k)} > 0$ and write

$$P(x) = \begin{vmatrix} s_{2k} & s_{2k+1} & \cdots & s_{2k+n} \\ \cdots & \cdots & \cdots & \cdots \\ s_{2k+n-1} & s_{2k+n} & \cdots & s_{2k+2n-1} \\ x^k & x^{k+1} & \cdots & x^{k+n} \end{vmatrix} = \Delta_{n-1}^{(k)} x^{k+n} + \ldots .$$

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It is easy to see that

$$\int_{-\infty}^{+\infty} P(x)\, x^{k+m}\, d\sigma(x) = 0 \qquad (m = 0, 1, \ldots, n-1)$$

and that

$$\int_{-\infty}^{+\infty} P(x)\, x^{k+n}\, d\sigma(x) = \Delta_n^{(k)}.$$

Therefore

$$\Delta_n^{(k)} = \frac{1}{\Delta_{n-1}^{(k)}} \int_{-\infty}^{+\infty} | P(x) |^2\, d\sigma(x) > 0,$$

ii) The condition is sufficient.

If $\{s_k\}_{-\infty}^{+\infty}$ is bi-positive, then, for each $h \geq 0$, the sequence

$$\{s_n^{(-h)}\}_{n=0}^{\infty}$$

where

$$s_n^{(-h)} = s_{n-2h} \qquad (n = 0, 1, 2, \ldots)$$

is "positive" in the above-mentioned sense. Therefore, there exists a solution of the classical moment problem, i.e., a distribution function σ_h such that:

$$s_n^{(-h)} = s_{n-2h} = \int_{-\infty}^{+\infty} x^n\, d\sigma_h(x) \qquad (n = 0, 1, 2, \ldots).$$

Let us write

$$\Phi_h(x) = \int_{-\infty}^{x} t^{2h}\, d\sigma_h(t) \ (x \in R)$$

which is clearly a distribution function satisfying moreover

$$d\Phi_h(x) = x^{2h}\, d\sigma_h(x)$$

and thus,

$$s_m = s_{2h+m}^{(-h)} = \int_{-\infty}^{+\infty} x^{2h+m}\, d\sigma_h(x) = \int_{-\infty}^{+\infty} x^m\, d\Phi_h(x)$$

(valid for $m \geq -2h$).

The sequence of functions $\{\Phi_h\}_{h=0}^{\infty}$ satisfies the conditions of Helly's Selection Principle (see [2] or [3]), since

$$\Phi_h(x) = \int_{-\infty}^{x} \mathrm{d}\Phi_h(t) \leq \int_{-\infty}^{+\infty} \mathrm{d}\Phi_h(t) = s_0 \qquad \forall\, x \in R,\ \forall\, h \geq 0;$$

therefore it has a subsequence $\{\psi_i(x)\}$ converging in R to a distribution function $\sigma(x)$. We shall prove that $\sigma(x) = \lim_{i\to\infty} \psi_i(x)$ satisfies the conditions of the theorem, i.e., that

$$\int_{-\infty}^{+\infty} x^n \, \mathrm{d}\sigma(x) = s_n \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

We will consider the following cases:

a) $n > 0$:
Given $\epsilon > 0$, let us take $A = (2\, s_{2n}/\epsilon)^{1/n}$.

If $A_1, A_2 > A$ we have because of Helly's Second Theorem (see [2], [3])

$$\lim_{i\to\infty} \int_{-A_1}^{A_2} x^n \, \mathrm{d}\psi_i(x) = \int_{-A_1}^{A_2} x^n \, \mathrm{d}\sigma(x)$$

thus we can take i such that

$$\left| \int_{-A_1}^{A_2} x^n \, \mathrm{d}\psi_i(x) - \int_{-A_1}^{A_2} x^n \, \mathrm{d}\sigma(x) \right| < \epsilon/2$$

and it results:

