COMPOSITIO MATHEMATICA

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Compositio Math. **154** (2018), 2586–2605.

doi:10.1112/S0010437X18007467

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 154 (2018) 2586–2605
doi:10.1112/S0010437X18007467

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Abstract

A general conjecture is stated on the cone of automorphic vector bundles admitting nonzero global sections on schemes endowed with a smooth, surjective morphism to a stack of G-zips of connected Hodge type; such schemes should include all Hodge-type Shimura varieties with hyperspecial level. We prove our conjecture for groups of type A_1^n, C_2, and $\mathbf{F}_p$-split groups of type A_2 (this includes all Hilbert–Blumenthal varieties and should also apply to Siegel modular 3-folds and Picard modular surfaces). An example is given to show that our conjecture can fail for zip data not of connected Hodge type.

Introduction

This paper develops a particular aspect of our general program launched in [GK]. Recall that the aim of the program is to connect (a) *automorphic algebraicity*, (b) *G-zip geometricity* and (c) *Griffiths–Schmid algebraicity*. This paper zooms in on (b); the basic question which guides us is as follows.

Question A. Let $\mathcal{Z}$ be a zip datum with underlying group G. Assume X is a scheme, endowed with a smooth surjective morphism $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. To what extent is the global geometry of X controlled by the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and the map ζ?

We briefly describe a general framework underlying Question A, before specializing to a concrete instance of it which is studied here. Let G be a connected, reductive $\mathbf{F}_p$-group and $\mu \in X_*(G)$ a cocharacter. By the work of Moonen and Wedhorn for $G = \mathrm{GL}(n)$ [MW04] and Pink, Wedhorn, and Ziegler for general G [PWZ15, PWZ11], the pair (G, μ) gives rise to a zip datum $\mathcal{Z}$ and a stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ (see § 1). To give some sense of what this stack is, and how it is related to more classical objects, let us recall two historical sources of motivation for it.

One source of motivation comes from Hodge theory; the other from the theory of Shimura varieties and their Ekedahl–Oort (EO) stratification in characteristic $p > 0$. Of course these two sources are not disjoint from one another, but they do help to shed light on two different points of view concerning G-zips and their applications. As explained below, the connection with Hodge theory shows that the theory of G-zips can be applied to a very *general* class of schemes in characteristic p, in the same way that classical Hodge theory applies (at least) to smooth complex projective varieties. By contrast, the connection with the EO stratification of Shimura varieties, also recalled below, gives particularly rich and *special* examples of G-zips and is fruitful for applications to automorphic forms (cf. our previous papers [GK, GK17]).

Received 13 February 2017, accepted in final form 4 April 2018, published online 31 October 2018.
2010 Mathematics Subject Classification 11F33, 11F46, 14L30 (primary).
Keywords: G-zips, Hasse invariants, Shimura varieties, automorphic vector bundles.

Motivation I: Hodge theory

As already observed in the introduction to the Moonen–Wedhorn paper [MW04], the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is a characteristic-p analogue of a period domain, or more generally a Mumford–Tate domain [GGK12]. Suppose $f : Y \to X$ is a proper, smooth morphism of schemes in characteristic p satisfying the Deligne–Illusie conditions: $\dim(Y/X) < p$ and f admits a smooth lift $\tilde{f} : \tilde{Y} \to \tilde{X}$ with $\tilde{X}$ flat over $\mathbf{Z}/p^2\mathbf{Z}$. Then the Hodge–de Rham spectral sequence for Y/X degenerates at E_1 and the conjugate spectral sequence degenerates at E_2, giving rise to the Hodge and conjugate filtrations, respectively. Given $i \geqslant 0$, the parabolic subgroups P and Q stabilizing the Hodge and conjugate filtrations of $H^i_{\mathrm{dR}}(Y/X)$ give rise to a zip datum $\mathcal{Z}$. Thus $H^i_{\mathrm{dR}}(Y/X)$ is a $\mathrm{GL}(n)$-zip of type $\mathcal{Z}$, where n is the rank of $H^i_{\mathrm{dR}}(Y/X)$. It yields a map $\zeta : X \to \mathrm{GL}(n)\text{-}\mathtt{Zip}^{\mathcal{Z}}$ [MW04, § 6].

The map ζ should be thought of as an analogue of the period map associated to a variation of Hodge structure (VHS) over a smooth, projective $\mathbf{C}$-scheme. This analogy will be the subject of forthcoming work with Y. Brunebarbe. The analogue of our guiding Question A in Hodge theory is one that has played a central role in algebraic geometry for the past 150 or so years, going back (at least) to the work of Abel and Riemann on periods of abelian integrals: To what extent does a period map control the global geometry of the base of a VHS?

If the Hodge and conjugate filtrations preserve certain tensors in $H^i_{\mathrm{dR}}(Y/X)^{\langle\otimes\rangle}$, then the above $\mathrm{GL}(n)$-zip arises from a G-zip, where G is the group stabilizing the tensors. For example, Moonen and Wedhorn explain how, when Y/X is a family of K3 surfaces, the primitive part of $H^2_{\mathrm{dR}}(Y/X)$ is naturally an $\mathrm{SO}(2,19)$-zip. They use this to provide a unified framework for previous works on stratifications of families of K3 surfaces by Artin, Katsura and van der Geer, and Ogus (see [MW04, Ogu01] and the references therein). In classical Hodge theory, the above brings to mind the Mumford–Tate group. We hope to return to the question of Mumford–Tate groups for G-zips in future work.

Motivation II: Shimura varieties

Let us now turn to motivation stemming from the theory of Shimura varieties and the EO stratification. Let X be the special fiber of the Kisin–Vasiu model of a Hodge-type Shimura variety at a place of good reduction, attached to a Shimura datum $(\mathbf{G}, \mathbf{X})$ and a hyperspecial level at p. Write $G := G_{\mathbf{Z}_p} \times \mathbf{F}_p$, where $G_{\mathbf{Z}_p}$ is a reductive $\mathbf{Z}_p$-model of $\mathbf{G}_{\mathbf{Q}_p}$. If $\mathcal{A}/X$ is a universal abelian scheme corresponding to some symplectic embedding of $(\mathbf{G}, \mathbf{X})$, then $H^1_{\mathrm{dR}}(\mathcal{A}/X)$ is naturally a G-zip; the classifying map $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is smooth by Zhang [Zha18] and surjective by Nie [Nie15] and Kisin, Madapusi Pera, and Shin [Kis15].

The EO stratification of X is given by the fibers of ζ. When X is of PEL type (respectively, Siegel type) this recovers the earlier definition of the EO stratification by Moonen [Moo99] (respectively, Ekedahl and Oort [Oor99]). Even in these special cases the scheme-theoretic structure of the strata and stratification property is most easily seen via the G-zip approach.

Specializing Question A to the Shimura variety X gives:

Question B. To what extent is the global geometry of the Shimura variety X controlled by the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and the morphism ζ?

Since the underlying set of the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is just a finite set of points, it may initially appear to the reader that a pair $(G\text{-}\mathtt{Zip}^{\mathcal{Z}}, \zeta)$ will capture little of the geometry of X. This would suggest that the answer to both Questions A and B should be 'minimal' and that $(G\text{-}\mathtt{Zip}^{\mathcal{Z}}, \zeta)$ retains much less geometric information than a period map in classical Hodge theory.

