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UNIVERSAL RADIAL LIMITS OF HOLOMORPHIC FUNCTIONS

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Abstract. We investigate the radial behavior of holomorphic functions in the unit ball B of $\mathbb{C}^n$. In particular, we prove the existence of universal holomorphic functions f in the following sense : given any measurable function φ on ∂B, there is a sequence $(r_n)_{n\geq 1}$, $0 < r_n < 1$, that converges to 1, such that $f(r_n\xi)$ converges to $\varphi(\xi)$ for almost every $\xi \in \partial B$.

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1. Introduction. Bagemihl and Seidel [**1**] have proved in 1954 the following result: given φ a measurable function on the unit circle $\mathbb{T}$, there exists a holomorphic function f in the unit disk $\mathbb{D}$ such that, for almost every ξ of $\mathbb{T}$, $f(r\xi) \to \varphi(\xi)$ as $r \to 1$. Generally, such a function cannot be bounded, but Kahane and Katznelson [**10**] have proved that we may control its growth to the boundary : precisely, given any growth rate $\omega : [0, 1[\to]0, +\infty[$ (i.e. ω is increasing and unbounded), it is possible to take f such that $|f(z)| \leq \omega(|z|)$ for any $z \in \mathbb{D}$.

The generalization of these statements for the unit ball B of $\mathbb{C}^n$ has been done by Hakim and Sibony [**7**] (without the growth rate condition) and by Iordan [**8**] (with this growth rate condition). We mention that their proofs are constructive.

On the other hand, it is well known that there exist holomorphic functions that have universal properties. This fact was first noticed by Birkhoff [**4**] in 1929 who proved the existence of a holomorphic function f such that its translates $[z \mapsto f(z+n)]$ are a dense set in $O(\mathbb{C})$, the set of holomorphic functions on the whole complex plane endowed with the compact-open topology. The study of universal properties of analytic functions has recently received an increasing interest. One of the most striking results in this direction is the following Theorem of Nestoridis [**11**].

There exists a function f analytic in $\mathbb{D}$ such that, given any compact set K with $\mathbb{C}\backslash K$ connected and $K \cap \mathbb{D} = \varnothing$, given any function g continuous on K and analytic inside K, there exists a subsequence of the partial sums of the Taylor series of f that converges uniformly to g on K.

For more information on universal series, we refer to the surveys [**6**] and [**9**].

Connecting together these two points of view, an intriguing question arises: does there exist a holomorphic function which is universal with respect to radial limits? Precisely, does there exist a *single* holomorphic function f such that, for *any* measurable function φ on ∂B, there exists a sequence $(r_n)_{n\geq 1}$, $0 < r_n < 1$, that converges to 1, such that $f(r_n\xi)$ converges to $\varphi(\xi)$ for almost every $\xi \in \partial B$? In this paper, we solve positively this problem, by stating Theorem 1 below. The proof is done by using a category argument. Hence, it is technically rather easier than that made in [**7**] or in [**8**] for a prescribed limit. This also means that quasi-all holomorphic functions in $O(B)$ (in the sense of categories) solve this problem. Moreover, it is also possible to control the growth near the boundary, and to obtain universal radial limits with respect to every center.

The paper is organized as follows. Section 2 contains the notations and preliminary results. In Section 3, we prove our main Theorem. Finally, in Section 4, we explain how recent results on approximation of holomorphic functions by Dirichlet series can be used to investigate radial limits of such series.

2. Preliminaries. In the following, let us denote by B the open unit ball of $\mathbb{C}^n$, by B_r the open ball of radius r, and by m the normalized Lebesgue measure on $S = \partial B$ such that $m(S) = 1$. For L a compact subset of $\mathbb{C}^n$, $\|.\|_L$ denotes the supremum norm on L, and $C(L)$ the set of continuous functions on L endowed with this norm. If E is a subset of S, E_L^* will stand for $E_L^* = \{r(\xi - z_0) + z_0; \xi \in E, z_0 \in L, 0 \leq r \leq 1\}$. $O(B)$ is the set of holomorphic functions on B, equipped with the topology of uniform convergence on compact subsets of B. Of course, for this topology, $O(B)$ is a Baire space. Let ω be any growth rate, and set $v = \frac{1}{\omega}$. To produce universal functions whose growth near the boundary is restricted, one has to introduce the following Banach space