$$\begin{aligned}
\left| s_n - \int_{-A_1}^{A_2} x^n \, \mathrm{d}\sigma(x) \right| &= \left| \int_{-\infty}^{+\infty} x^n \, \mathrm{d}\psi_i(x) - \int_{-A_1}^{A_2} x^n \, \mathrm{d}\sigma(x) \right| \\
&\leq \left| \int_{-A_1}^{A_2} x^n \, \mathrm{d}\psi_i(x) - \int_{-A_1}^{A_2} x^n \, \mathrm{d}\sigma(x) \right| + \left| \left(\int_{A_2}^{+\infty} + \int_{-\infty}^{-A_1} \right) x^n \, \mathrm{d}\psi_i(x) \right| \\
&< \epsilon/2 + \frac{1}{A^n} \left(\int_{A_2}^{+\infty} + \int_{-\infty}^{-A_1} \right) x^{2n} \, \mathrm{d}\psi_i(x) \leq \epsilon/2 + \frac{1}{A^n} \int_{-\infty}^{+\infty} x^{2n} \, \mathrm{d}\psi_i(x) \\
&= \epsilon/2 + \frac{s_{2n}}{A^n} = \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Therefore the integral

$$\int_{-\infty}^{+\infty} x^n \, \mathrm{d}\sigma(x)$$

converges to s_n.

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b) $n < 0$:
Given $\epsilon > 0$, let

$$a = (3\, s_{2n}/\epsilon)^{1/n} = (\epsilon/3s_{2n})^{-1/n};$$
$$A = (\epsilon/3s_0)^{1/n} = (3s_0/\epsilon)^{-1/n}$$

(note that now $-1/n > 0$).

If $A_1, A_2 > A$; $0 < a_1, a_2 < a$, because of Helly's Second Theorem, being x^n continuous in $[-A_1, -a_1]$ and $[a_2, A_2]$, we have

$$\lim_{i\to\infty} \int_{-A_1}^{-a_1} x^n\, d\psi_i(x) = \int_{-A_1}^{-a_1} x^n\, d\sigma(x)$$

and

$$\lim_{i\to\infty} \int_{a_2}^{A_2} x^n\, d\psi_i(x) = \int_{a_2}^{A_2} x^n\, d\sigma(x).$$

Thus, we can take i such that, on the one hand $i \geq -n$ and, on the other,

$$\left| \int_{-A_1}^{-a_1} x^n\, d\psi_i(x) + \int_{a_2}^{A_2} x^n\, d\psi_i(x) - \int_{-A_1}^{-a_1} x^n\, d\sigma(x) - \int_{a_2}^{A_2} x^n\, d\sigma(x) \right| < \epsilon/3.$$

We have:

$$\begin{aligned}
&\left| s_n - \int_{-A_1}^{-a_1} x^n\, d\sigma(x) - \int_{a_2}^{A_2} x^n\, d\sigma(x) \right| \\
&\leq \epsilon/3 + \left| \left(\int_{-\infty}^{-A_1} + \int_{A_2}^{+\infty} \right) x^n\, d\psi_i(x) + \int_{-a_1}^{a_2} x^n\, d\psi_i(x) \right| \\
&\leq \epsilon/3 + A^n \int_{-\infty}^{+\infty} d\psi_i(x) + a^{-n} \int_{-a_1}^{a_2} x^{2n}\, d\psi_i(x) \\
&\leq \epsilon/3 + A^n\, s_0 + s_{2n}\, a^{-n} = \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

which proves that

$$\int_{-\infty}^{+\infty} x^n\, d\sigma(x) = s_n.$$

c) $n = 0$. Evident.

4. COROLLARY. *(Stieltjes's Condition). Given a sequence $\{s_k\}_{-\infty}^{+\infty}$ of real numbers, the necessary and sufficient condition for the existence of a distribution function $\sigma : R^+ = [0, +\infty) \to R$ satisfying*

$$s_k = \int_0^{+\infty} x^k\, d\sigma(x) \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

is that both $\{s_k\}_{-\infty}^{+\infty}$ and $\{s_{k+1}\}_{-\infty}^{+\infty}$ should be bi-positive. (Note that the condition is equivalent to asking that all the minors of the matrix $[s_{i+j}]_{i,j=-\infty}^{+\infty}$ should be positive).

PROOF. The condition is obviously necessary, since both $\sigma(x)$ and $\zeta(x) = \int_0^x t\, d\sigma(t)$ satisfy the conditions of the theorem and thus $\{s_k\}_{-\infty}^{+\infty}$ and $\{s_{k+1}\}_{-\infty}^{+\infty}$ are bi-positive.