One of the key aims of this paper is to provide evidence to the contrary. The previous papers [KW18, Kos16, GK, GK17] already deduced nontrivial information about the global geometry

https://doi.org/10.1112/S0010437X18007467 Published online by Cambridge University Press

of X from a study of group-theoretical Hasse invariants on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ (and the closely related stacks of zip flags). For example, we showed by this method that the EO stratification of X is uniformly principally pure [GK, Corollary 4.3.5] and that all EO strata of the minimal compactification $X^{\min}$ are affine [GK, Proposition 6.3.1].

The global sections cone

We return to the general setting of Question A. Set $\mathcal{X} := G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and consider a characteristic p scheme X equipped with a morphism $\zeta : X \to \mathcal{X}$.

This note is concerned with an example where some global geometry of X may be understood purely in terms of $\mathcal{X}$: the question of which automorphic vector bundles $\mathscr{V}_X(\lambda)$ admit global sections on X. Our forthcoming joint work with Stroh and Brunebarbe will study the closely related question of which $\mathscr{V}_X(\lambda)$ are ample on X and on its partial flag spaces (see § 1.3 below and [GK17]).

Let L be the Levi subgroup of G given by $\mathcal{Z}$ and choose a Borel pair (B, T) appropriately adapted to $\mathcal{Z}$ (see § 1.1). A $B \cap L$-dominant character $\lambda \in X^*(T)$ gives rise to a vector bundle $\mathscr{V}_{\mathcal{X}}(\lambda)$ on $\mathcal{X}$ (§ 1.4). Put $\mathscr{V}_X(\lambda) := \zeta^*(\mathscr{V}_{\mathcal{X}}(\lambda))$.

We call the $\mathscr{V}_X(\lambda)$ the *automorphic vector bundles* associated to ζ. When X is the special fiber of a Hodge-type Shimura variety, the $\mathscr{V}_X(\lambda)$ recover the usual automorphic vector bundles on X. For a general X, it is *a priori* unclear what, if any, relationship the $\mathscr{V}_X(\lambda)$ bear to automorphic forms.

Let $C_{\mathcal{X}}$ (respectively, C_X) denote the (saturated) cones of $\lambda \in X^*(T)$ such that $\mathscr{V}_{\mathcal{X}}(n\lambda)$ (respectively, $\mathscr{V}_X(n\lambda)$) admits a nonzero global section for some $n \geqslant 1$ (§ 2.1). The inclusion $C_{\mathcal{X}} \subset C_X$ holds in general simply by pulling back sections. Below we propose a conjecture that, under certain hypotheses, the global sections cones of $\mathcal{X}$ and X are equal. This is surprising because one expects bundles on X to admit many more sections than bundles on $\mathcal{X}$; nevertheless our conjecture predicts that the mere existence of sections is to a large extent controlled by $\mathcal{X}$.

Our approach to the conjecture, as well as one of the hypotheses, will be in terms of the stack of zip flags $\mathcal{Y} \to \mathcal{X}$ and the flag space $Y = X \times_{\mathcal{X}} \mathcal{Y}$, both recalled in § 1.3. The stack $\mathcal{Y}$ admits a stratification parameterized by the Weyl group of T in G; if ζ is smooth then the same is true of Y by pullback.

We will say that a reduced scheme S is pseudo-complete if every $h \in H^0(S, \mathcal{O}_S)$ is locally constant. For example, a proper, reduced scheme is pseudo-complete.

Conjecture C (Conjecture 2.1.6). Let $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. Assume that:

(a) the zip datum $\mathcal{Z}$ is of connected Hodge type (Definition 1.1.2);

(b) for all connected components $X^\circ \subset X$, the map $\zeta : X^\circ \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is smooth and surjective;

(c) all strata closures in Y are pseudo-complete.

Then the global sections cones of X and $\mathcal{X}$ coincide: $C_{\mathcal{X}} = C_X$.

The following result establishes the conjecture in some special cases.

Theorem D (Theorems 4.2.3, 5.1.1). *Suppose that either:*

(a) *G is of type A_1^n (i.e., $G^{\mathrm{ad}}_{\overline{\mathbf{F}}_p} \cong \mathrm{PGL}(2)^n_{\overline{\mathbf{F}}_p}$) and $\mathcal{Z}$ is attached to a Borel of G;*

(b) *G is of type C_2 and $\mathcal{Z}$ is proper of connected Hodge type (Definition 1.1.2);*

(c) *G is $\mathbf{F}_p$-split of type A_2 and $\mathcal{Z}$ is proper of connected Hodge type.*

Then Conjecture C holds for $\mathcal{Z}$.

In the three cases of Theorem D, $C_{\mathcal{X}}$ is given explicitly in Corollary 4.2.4 and Figures 1–2, respectively.

For example, Conjecture C applies to a proper smooth k-scheme X endowed with a smooth, surjective map $X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. It should also apply when X is the special fiber at p of a Shimura variety of Hodge type with hyperspecial level at p (see § 2.2). Specializing to this case, Theorem D(a) applies to Hilbert modular varieties. Modulo a technical assumption on toroidal compactifications, part (b) applies to Siegel modular 3-folds (Shimura varieties of type GSp(4)), and part (c) applies to Picard modular surfaces at a split prime (GU(2, 1)-Shimura varieties at a split prime). However, as emphasized above, the range of applications of both the conjecture and the theorem is much broader than just Shimura varieties.

Both $\mathcal{X}, \mathcal{Y}$ are stratified (§ 1.3), and when ζ is smooth, so are X, Y. They are then the Zariski closures of their top-dimensional strata. A natural generalization of Conjecture C is to ask when the global sections cone $C_{\mathcal{X},w}$ of a stratum $\mathcal{X}_w$ of $\mathcal{X}$ coincides with the cone $C_{X,w}$ of the corresponding stratum X_w of X. We find it more natural to study the analogous question on the flag space $Y \to X$; see Question 2.1.4. The situation for general strata seems more complicated than for X itself; see Remark 2.1.7(b). Nevertheless, when G is of type A_1^n, we define a notion of 'admissible stratum' (Definition 4.2.1) and prove the following theorem.

Theorem E (Theorem 4.2.3). *Let G be a group of type A_1^n and $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ as in Theorem D(a). Then $C_{\mathcal{X},w} = C_{X,w}$ holds for each admissible stratum $\mathcal{X}_w$. Moreover, if G is $\mathbf{F}_p$-split, then all strata are admissible.*

Note that $X = Y$ in the context of Theorem 4.2.3. See Theorem 5.1.1 for related results about the equality of cones of flag strata when G is of type C_2 or split of type A_2.

Diamond shared with us his conjecture that when X is a Hilbert modular variety, the cone C_X is equal to that spanned by Goren's partial Hasse invariants [Gor01]. As Diamond later informed us, a related question of determining the 'minimal cone' of mod p Hilbert modular forms had been raised earlier by Andreatta and Goren [AG04, Question 15.8]. Inspired by Diamond's conjecture and the observation that Goren's cone can be reinterpreted as the zip cone $C_{\mathcal{X}}$, we were led to study Conjecture C, first for groups of type A_1^n and then more generally.

After we announced the results of this paper and communicated them to Diamond, we received the preprint of Diamond and Kassaei [DK17]. It determines the 'minimal cone' of Hilbert modular forms mod p. As a corollary Diamond and Kassaei deduce a different proof that $C_{\mathcal{X}} = C_X$ in the special case when X is the special fiber of a Hilbert modular variety at a place of good reduction [DK17, Corollary 1.3].