$$Hv_0(B) = \left\{ f \in O(B);\ \sup_{z \in B} |f(z)|v(|z|) < \infty \text{ and } \lim_{r \to 1} \max_{|z|=r} |f(z)|v(|z|) = 0 \right\}$$

endowed with the norm $\|f\|_{Hv_0} = \sup_{z \in B} |f(z)|v(|z|)$. This Banach space has been studied for instance in [**3**]. In particular, it is proved in [**3**, Theorem 1.5] that the polynomials are dense in $Hv_0(B)$.

The proof of the Bagemihl-Seidel Theorem in $\mathbb{C}$ depends in an essential way on Mergelyan's Theorem for appropriate compact subsets of $\overline{\mathbb{D}}$. Mergelyan's Theorem is not fully available in $\mathbb{C}^n$. Nevertheless, a weak extension has been proved by Hakim and Sibony in [**7**]. Using this result, Iordan [**8**, Lemma 3] has deduced a Lemma of approximation near the boundary for functions with restricted growth. To prove the universality of radial limits with respect to every center, we will need a slightly different version of this Lemma. It is convenient to recall first some basic facts about K-limits of functions defined in B (all details can be found in [**12**]). For $\xi \in S$ and $\alpha > 1$, the Korányi region $D_\alpha(\xi)$ is defined by

$$D_\alpha(\xi) = \left\{ z \in \mathbb{C}^n;\ |1 - \langle z, \xi \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\} \subset B.$$

A continuous function $F : B \to \mathbb{C}$ has *K-limit λ at ξ* if the following is true: for every $\alpha > 1$ and every sequence (z_i) in $D_\alpha(\xi)$ that converges to ξ, $F(z_i) \to \lambda$ as $i \to +\infty$. If L denotes a compact subset of B, one may observe that there exists $\alpha > 1$ and $0 < r < 1$ such that $\{z_0 + \rho(\xi - z_0);\ 0 < \rho < r,\ z_0 \in L\} \subset D_\alpha(\xi)$. Therefore, one gets the following result.

LEMMA 1. *Let L be a compact subset of B and h a continuous function on B such that h has K-limit almost everywhere on S. For every $0 < \varepsilon < 1$, there exists a subset E of S and an extension $\tilde{h}$ of h to $B \cup E$ such that $m(E) \geq 1 - \varepsilon$ and $\tilde{h}$ is continuous on E_L^*.*

Proof. The proof is like the proof of Egorov's Theorem. For the sake of completeness, we sketch it. Consider

$$S(n,k) = \{\xi \in S;\ \forall z_0 \in L,\ \forall \rho, \rho' \text{ with } 1 - 1/n < \rho, \rho' < 1,$$
$$|h(z_0 + \rho(\xi - z_0)) - h(z_0 + \rho'(\xi - z_0))| < 1/k\}.$$

Because of the remark before the Lemma, for any k, $\bigcup_n S(n,k)$ has full measure. Therefore, there exists a sequence (n_k) such that $m(S(n_k,k)) \geq 1 - \varepsilon/2^k$ for any k. If we define E by $E = \bigcap_{k\geq 1} S(n_k,k)$ and if we extend h to E by setting $\tilde{h}(\xi) = K - \lim_\xi h$, then it is easy to check that $\tilde{h}$ is continuous on E_L^*, as well as $m(E) \geq 1 - \varepsilon$. □

We are now able to state the promised Lemma.

LEMMA 2. *Let ω be a growth rate, φ a continuous function on B and L a compact subset of B. For every $\varepsilon > 0$, there exists a compact subset E of S and a holomorphic function f in B, continuous on E_L^*, such that*

(i) $m(E) > 1 - \varepsilon$,
(ii) $\|\varphi - f\|_E < \varepsilon$,
(iii) $|f(z)| \leq \omega(|z|)$ *for every* $z \in B$.