In order to prove sufficiency it is enough to bear in mind that all the distributions appearing in the proof of the theorem have, in this case, their support in $[0, +\infty)$. (see [6]).

5. COROLLARY. *Let $\{s_k\}_{-\infty}^{+\infty} \subset R$ be arbitrary. Then there exists a function $\sigma : R \to R$ of bounded variation such that*

$$s_k = \int_{-\infty}^{+\infty} x^k\, d\sigma(x), \; (k = 0, \pm 1, \pm 2, \ldots).$$

PROOF. We shall prove that there exist two bi-positive sequences $\{s'_k\}_{-\infty}^{+\infty}$, $\{s''_k\}_{-\infty}^{+\infty}$ such that

$$s_k = s'_k - s''_k \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

It is enough then to write $\sigma = \sigma' - \sigma''$, being σ' and σ'' the corresponding distribution functions.

We begin by taking s'_0, s''_0 positive such that $s'_0 - s''_0 = s_0$. If we assume that $\{s''_k\}_{-2n}^{+2n}$ $\{s'_k\}_{k=-2n}^{2n}$ are already built in such a way that $s_k = s'_k - s''_k$ and both truncated sequences be bi-positive, we take:

i) s'_{2n+1}, s''_{2n+1} arbitrary, except that $s'_{2n+1} - s''_{2n+1} = s_{2n+1}$.

ii) Then s'_{2n+2}, s''_{2n+2} such that

$$\Delta'^{(-n)}_{2n+1} = \begin{vmatrix} s'_{-2n} & \cdots & s'_1 \\ \cdots & \cdots & \cdots \\ s'_1 & \cdots & s'_{2n+2} \end{vmatrix} > 0; \; \Delta''^{(-n)}_{2n+1} = \begin{vmatrix} s''_{-2n} & \cdots & s''_1 \\ \cdots & \cdots & \cdots \\ s''_1 & \cdots & s''_{2n+2} \end{vmatrix} > 0$$

and $s'_{2n+2} - s''_{2n+2} = s_{2n+2}$.

iii) s'_{-2n-1}, s''_{-2n-1} again arbitrary except that $s'_{-2n-1} - s''_{-2n-1} = s_{-2n-1}$

iv) Finally s'_{-2n-2}, s''_{-2n-2} verifying $\Delta'^{(-n-1)}_{2n+2} > 0$, $\Delta''^{(-n-1)}_{2n+2} > 0$; $s'_{-2n-2} - s''_{-2n-2} = s_{-2n-2}$.

The sequences $\{s'_k\}_{-\infty}^{+\infty}$ $\{s''_k\}_{-\infty}^{+\infty}$ built in this way are bi-positive.

6. COROLLARY. *The function σ of the last corollary can be built with support in $R^+ = [0, +\infty)$.*

PROOF. It is enough to take in the construction just made the moments of odd subindex subject to the condition that the corresponding determinants should be positive:

$$\begin{vmatrix} s'_{-2n+1} & \cdots & s'_1 \\ \cdots & \cdots & \cdots \\ s'_1 & \cdots & s'_{2n+1} \end{vmatrix} > 0 \qquad \begin{vmatrix} s''_{-2n+1} & \cdots & s''_1 \\ \cdots & \cdots & \cdots \\ s''_1 & \cdots & s''_{2n+1} \end{vmatrix} > 0$$

$$\begin{vmatrix} s'_{-2n-1} & \cdots & s'_0 \\ \cdots & \cdots & \cdots \\ s'_0 & \cdots & s'_{2n+1} \end{vmatrix} > 0 \qquad \begin{vmatrix} s''_{-2n-1} & \cdots & s''_0 \\ \cdots & \cdots & \cdots \\ s''_0 & \cdots & s''_{2n+1} \end{vmatrix} > 0$$

7. **Trivial observation**. Given a sequence $\{w_k\}_{-\infty}^{+\infty}$ of complex numbers, there exists a function $\Omega : (-\infty, +\infty) \to C$ (resp. $\Omega : [0, +\infty) \to C$) of bounded variation, such that

$$w_k = \int_{-\infty}^{+\infty} x^k \, d\Omega(x) \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

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FACULTAD DE CIENCIAS
SANTANDER, SPAIN

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