The approach of Diamond and Kassaei [DK17] uses special properties of Hilbert modular varieties (e.g., the results of Tian and Xiao that in the Hilbert case EO strata are $\mathbf{P}^1$-bundles over quaternionic Shimura varieties). By contrast, our methods only use the map ζ and its basic properties, which hold for all Hodge-type Shimura varieties and even more general $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$-schemes. It remains to be seen whether the Diamond–Kassaei result on the 'minimal cone' holds in our more general setting, or whether this finer information is special to Hilbert modular varieties.

Outline

Section 1 recalls the theory of G-zips, G-zip-flags and automorphic vector bundles in this context. In § 2 we define the global sections cones in $X^*(T)$ and formulate our conjectural generalization of Theorem D to zip data of Hodge type; see Conjecture 2.1.6. Section 3 gives some general results which form the basic strategy for proving Theorem D. The proof of Theorem D(a) is the subject of § 4. Our results on groups of type A_2, C_2, and C_3 are given in § 5.

1. Review of zip data, flag spaces and automorphic bundles

1.1 Zip data [PWZ15, PWZ11]

Fix an algebraic closure k of $\mathbf{F}_p$. Let G be a connected reductive $\mathbf{F}_p$-group. Denote by $\varphi : G \to G$ the Frobenius morphism. Let $\mathcal{Z} := (G, P, L, Q, M, \varphi)$ be a Frobenius zip datum. Recall that this means P, Q are parabolic subgroups of G_k and $L \subset P$, $M \subset Q$ are Levi subgroups, with the property that $\varphi(L) = M$. We say that $\mathcal{Z}$ is a zip datum of Borel type if P is a Borel subgroup of G (this implies that Q is a Borel too). The zip group E is the subgroup of $P \times Q$ defined by

$$E := \{(x, y) \in P \times Q, \varphi(\overline{x}) = \overline{y}\} \tag{1.1.1}$$

where $\overline{x} \in L$ and $\overline{y} \in M$ denote the Levi components of x and y, respectively. Let $G \times G$ act on G by $(a, b) \cdot g := agb^{-1}$; restriction yields an action of E on G. The stack of G-zips of type $\mathcal{Z}$ is isomorphic to the quotient stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}} \simeq [E\backslash G]$. We say that $\mathcal{Z}$ is *proper* if P is a proper parabolic subgroup of G.

For convenience, we assume that there exists a Borel pair (B, T) defined over $\mathbf{F}_p$ such that $B \subset P$. Then there exists an element $z \in W$ such that ${}^zB \subset Q$, and (B, T, z) defines a W-frame for $\mathcal{Z}$ [GK17, Definition 2.3.1].

A cocharacter datum (G, μ) is a connected, reductive $\mathbf{F}_p$-group G together with $\mu \in X_*(G)$. Every such (G, μ) gives rise to a zip datum $\mathcal{Z}_\mu$ [GK17, § 2.2]. Given a cocharacter datum (G, μ), one has the associated adjoint datum $(G^{\mathrm{ad}}, \mu^{\mathrm{ad}})$, where G^{ad} is the adjoint group of G and μ^{ad} is the composition of μ with $G \twoheadrightarrow G^{\mathrm{ad}}$. A morphism $\varphi : (G_1, \mu_1) \to (G_2, \mu_2)$ between cocharacter data is a morphism of groups $\varphi : G_1 \to G_2$ such that $\mu_2 = \varphi \circ \mu_1$.

Definition 1.1.2. Let (G, μ) be a cocharacter datum. Let $\mathrm{PGSp}(2g)$ be the split adjoint group of type C_g and $\mu_g \in X_*(\mathrm{PGSp}(2g))$ a minuscule cocharacter. We say that (G, μ) is of *connected Hodge type* if for some $g \geqslant 1$ there exists a morphism $(G, \mu) \to (\mathrm{PGSp}(2g), \mu_g)$ such that $G \to \mathrm{PGSp}(2g)$ has central kernel. A zip datum $\mathcal{Z}$ is of connected Hodge type if $\mathcal{Z} = \mathcal{Z}_\mu$ for some (G, μ) of connected Hodge type.

1.2 Notation

Let $\Phi \subset X^*(T)$ (respectively, Φ_L) be the set of T-roots in G (respectively L). Let Φ^+ (respectively, Φ_L^+) be the system of positive roots given by putting $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi_L^+$) when the $(-\alpha)$-root group $U_{-\alpha}$ is contained in B (respectively, $B_L := B \cap L$). Write $\Delta \subset \Phi^+$ (respectively, $I \subset \Phi_L^+$) for the subset of simple roots.

For $\alpha \in \Phi$, let s_α be the corresponding root reflection. Let W (respectively, W_L) be the Weyl group of Φ (respectively, Φ_L). Then $(W, \{s_\alpha | \alpha \in \Delta\})$ is a Coxeter system; denote by $\ell : W \to \mathbf{N}$ its length function and by $\leqslant$ the Bruhat–Chevalley order. Write w_0 for the longest element of W. The *lower neighbors* of $w \in W$ are the $w' \in W$ satisfying $w' \leqslant w$ and $\ell(w') = \ell(w) - 1$. Let ${}^IW \subset W$ be the subset of elements $w \in W$ which are minimal in the coset W_Lw. The I-dominant characters of T are denoted $X^*_{+,I}(T)$.

The Zariski closure of a subscheme or substack Z is denoted $\overline{Z}$; it is always endowed with the reduced structure.

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1.3 Review of flag spaces and their stratification [GK, GK17]

Let (B,T) be an $\mathbf{F}_p$-Borel pair of G such that $B \subset P$ (this can be assumed after possibly conjugating $\mathcal{Z}$; see [GK17, Remark 2.3.2(2)]). Write $\mathcal{X} := G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. The stack of zip flags $\mathcal{Y} := G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}$ was defined in [GK, § 2.1]; see also [GK17, § 3]. It is isomorphic to $[E'\backslash G]$, where $E' = E \cap (B \times G)$. Recall that $\mathcal{Y}$ parametrizes G-zips with an additional compatible B-torsor. Thus $\mathcal{Y}$ is naturally a P/B-bundle $\pi : \mathcal{Y} \to \mathcal{X}$.

Consider a morphism of stacks $\zeta : X \to \mathcal{X}$. Form the fiber product

$$\begin{tikzcd} Y \arrow[r, "\zeta_Y"] \arrow[d, "\pi_{Y/X}"'] & \mathcal{Y} \arrow[d, "\pi"] \\ X \arrow[r, "\zeta"'] & \mathcal{X} \end{tikzcd} \tag{1.3.1}$$

We call Y the (full) flag space of X attached to B [GK, § 9.1]. In [GK17, § 7.2], we also defined partial flag spaces for intermediate parabolics $B \subset P_0 \subset G$, but these are not used here. By [GK17, § 4.1], there is a zip datum $\mathcal{Z}_B := (G, B, T, {}^zB, T, \varphi)$ and natural smooth morphisms of stacks Ψ and β as follows:

$$\mathcal{Y} \xrightarrow{\Psi} G\text{-}\mathtt{Zip}^{\mathcal{Z}_B} \xrightarrow{\beta} [B\backslash G/B]. \tag{1.3.2}$$

The stacks $\mathcal{X}_B := G\text{-}\mathtt{Zip}^{\mathcal{Z}_B}$ and $\mathrm{Sbt} := [B\backslash G/B]$ are finite; their points are both parametrized by the Weyl group W. The stack Sbt admits the *Schubert stratification* by locally closed substacks Sbt_w for $w \in W$ ordered by the Bruhat–Chevalley order. The morphism β is bijective, but not an isomorphism. By pullback, the fibers of Ψ define a stratification of $\mathcal{Y}$ by locally closed substacks $\mathcal{Y}_w$, with the same closure relations.