Proof. The proof is exactly the same as Lemma 3 in **[8]**, replacing Lemma 2 of **[8]** by our Lemma 1, and using the fact that a bounded holomorphic function in B has K-limit almost everywhere on S. □

3. Radial limits in the Unit Ball. The aim of this section is to prove our main Theorem.

THEOREM 1. *The set of functions $f \in Hv_0(B)$ such that, given any measurable function φ on S, there exists an increasing sequence (r_j), $0 < r_j < 1$, $\lim_{j\to\infty} r_j = 1$, with*

$$\forall z_0 \in B,\ \lim_{j\to+\infty} f(r_j(\xi - z_0) + z_0) = \varphi(\xi) \text{ for almost every } \xi \in S,$$

is residual in $Hv_0(B)$.

Proof. The proof is split into three steps:

Step 1. *A useful lemma.* In order to produce a universal function by a Category argument, we need to approximate simultaneously any holomorphic function in $Hv_0(B)$ (for the denseness when we apply Baire's Theorem), and any continuous function on S (for the universality). This is the content of the following Lemma.

LEMMA 3. *Let φ be a continous function on S, $g \in Hv_0(B)$, $\varepsilon > 0$ and L a compact subset of B. There exists a compact subset E of S and a function f holomorphic in B, continuous on E_L^*, such that*

(i) $m(E) > 1 - \varepsilon$,
(ii) $\|f - \varphi\|_E < \varepsilon$,
(iii) $\|f - g\|_{Hv_0(B)} < \varepsilon$.

Proof. First, we approximate g in $Hv_0(B)$ by a polynomial P, so that $\|g - P\|_{Hv_0(B)} < \varepsilon/2$. Then, apply Lemma 2 to the continuous function $\varphi - P$ and to the growth rate $(\varepsilon/2)\omega$ to produce a compact subset E of S and a function f

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holomorphic in B and continuous on E_L^* such that $m(E) > 1 - \varepsilon$, $\|f - (\varphi - P)\|_E < \varepsilon$ and $\|f\|_{Hv_0(B)} < \varepsilon/2$. The function $\tilde{f} = f + P$ solves the problem. □

Step 2. *A set of strange functions.* Let us fix:

- (φ_j) a dense sequence in $C(S)$,
- (r_l) an increasing sequence of $[0, 1[$, with $r_l \to 1$,
- (L_s) an increasing sequence of compact subsets of B such that $\bigcup_{s\geq 1} L_s = B$.

We then set, for $j, k, l, s \geq 1$:

$$U(j,k,l,s) = \Big\{f \in Hv_0(B);\ \text{there exists } \rho \geq r_l,\ \rho < 1,\ \text{there exists } E \text{ a compact subset of } S \text{ with } m(E) \geq 1 - 1/2^k \text{ such that, for any } \xi \in E, \text{ for any } z_0 \in L_s,\ |f(\rho(\xi - z_0) + z_0) - \varphi_j(\xi)| < \tfrac{1}{2^k}\Big\}.$$

We claim that each $U(j,k,l,s)$ is an open dense subset of $Hv_0(B)$. Indeed, if $f \in U(j,k,l,s)$, there exist ρ and E which are associated to f. Set $K = \{\rho(\xi - z_0) + z_0;\ z_0 \in L_s,\ \xi \in E\}$. By compactness of K, there exists $\eta > 0$ such that:

$$\forall \xi \in E,\ \forall z_0 \in L_s,\ |f(\rho(\xi - z_0) + z_0) - \varphi_j(\xi)| < \frac{1}{2^k} - \eta.$$

If $g \in Hv_0(B)$ is sufficiently close to f, one has $\|f - g\|_K < \eta$. Hence, g belongs to $U(j,k,l,s)$, which proves that this last set is open for the topology of $Hv_0(B)$.

On the other hand, take any $g \in Hv_0(B)$ and any $\eta > 0$. Applying Lemma 3 to φ_j, g, $\varepsilon = \min(\eta, \frac{1}{2^k})$ and L_s, one obtains a compact subset E of S and f a function holomorphic in B and continuous on $E_{L_s}^*$ with $m(E) > 1 - 1/2^k$, $\|f - g\|_{Hv_0(B)} < \eta$ and $\|f - \varphi_j\|_E < 1/2^k$. By uniform continuity of f on $E_{L_s}^*$, there exists $\rho \geq r_l$, $\rho < 1$, such that, $\forall \xi \in E$, $\forall z_0 \in L_s$,

$$|f(\rho(\xi - z_0) + z_0) - \varphi_j(\xi)| < \frac{1}{2^k}.$$

Therefore, f belongs to $U(j,k,l,s)$, and is close enough to g : this proves that $U(j,k,l,s)$ is dense in $Hv_0(B)$.