Let $Y_w := \zeta_Y^{-1}(\mathcal{Y}_w)$, the corresponding flag stratum in Y. Both $\mathcal{Y}_w$ and Y_w are endowed with the reduced structure. The Zariski closure $\overline{\mathcal{Y}}_w$ of $\mathcal{Y}_w$ is normal [GK17, § 4]. Let $Y_w^* := \zeta_Y^{-1}(\overline{\mathcal{Y}}_w)$. If ζ is smooth, then so is ζ_Y and then $Y_w^* = \overline{Y}_w$. If ζ is not smooth, Y_w^* may not be the Zariski closure of Y_w.

Although we shall not need it explicitly in this paper, recall that $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ also admits a 'zip stratification', whose strata are parameterized by ${}^IW$ [PWZ15, PWZ11]. When G is of type A_1^n and $\mathcal{Z}$ is of Borel type, $G\text{-}\mathtt{Zip}^{\mathcal{Z}} = G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}$ and the zip stratification agrees with that of $G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}$ recalled above.

1.4 Automorphic vector bundles

All of the automorphic bundles studied in this paper arise from the general associated sheaves construction. If a k-group H acts on a k-scheme X, then every H-representation ρ on a k-vector space yields a vector bundle $\mathscr{V}(\rho)$ on the quotient stack $[H\backslash X]$; cf. [GK, § N.4] and [Jan03, § 5.8]. In particular, every representation of E (respectively E', B, $B \times B$) yields an associated vector bundle on $\mathcal{X} = [E\backslash G]$ (respectively, $\mathcal{Y} = [E'\backslash G]$, P/B, $[B\backslash G/B]$).

A character $\lambda \in X^*(T)$ gives a P-equivariant line bundle $\mathscr{L}_\lambda$ on the flag variety P/B. The P-module $H^0(\lambda) := H^0(P/B, \mathscr{L}_\lambda)$ gives an E-module via the first projection $E \to P$. Denote by $\mathscr{V}_{\mathcal{X}}(\lambda)$ the associated vector bundle on $\mathcal{X}$. If $\lambda \in X^*(T)$ is not I-dominant, then $\mathscr{V}_{\mathcal{X}}(\lambda) = 0$ by definition. Given a stack X and a morphism $\zeta : X \to \mathcal{X}$, set $\mathscr{V}_X(\lambda) := \zeta^*(\mathscr{V}(\lambda))$. We call the $\mathscr{V}_{\mathcal{X}}(\lambda)$ *automorphic vector bundles.*

Let $\mathscr{L}_{\mathcal{Y}}(\lambda)$ be the line bundle on $\mathcal{Y}$ associated to λ via the first projection $E' \to B$. Set $\mathscr{L}_Y(\lambda) := \zeta_Y^*(\mathscr{L}_{\mathcal{Y}}(\lambda))$. One has the direct image formulas:

$$(\pi_{Y/X})_* \mathscr{L}_Y(\lambda) = \mathscr{V}_X(\lambda). \tag{1.4.1}$$

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2. The conjecture

2.1 Cones

Let G be a connected, reductive $\mathbf{F}_p$-group. Fix a zip datum $\mathcal{Z} := (G, P, L, Q, M, \varphi)$, with an $\mathbf{F}_p$-Borel pair (B, T) such that $B \subset P$. Recall that $\mathcal{X} := G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and $\mathcal{Y} = G\text{-}\mathtt{ZipFlag}^{\mathcal{Z}}$ denote the associated stacks of G-zips and G-zip flags. Let X be a stack together with a map $\zeta : X \to \mathcal{X}$. The trivial example $X = \mathcal{X}$ is allowed here. Let $\zeta_Y : Y \to \mathcal{Y}$ be the base change of ζ by $\pi : \mathcal{Y} \to \mathcal{X}$.

Definition 2.1.1. For $w \in W$, the global sections cones of X and Y_w are

$$C_X := \{\lambda \in X^*(T) \mid H^0(X, \mathscr{V}_X(n\lambda)) \neq 0 \text{ for some } n \geqslant 1\}, \tag{2.1.2}$$
$$C_{Y,w} := \{\lambda \in X^*(T) \mid H^0(Y_w^*, \mathscr{L}_Y(n\lambda)) \neq 0 \text{ for some } n \geqslant 1\}. \tag{2.1.3}$$

Put $C_Y = C_{Y,w_0}$. By (1.4.1), one has $H^0(Y, \mathscr{L}_Y(\lambda)) = H^0(X, \mathscr{V}_X(\lambda))$; thus $C_X = C_Y$. If ζ is surjective, so is ζ_Y and then $C_{\mathcal{Y},w} \subset C_{Y,w}$ for all $w \in W$.

The main focus of this paper is the following instance of Question A.

Question 2.1.4. For which $w \in W$ is $C_{\mathcal{Y},w} = C_{Y,w}$?

Definition 2.1.5. A reduced scheme Z is *pseudo-complete* if every $h \in H^0(Z, \mathcal{O}_Z)$ is locally constant.

Concerning the cone $C_X = C_Y = C_{Y,w_0}$, our principal conjecture is as follows.

Conjecture 2.1.6. Let X be a k-scheme and $\zeta : X \to \mathcal{X}$. Assume that:

(a) the zip datum $\mathcal{Z}$ is of connected Hodge type (Definition 1.1.2);
(b) for any connected component $X^\circ \subset X$, the map $\zeta : X^\circ \to \mathcal{X}$ is smooth and surjective;
(c) for all $w \in W$, Y_w^* is pseudo-complete.

Then the global sections cones of X and $\mathcal{X}$ coincide, that is, $C_X = C_{\mathcal{X}}$.

Remark 2.1.7. As motivation for the assumptions of the conjecture, we note how some variants fail to hold.

(a) A multiple $n \geqslant 1$ as in Definition 2.1.1 is necessary. For example, the special fiber of the (compactified) modular curve satisfies the assumptions of Conjecture 2.1.6. In this case, the Hodge line bundle ω satisfies $H^0(\mathcal{X}, \omega^n) \neq 0 \iff n = (p-1)m, m \geqslant 0$.
(b) We conjecture that $C_{\mathcal{Y},w} \neq C_{Y,w}$ when Y is the mod p special fiber of a Hilbert modular 3-fold at a totally inert prime and $w \in W$ has length two. This conjecture would show that the answer to Question 2.1.4 can be 'not all'.
(c) In § 5.6 we give an example of a pair (X, ζ) satisfying assumptions 2.1.6(b)–(c), but not (a), for which $C_{\mathcal{X}} \neq C_X$.

Remark 2.1.8. In contrast to $C_X = C_Y$, when $\mathcal{Y} \neq \mathcal{X}$ it seems difficult to relate the cones of flag strata in Y (respectively, $\mathcal{Y}$) with cones of zip strata in X (respectively, $\mathcal{X}$). For this reason, we do not know if it is reasonable to expect a variant of Conjecture 2.1.6 where (c) is replaced by the analogous condition for strata of X.

2.2 Shimura varieties

Let X be the special fiber of a Hodge-type Shimura variety with hyperspecial level at p. Let G be the corresponding reductive $\mathbf{F}_p$-group and $\mathcal{Z}$ the zip datum of X. By [Zha18], there is a smooth morphism of stacks $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. Let X^{tor} be a proper, smooth toroidal compactification of X afforded by [Mad18]. In [GK, § 6.2], we constructed an extension $\zeta^{\mathrm{tor}} : X^{\mathrm{tor}} \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ of ζ to X^{tor}.

By definition, both ζ and ζ^{tor} satisfy assumption 2.1.6(a). A number of works have recently shown that ζ is surjective on every connected component X° of X; cf. [Lee18, Kis15]. Thus ζ satisfies assumption 2.1.6(b). Since X^{tor} is reduced and proper, $(X^{\mathrm{tor}}, \zeta^{\mathrm{tor}})$ satisfies assumption 2.1.6(c).

Therefore, if ζ^{tor} is smooth, then Conjecture 2.1.6 applies to $(X^{\mathrm{tor}}, \zeta^{\mathrm{tor}})$. Moreover, provided the usual hypotheses are satisfied, the classical Koecher principle implies that $C_X = C_{X^{\mathrm{tor}}}$, so the conjecture for X^{tor} is equivalent to that for X.

The smoothness of ζ^{tor} should follow from the work of Lan and Stroh [LS]. For the special groups appearing in Theorem D, the smoothness of ζ^{tor} may also follow from Boxer's thesis [Box15], once it is suitably reinterpreted in the language of G-zips. We expect that assumption 2.1.6(c) for X itself also follows from a version of Lan and Stroh's Koecher principle for strata [LS, Theorem 2.5.10], but have not checked this.

In any case, assumption 2.1.6(c) certainly holds for Hilbert modular varieties X of dimension >1 by the classical Koecher principle, because then $X = Y$ is its own flag space and the proper strata of X are proper (they do not intersect the toroidal boundary).

3. Strategy of proof

3.1 Some general remarks

Assume X is a k-scheme satisfying assumptions 2.1.6(b)–(c). Proposition 3.4.4 and Corollary 3.4.6 below provide a simple strategy to prove the equality of cones $C_{\mathcal{Y},w} = C_{Y,w}$ for all $w \in W$. This strategy assumes that the stratification of $\mathcal{Y}$ has some particularly nice properties. *A priori*, it supposes neither that $\mathcal{Z}$ is of connected Hodge type, nor uses the fact that Y arises as a fiber product of X and $\mathcal{Y}$ over $\mathcal{X}$.

The problem is then that the hypotheses of Proposition 3.4.4 will usually not be satisfied by all strata. In Theorem D, the only cases where the hypotheses below are satisfied for all $w \in W$ are $\mathbf{F}_p$-split groups of type A_1^n and arbitrary groups of type $A_1 \times A_1$. The work to prove Theorem D in the other cases consists of weakening the hypotheses of Proposition 3.4.4 and using additional knowledge about the cone C_Y. The former leads to the notions of admissibility in § 4.2; the latter uses the fact that, since $Y \to X$ is a flag variety bundle, $C_Y \subset X^*_{+,I}(T)$. Proposition 3.4.7 gives a simple but useful extension of this kind.

3.2 One-dimensional strata

The following proposition will serve as the first step of many inductive arguments later on.

Proposition 3.2.1. *Assume $\zeta : X \to \mathcal{X}$ satisfies assumptions 2.1.6(b)–(c). If $w \in W$ and $\ell(w) = 1$, then $C_{\mathcal{Y},w} = C_{Y,w}$.*

Proof. Let $\lambda \in C_{Y,w}$ and assume $\lambda \notin C_{\mathcal{Y},w}$. Since $\ell(w) = 1$, we have $-\lambda \in C_{\mathcal{Y},w}$. Hence for some $m \geqslant 1$, there exist nonzero $h \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(-m\lambda))$ and $f \in H^0(Y_w^*, \mathscr{L}_Y(m\lambda))$.

Since ζ_Y is smooth, Y_w^* is reduced, so there is an irreducible component $Y_w' \subset Y_w$ where $f|_{Y_w'} \neq 0$. Since h is nowhere vanishing on $\mathcal{Y}_w$, the pullback $\zeta_Y^*(h)$ is nowhere vanishing on Y_w.

In particular, $\zeta_Y^*(h)$ is nowhere zero on Y'_w. So $\zeta_Y^*(h)f \in H^0(Y'_w, \mathcal{O}_{Y'_w})$ is nonzero too. By assumption 2.1.6(c), $\zeta_Y^*(h)f$ is constant. Thus h is nowhere zero on $\mathcal{Y}_w$ and $\mathscr{L}_{\mathcal{Y}}(m\lambda)|_{\mathcal{Y}_w} \simeq \mathcal{O}_{\mathcal{Y}_w}$; this contradicts $\lambda \notin C_{\mathcal{Y},w}$. □

3.3 Changing the center of G

Let $\tilde{G}$ be the simply-connected covering of the derived group of G (in the sense of root data of reductive algebraic groups). Write $\iota : \tilde{G} \to G$ for the natural map. Pulling back $\mathcal{Z}$ to $\tilde{G}$ along ι yields a zip datum $\tilde{\mathcal{Z}}$ for $\tilde{G}$. Write $\tilde{\mathcal{X}} = \tilde{G}\text{-}\mathtt{Zip}^{\tilde{\mathcal{Z}}}$ and $\tilde{\mathcal{Y}}$ for the corresponding stack of zip flags. The map $\iota : \tilde{G} \to G$ induces a homeomorphism $\tilde{\mathcal{X}} \to \mathcal{X}$. Consider the fiber product

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\zeta}} & \tilde{\mathcal{X}} \\ \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota} \\ X & \xrightarrow{\zeta} & \mathcal{X}. \end{array}$$

Let $\tilde{T}$ be a maximal torus in $\tilde{G}$ such that $\iota(\tilde{T}) \subset T$.

LEMMA 3.3.1. *Let X be a stack and $\zeta : X \to \mathcal{X}$ arbitrary ($X = \mathcal{X}$ allowed). For all $w \in W$, one has $\iota^* C_{Y,w} = C_{\tilde{Y},w}$. In particular, $C_{\mathcal{Y},w} = C_{Y,w} \Longleftrightarrow C_{\tilde{\mathcal{Y}},w} = C_{\tilde{Y},w}$.*

Proof. By pullback of sections, $\iota^* C_{Y,w} \subset C_{\tilde{Y},w}$ for all $w \in W$. The reverse inclusions follow from the descent lemma [GK, Lemma 3.2.2]. □

Consequently, the equality of global sections cones for (X, ζ) depends only on the type of G, not on G itself. By contrast, it may happen that $H^0(Y_w^*, \mathscr{L}_Y(\lambda)) = 0$ while $H^0(\tilde{Y}_w^*, \mathscr{L}_{\tilde{Y}}(\iota^*\lambda)) \neq 0$ (for example this is the case for $G = \mathrm{GL}(2)$, $w = e$ and λ corresponding to the $(p+1)$st power of the Hodge line bundle).

3.4 The basic strategy

DEFINITION 3.4.1. Let $w \in W$ and $\lambda \in X^*(T)$.

(a) A *partial Hasse invariant* of $\mathscr{L}_{\mathcal{Y}}(\lambda)$ on $\mathcal{Y}_w^*$ is a section $s \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\lambda))$ which is pulled back from the Schubert stratum Sbt_w^* (§ 1.3).

(b) The *Schubert cone* $C_{\mathrm{Sbt},w} \subset C_{\mathcal{Y},w}$ of w is the cone of $\lambda \in X^*(T)$ such that $\mathscr{L}_{\mathcal{Y}}(N\lambda)$ admits a partial Hasse invariant on $\mathcal{Y}_w$ for some $N \geqslant 1$.