Now, by Baire's Theorem, $\bigcap_{j,k,l,s} U(j,k,l,s)$ is a dense G_δ subset of $Hv_0(B)$. In particular, it is non-empty. Hence Theorem 1 follows immediately from the result of our last step.

Step 3. *Each function f in $\bigcap_{j,k,l,s} U(j,k,l,s)$ is a solution.*

Pick φ any measurable function on S. By induction on k, we should build an increasing sequence (ρ_k), $0 < \rho_k < 1$, $\lim \rho_k = 1$, and a measurable subset E_k of S such that:

1. $m(E_k) \geq 1 - 1/2^k$,
2. $\forall \xi \in E_k, \forall z_0 \in L_k, |f(\rho_k(\xi - z_0) + z_0) - \varphi(\xi)| < \frac{1}{2^k}$.

Suppose that this has been done. Then we can define $F_N = \bigcap_{k\geq N} E_k$ and $E = \bigcup_{N\geq 1} F_N$. It is easy to check that $m(E) = 1$. Let us consider any $\xi \in E$ and any $z_0 \in B$; there exists $N_0 \in \mathbb{N}$ such that $k \geq N_0 \Rightarrow \xi \in E_k$ and $z_0 \in L_k$. Hence, one has

$$|f(\rho_k(\xi - z_0) + z_0) - \varphi(\xi)| < \frac{1}{2^k}.$$

Therefore, $f(\rho_k(\xi - z_0) + z_0) \to \varphi(\xi)$ if $k \to +\infty$.

It remains to construct ρ_k and E_k. Suppose that the construction has been carried out until step $k-1$ (it is trivial for $k=0$). Fix $l \geq 1$ such that $r_l \geq \rho_{k-1}$. By Lusin's Theorem, there exists $G_k \subset S$, $m(G_k) \geq 1 - 1/2^{k+1}$ and $j_k \in \mathbb{N}$ such that

$$\|\varphi_{j_k} - \varphi\|_{G_k} < \frac{1}{2^{k+1}}.$$

Now, since f belongs to $U(j_k, k+1, l, k)$, there exists $\rho_k > \rho_{k-1}$ and E with $m(E) \geq 1 - 1/2^{k+1}$ such that

$$\forall \xi \in E,\ \forall z_0 \in L_k,\ |f(\rho_k(\xi - z_0) + z_0) - \varphi_{j_k}(\xi)| < \frac{1}{2^{k+1}}.$$

It is now sufficient to set $E_k = E \cap G_k$. □

REMARKS. If we are not interested by the growth near the boundary, we keep the residuality in $O(B)$ of holomorphic functions with universal radial limits. The proof remains the same, except that we replace Lemma 3 by the following result.

LEMMA 4. *Let φ be a continuous function on S, $g \in O(B)$, $\varepsilon > 0$ and $0 < r < 1$. There exists a compact subset E of S and a holomorphic polynomial f such that*

(i) $m(E) > 1 - \varepsilon$,
(ii) $\|f - \varphi\|_E < \varepsilon$,
(iii) $\|f - g\|_{\overline{B_r}} < \varepsilon$.

Proof. We apply Lemma 3 of [**7**] to produce a compact subset G of $\overline{B}$ with interior V such that $\overline{B_r} \subset V \subset B_\rho$ (ρ is any positive number satisfying $r < \rho < 1$), $E = G \cap S$ has measure $m(E) > 1 - \varepsilon$, and Mergelyan's Theorem is true on G. Let $\tilde{\varphi}$ be a continuous extension of φ to $\overline{B}$, with $\tilde{\varphi} = 0$ on $\overline{B_\rho}$. $\tilde{\varphi}$ is continuous on G and holomorphic in V. Hence, one can find a polynomial Q such that $\|Q - \tilde{\varphi}\|_G < \frac{\varepsilon}{2}$. We also approximate g by a holomorphic polynomial R such that $\|g - R\|_{\overline{B_r}} < \varepsilon/2$.