Recall that $\mathcal{Y}_w^*$ and Y_w^* are normal, so we may consider Weil divisors. If $s \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\lambda))$ is a partial Hasse invariant, its divisor will be supported on a (possibly empty) union of codimension-one strata closures in $\mathcal{Y}_w^*$. If ζ is smooth, the multiplicities in $\mathrm{div}(\zeta_Y^*(s))$ equal those of $\mathrm{div}(s)$.

DEFINITION 3.4.2. Let $w \in W$ and $\{w_i\}_{i=1}^n$ the set of lower neighbors of $w \in W$. A *separating system* of partial Hasse invariants for $\mathcal{Y}_w$ is a set of partial Hasse invariants $\{s_i\}_{i=1}^n$ with $s_i \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\lambda_i))$ such that $\mathrm{div}(s_i) = \mathcal{Y}_{w_i}^*$.

Remark 3.4.3.

(a) For groups of type A_1^n, there always exists a particularly simple separating system of partial Hasse invariants; see § 4.1.

(b) In general, many strata do not admit a separating system, as the number of lower neighbors of $w \in W$ can exceed the semisimple rank of G.

(c) If $\mathrm{Pic}(G) = 0$, then clearly the element $w_0 \in W$ admits a separating system.

PROPOSITION 3.4.4. *Let $w \in W$ with lower neighbors $\{w_i\}_{i=1}^n$. Assume that:*

(a) *there exists a separating system of partial Hasse invariants for $\mathcal{Y}_w$;*
(b) *one has $\bigcap_{i=1}^n C_{\mathcal{Y},w_i} \subset C_{\mathcal{Y},w}$;*
(c) *each w_i satisfies the equality of cones $C_{Y,w_i} = C_{\mathcal{Y},w_i}$.*

Then w satisfies the equality of cones $C_{Y,w} = C_{\mathcal{Y},w}$.

Proof. Let $\lambda \in C_{Y,w}$ and assume that $\lambda \notin C_{\mathcal{Y},w}$. Choose a nonzero $f \in H^0(Y_w^*, \mathscr{L}_Y(N\lambda))$ for some $N \geqslant 1$. Let $\{s_i \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\lambda_i))\}_{i=1}^n$ be a separating system of partial Hasse invariants for $\mathcal{Y}_w$. By (b), there exists $i \in \{1, \ldots, n\}$ such that $\lambda \notin C_{\mathcal{Y},w_i}$. By (c), $\lambda \notin C_{Y,w_i}$. Hence $H^0(Y_{w_i}^*, \mathscr{L}_Y(N\lambda)) = 0$. Multiplication by $\zeta_Y^*(s_i)$ gives an exact sequence

$$0 \to H^0(Y_w^*, \mathscr{L}_Y(N\lambda - \lambda_i)) \to H^0(Y_w^*, \mathscr{L}_Y(N\lambda)) \to H^0(Y_{w_i}^*, \mathscr{L}_Y(N\lambda)).$$

Thus $H^0(Y_w^*, \mathscr{L}_Y(N\lambda - \lambda_i)) \simeq H^0(Y_w^*, \mathscr{L}_Y(N\lambda))$. In particular, $N\lambda - \lambda_i \in C_{Y,w}$.

It is clear that $N\lambda - \lambda_i \notin C_{\mathcal{Y},w}$; otherwise λ would also lie in $C_{\mathcal{Y},w}$. So we may repeat the same procedure to $N\lambda - \lambda_i$, but with possibly a different $i' \in \{1, \ldots, n\}$. Hence there exists a sequence $(i_d)_{d\geqslant 1}$ with values in $\{1, \ldots, n\}$ such that $N\lambda - \sum_{d=1}^m \lambda_{i_d} \in C_{Y,w}$ for all $m \geqslant 1$ and such that multiplication by $\prod_{d=1}^m s_{i_d}$ gives an isomorphism

$$H^0\left(Y_w^*, \mathscr{L}_Y\left(N\lambda - \sum_{d=1}^m \lambda_{i_d}\right)\right) \xrightarrow{\sim} H^0(Y_w^*, \mathscr{L}_Y(N\lambda)). \tag{3.4.5}$$

There exists $j \in \{1, \ldots, n\}$ such that $i_d = j$ for infinitely many $d \geqslant 1$. We have shown that f is divisible by s_j^m for all $m \geqslant 1$. This implies that s_j is nowhere vanishing, a contradiction. □

COROLLARY 3.4.6. *Assume that assumptions 3.4.4(a)–(b) hold for all $w \in W$. Then $C_{Y,w} = C_{\mathcal{Y},w}$ for all $w \in W$. In particular, $C_Y = C_{\mathcal{Y}}$.*

PROPOSITION 3.4.7. *Let $w \in W$ be a lower neighbor of w_0. Assume that:*

(a) *the Picard group of G is trivial;*
(b) *$X_{+,I}(T) \cap C_{\mathcal{Y},w} \subset C_{\mathcal{Y}}$;*
(c) *one has $C_{Y,w} = C_{\mathcal{Y},w}$.*

Then one has the equality of cones $C_Y = C_{\mathcal{Y}}$.

Proof. Let $\lambda \in C_Y$ and assume that $\lambda \notin C_{\mathcal{Y}}$. Fix a nonzero $f \in H^0(Y, \mathscr{L}_Y(N\lambda))$ for some $N \geqslant 1$. Since $C_Y \subset X_{+,I}(T)$, we deduce that $\lambda \notin C_{\mathcal{Y},w} = C_{Y,w}$. By (a), we can find $\mu \in X^*(T)$ and a partial Hasse invariant $s \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\mu))$ such that $\operatorname{div}(s) = \mathcal{Y}_w^*$. Since $N\lambda \notin C_{Y,w}$, we have $H^0(Y_w^*, \mathscr{L}_Y(N\lambda)) = 0$. Hence f restricts to zero along Y_w^*, and thus f is divisible by $s' := \zeta_Y^*(s)$; there exists $g \in H^0(Y, \mathscr{L}_Y(N\lambda - \mu))$ such that $f = s'g$. We have shown that $N\lambda - \mu \in C_Y$, hence $\lambda - \mu/N \in C_Y$.

It is clear that $\lambda - \mu/N \notin C_{\mathcal{Y}}$, because otherwise λ would also lie in $C_{\mathcal{Y}}$. Repeating this argument, we deduce that f is divisible by s'^m for all $m \geqslant 1$, which is a contradiction, as in the proof of Proposition 3.4.4. □

https://doi.org/10.1112/S0010437X18007467 Published online by Cambridge University Press

4. Example 1: Groups of type A_1^n

In this section, we study Question 2.1.4 for $\mathbf{F}_p$-groups G of type A_1^n. As a corollary, we deduce results about Hilbert modular varieties.

We prove Conjecture 2.1.6 when G is of type A_1^n (and $\mathcal{Z}$ of Borel type). Therefore, throughout this section one has $X = Y$. If G is $\mathbf{F}_p$-split or if G splits over the quadratic extension $\mathbf{F}_{p^2}$, then we show that $C_{\mathcal{Y},w} = C_{Y,w}$ holds for all $w \in W$, which gives a complete answer to Question 2.1.4.

In the general case, we define a set of admissible strata for which one has $C_{\mathcal{Y},w} = C_{Y,w}$. However, we conjecture that not all w satisfy this equality of cones.

4.1 Notation

Let $n \geqslant 1$ be an integer; let $n = n_1 + \cdots + n_r$, be a partition of n with $n_i \geqslant 1$ for all $i = 1, \ldots, r$. Consider the $\mathbf{F}_p$-reductive group G defined by

$$G := G_1 \times \cdots \times G_r, \quad G_i := \mathrm{Res}_{\mathbf{F}_{p^{n_i}}/\mathbf{F}_p}(\mathrm{SL}_{2,\mathbf{F}_{p^{n_i}}}). \tag{4.1.1}$$

Define $N_m = \sum_{i=1}^m n_i$ for all $1 \leqslant m \leqslant r$ and $N_0 := 0$. Denote again by σ the permutation of $\{1, \ldots, n\}$ defined as a product $\sigma = c_1 \cdots c_r$ where c_i is the n_i-cycle $c_i = (N_i\ (N_i - 1) \cdots (N_{i-1} + 1))$ for $i = 1, \ldots, r$. There is an isomorphism

$$G_k \simeq \mathrm{SL}_{2,k}^n \tag{4.1.2}$$

such that the action of $\sigma \in \mathrm{Gal}(k/\mathbf{F}_p)$ on $G(k) \simeq \mathrm{SL}_2(k)^n$ is given by

$${}^{\sigma}(x_1, \ldots, x_n) := (\varphi(x_{\sigma(1)}), \varphi(x_{\sigma(2)}), \ldots, \varphi(x_{\sigma(n)})). \tag{4.1.3}$$

Let $T \subset \mathrm{SL}_{2,k}$ be the diagonal torus. We identify $X^*(T) = \mathbf{Z}$ by sending $m \in \mathbf{Z}$ to the character $\mathrm{diag}(x, x^{-1}) \mapsto x^m$. Define $\widetilde{T} := T \times \cdots \times T \subset G_k$ and identify similarly $X^*(\widetilde{T}) = \mathbf{Z}^n$. Let $B \subset \mathrm{SL}_{2,k}$ be the Borel subgroup of lower-triangular matrices, and define $\widetilde{B} := B \times \cdots \times B \subset G$. Denote by $\widetilde{B}_-$ the opposite Borel. The Weyl group of G is $W = \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2$, where $\mathfrak{S}_2$ is the permutation group on two elements.

Let $\mathcal{Z}$ be the Borel-type zip datum $(G, \widetilde{B}, \widetilde{T}, \widetilde{B}_-, \widetilde{T}, \varphi)$. Denote by $\mathcal{X}$ the corresponding stack of G-zips. Fix a map $\zeta : X \to \mathcal{X}$ satisfying the assumptions of Conjecture 2.1.6. Define a Zariski open subset $U \subset \mathrm{SL}_2$ as the non-vanishing locus of the function

$$h : \mathrm{SL}_{2,k} \to \mathbf{A}_k^1, \quad h : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a. \tag{4.1.4}$$

Denote by $Z \subset \mathrm{SL}_{2,k}$ the zero locus of h (note that Z is a reduced subscheme). Identify the elements of W with subsets $S \subset \{1, \ldots, n\}$ by the map

$$W \to \mathcal{P}(\{1, \ldots, n\}), \quad \tau = (\tau_1, \ldots, \tau_n) \mapsto \{i \in \{1, \ldots, n\} : \tau_i = 1\}. \tag{4.1.5}$$

For a subset $S \subset \{1, \ldots, n\}$, write

$$S = S_1 \sqcup \cdots \sqcup S_r, \quad S_i := S \cap \{N_{i-1} + 1, \ldots, N_i\}. \tag{4.1.6}$$

The zip stratum corresponding to a subset $S \subset \{1, \ldots, n\}$ is defined by

$$G_S := \prod_{i=1}^n G_{S,i}, \tag{4.1.7}$$

where $G_{S,i} := U$ if $i \in S$ and $G_{S,i} := Z$ if $i \notin S$. For a subset $S \subset \{1, \ldots, n\}$, denote by $\mathcal{X}_S := [E \backslash G_S] \subset G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and $X_S \subset X$ the corresponding locally closed subsets, endowed with the reduced structure, and define similarly $\mathcal{X}_S^*$ and X_S^* as their respective Zariski closures.

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Write C_S and $C_{X,S}$ for the cones corresponding to the zip stratum G_S, as defined in § 2.1. Denote by $e_1, \ldots, e_n \in \mathbf{Z}^n$ the natural basis of $\mathbf{Z}^n$. For any subset $S \subset \{1, \ldots, n\}$, we define a $\mathbf{Q}$-basis $\mathcal{B}_S = (\beta_{1,S}, \ldots, \beta_{n,S})$ of $\mathbf{Q}^n$ by

$$\beta_{i,S} := \begin{cases} e_i - pe_{\sigma(i)} & \text{if } i \in S, \\ -e_i - pe_{\sigma(i)} & \text{if } i \notin S. \end{cases} \tag{4.1.8}$$

The cone C_S is the set of characters $\lambda \in X^*(\widetilde{T})$ such that

$$\lambda = \sum_{i \in S} a_i \beta_{i,S}, \tag{4.1.9}$$

where $a_i \in \mathbf{N}$ for all $i \in S$ and $a_i \in \mathbf{Z}$ for all $i \notin S$.

4.2 The result

Let $d \in \mathbf{Z}_{\geqslant 1}$ be an integer. For any subset $R \subset \mathbf{Z}$ consisting of d consecutive integers, the map $\phi_R : R \to \mathbf{Z}/d\mathbf{Z}$, $k \mapsto \overline{k}$ is a bijection. Let $\mathbf{Z}/d\mathbf{Z}$ act on itself by addition. Then ϕ_R yields a natural action of $\mathbf{Z}/d\mathbf{Z}$ on the following objects:

(a) the set R itself;
(b) the powerset $\mathcal{P}(R)$;
(c) the set of pairs (S, j) where $S \subset R$ and $j \in S$.

Definition 4.2.1. Let $d \geqslant 1$ be an integer and $R \subset \mathbf{Z}$ a subset consisting of d consecutive numbers.

(1) A normalized admissible pair of R is a pair (S, x_r) such that $S \subset R$ is of the form $S = \{x_1, \ldots, x_r\}$ with $x_1 < \cdots < x_r$ and $x_{i+1} - x_i$ odd for all $1 \leqslant i \leqslant r - 1$.
(2) An admissible pair of R is a pair (S, j) that is in the $\mathbf{Z}/d\mathbf{Z}$-orbit of a normalized admissible pair for R.
(3) A G-admissible pair is a pair (S, j) such that $j \in S_m$ for some $1 \leqslant m \leqslant r$ (notation as in (4.1.6)) and (S_m, j) is an admissible pair of $\{N_{m-1} + 1, \ldots, N_m\}$.
(4) A subset $S \subset \{1, \ldots, n\}$ is G-admissible if the pair (S, j) is G-admissible for all $j \in S$.

Remark 4.2.2. The following give examples of G-admissible subsets of $\{1, 2, \ldots, n\}$:

(a) the singleton $\{i\}$ for all $i \in \{1, \ldots, n\}$;
(b) the set $\{1, \ldots, n\}$ itself;
(c) if $\sigma(i) = i$ for all $i \in \{1, \ldots, n\}$ (equivalently if $r = n$), then every subset S of $\{1, 2, \ldots, n\}$ is G-admissible.

The cases (a), (b), (c) correspond respectively to one-dimensional strata, the top-dimensional stratum, and the case that G is $\mathbf{F}_p$-split.