Let θ be a continuous function on $\overline{B}$ with $\theta = 0$ on S and $\theta = 1$ on $\overline{B_\rho}$. Applying Lemma 3 of [**7**] again, one obtains a holomorphic polynomial which approximates θ on G. In particular, for $\eta = \frac{\varepsilon}{2\|Q\|_{\overline{B}} + 2\|R\|_{\overline{B}}}$, one can find such a polynomial h with $\|h\|_E < \eta$ and $\|h - 1\|_{\overline{B_r}} < \eta$.

We finally set $f = Q(1-h) + hR$. Observe that if z belongs to E, then one has

$$|f(z) - \varphi(z)| \leq |Q(z) - \varphi(z)| + |h(z)|(|Q(z)| + |R(z)|) < \varepsilon.$$

On the other hand, if z belongs to $\overline{B_r}$, one obtains

$$|f(z) - g(z)| \leq |R(z) - g(z)| + |1 - h(z)|(|Q(z)| + |R(z)|) < \varepsilon.$$

This completes the proof of Lemma 4. □

Note added in proof. After this work was completed, Y. Dupain informed us that part of the result of our Theorem 1 could also be deduced from the main Theorem of his paper [**5**]. We mention that the work of Dupain implies neither the existence of holomorphic functions with respect to every center nor the residuality of such functions. The proofs used here (a category argument) and in his paper (a clever construction) are totally different.

4. Radial limits of Dirichlet series. Let $f(s)=\sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series, and $\sigma_a(f)$ its abscissa of absolute convergence:

$$\sigma_a(f)=\inf\left\{\sigma\in\mathbb{R};\ \sum_{n\geq 1}|a_n|n^{-\sigma}\text{ converges}\right\}.$$

We are interested in radial limits of Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$, the space of all Dirichlet series which are absolutely convergent in the right half-plane $\mathbb{C}_+$. $\mathcal{D}_a(\mathbb{C}_+)$ is a Fréchet space, endowed with the topology given by the family of semi-norms

$$\left\|\sum_{n\geq 1} a_n n^{-s}\right\|_\sigma=\sum_{n\geq 1}|a_n|n^{-\sigma},\ \text{defined for } \sigma>0.$$

This space is analogous to $O(B)$, but for Dirichlet series. Although there are very few Dirichlet series among holomorphic functions, it was proved in [**2**] that one can find a universal Dirichlet series "à la Nestoridis". The key point of the proof is a weak version of Mergelyan's Theorem for Dirichlet series. Since Mergelyan's Theorem is the heart of the argument of the previous section, the following result is not surprising.

THEOREM 2. *The set of Dirichlet series* $f\in\mathcal{D}_a(\mathbb{C}_+)$ *such that, given any measurable function* φ *on* $i\mathbb{R}$*, there exists a decreasing sequence* (σ_j), $\sigma_j\to 0$, *with the property that* $\lim_{j\to+\infty} f(\sigma_j+it)=\varphi(it)$ *for almost every* $t\in\mathbb{R}$*, is a dense* G_δ *subset of* $\mathcal{D}_a(\mathbb{C}_+)$.

Proof. First of all, we recall the following result from [**2**].

LEMMA 5. *Let* $K\subset\{s\in\mathbb{C};\ -1/2<\Re(s)\leq 0\}$ *be a compact set such that* $\mathbb{C}\backslash K$ *is connected. Given any* $f\in\mathcal{D}_a(\mathbb{C}_+)$*, any* $g\in C(K)\cap O(K)$*, and any* $\sigma,\varepsilon>0$*, there exists a Dirichlet polynomial* $P(s)=\sum_{n=1}^N a_n n^{-s}$ *such that:*

$$\begin{cases}\|P-g\|_{C(K)}<\varepsilon,\\ \|P-f\|_\sigma<\varepsilon.\end{cases}$$

If we now endow $\mathbb{R}$ with the measure $dm=\frac{1}{\pi}\frac{dt}{1+t^2}$, it is easy to deduce from this Lemma the same result as stated in Lemma 4, but for Dirichlet series. The rest of the proof follows, mutatis mutandis, that of Theorem 1. □

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