Theorem 4.2.3. *Let $\zeta : X \to \mathcal{X}$ be a map satisfying assumptions 2.1.6(a)–(c). For each G-admissible $S \subset \{1, \ldots, n\}$, one has $C_{Y,S} = C_{\mathcal{Y},S}$. In particular, $C_X = C_{\mathcal{X}}$.*

Theorem 4.2.3 will be proved at the end of § 4 as a corollary of Proposition 4.3.9 below. As an application, let F be a totally real number field and let X be the special fiber of a Hilbert modular variety attached to F. Using Lemma 3.3.1, we may reduce to Theorem 4.2.3. Hence we have the following result.

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Corollary 4.2.4. *If $[F:\mathbf{Q}]>1$, then Conjecture 2.1.6 holds for X. Explicitly,*

$$C_X = C_{\mathcal{X}} = \left\{\sum_{i=1}^{n} a_i(e_i - pe_{\sigma(i)}), a_i \in \mathbf{N}\right\}. \tag{4.2.5}$$

Remark 4.2.6. When X is a modular curve (so that $F=\mathbf{Q}$), X fails to satisfy Koecher's principle and Corollary 4.2.4 is false, because X is affine and the Hodge line bundle is both ample and anti-ample on X. However, Corollary 4.2.4 trivially holds for the compactified modular curve X^{tor}.

Remark 4.2.7. When $[F:\mathbf{Q}]>1$, all EO strata of X of codimension >0 are proper. Hence, in this case, assumption 2.1.6(c) already follows from the classical Koecher principle for X (independently of [LS]).

4.3 Proof of Theorem 4.2.3

Assume $\lambda \in \mathbf{Q}^n$ is a quasi-character expressed in the basis $\mathcal{B}_S$. Choose an element $j \in S$, and consider the subset $S\backslash\{j\}$, and the corresponding basis $\mathcal{B}_{S\backslash\{j\}}$ of $\mathbf{Q}^n$. We want to decompose λ in the basis $\mathcal{B}_{S\backslash\{j\}}$.

For this, it suffices to determine the decomposition of the vector $\beta_{j,S} = e_j - pe_{\sigma(j)}$ in $\mathcal{B}_{S\backslash\{j\}}$. Write $s_i := |S_i|$, for $i=1,\dots,r$, and $s := |S| = \sum_{i=1}^{r} s_i$. Let $m \in \{1,\dots,r\}$ such that $j \in S_m$. Let $a_1,\dots,a_n \in \mathbf{Q}$ be the unique rational numbers such that

$$\beta_{j,S} = \sum_{i=1}^{n} a_i \beta_{i,S\backslash\{j\}}. \tag{4.3.1}$$

One sees immediately that $a_i = 0$ for $j \notin \{N_{m-1}+1,\dots,N_m\}$. For $1 \leqslant a \leqslant b \leqslant n$, we define $\gamma(a,b,S) \in \{\pm 1\}$ by the formula

$$\gamma(a,b,S) := (-1)^{|\{x \in S,\ a \leqslant x \leqslant b\}|}. \tag{4.3.2}$$

For $d \geqslant 1$, $x=(x_1,\dots,x_d) \in \mathbf{Q}^d$ and $y=(y_1,\dots,y_d) \in \mathbf{Q}^d$, define $x * y$ as the vector $(x_1y_1,\dots, x_dy_d)$. Then we have the formula

$$\delta \begin{pmatrix} a_{N_{m-1}+1} \\ a_{N_{m-1}+2} \\ \vdots \\ a_{j-2} \\ a_{j-1} \\ a_j \\ a_{j+1} \\ a_{j+2} \\ \vdots \\ a_{N_m} \end{pmatrix} = \begin{pmatrix} (-1)^{s_m+N_m+j} \\ (-1)^{s_m+N_m+j+1} \\ \vdots \\ (-1)^{n_m+s_m-1} \\ (-1)^{n_m+s_m} \\ 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{N_m+j} \end{pmatrix} * \begin{pmatrix} \gamma(N_{m-1}+1, j-1, S) \\ \gamma(N_{m-1}+2, j-1, S) \\ \vdots \\ \gamma(j-2, j-1, S) \\ \gamma(j-1, j-1, S) \\ 1 \\ 1 \\ \gamma(j+1, j+1, S) \\ \vdots \\ \gamma(j+1, N_m-1, S) \end{pmatrix} * \begin{pmatrix} 2p^{j-N_{m-1}-1} \\ 2p^{j-N_{m-1}-2} \\ \vdots \\ 2p^2 \\ 2p \\ p^{n_m} + (-1)^{s_m+n_m+1} \\ 2p^{n_m-1} \\ 2p^{n_m-2} \\ \vdots \\ 2p^{j-N_{m-1}} \end{pmatrix} \tag{4.3.3}$$

where $\delta = p^{n_m} + (-1)^{s_m+n_m}$.

For the rest of the proof, we abbreviate $H^0(S,\lambda) := H^0(Y_S^*, \mathscr{L}_Y(\lambda))$.

Lemma 4.3.4. *Let $S \neq \emptyset$ be a subset and $i \in S$. Let $\lambda \in \mathbf{Z}^n$ with coordinates $(x_1,\dots,x_n)$ in $\mathcal{B}_S$. Then there exists M (depending on S and λ) such that, for $m \geqslant M$, one has $H^0(S, \lambda - m\beta_{i,S}) = 0$.*

Proof. By induction on $s = |S|$. Let $m \in \{1, \dots, r\}$ such that $i \in S_m$. By assumption 2.1.6(c), the result is clear if $s = 1$. So assume $s > 1$. Also, we may assume $S_m = \{x_1, \dots, x_{s_m}\}$ with $x_1 < \cdots < x_{s_m}$ and $i = x_1$. Let $j \in S - \{i\}$. By induction, there exists $M(\lambda, j) \geqslant 1$ such that $H^0(S - \{j\}, \lambda - m\beta_{i,S}) = 0$ for $m \geqslant M(\lambda, j)$.

Consider the unique (up to scalar) nonzero $h_j \in H^0(\mathcal{Y}_S^*, \beta_{j,S})$. The vanishing locus of h_j is exactly $\mathcal{Y}_{S-\{j\}}^*$, and that of $\zeta^*(h_j)$ is $X_{S-\{j\}}^*$ by smoothness of ζ. Multiplication by $\zeta^*(h_j)$ yields a short exact sequence of sheaves

$$0 \to \mathscr{L}_Y(\lambda - m\beta_{i,S} - \beta_{j,S})|_{Y_S^*} \to \mathscr{L}_Y(\lambda - m\beta_{i,S})|_{Y_S^*} \to \mathscr{L}_Y(\lambda - m\beta_{i,S})|_{Y_{S-\{j\}}^*} \to 0$$

and a long exact sequence of cohomology

$$0 \to H^0(S, \lambda - m\beta_{i,S} - \beta_{j,S}) \to H^0(S, \lambda - m\beta_{i,S}) \to H^0(S - \{j\}, \lambda - m\beta_{i,S}) \to \cdots.$$

Hence, for $m \geqslant M(\lambda, j)$, one has an isomorphism

$$H^0(S, \lambda - m\beta_{i,S} - \beta_{j,S}) \simeq H^0(S, \lambda - m\beta_{i,S}). \tag{4.3.5}$$

Now there exists an integer $M(\lambda - \beta_{j,S}, j) \geqslant M(\lambda, j)$ such that

$$H^0(S - \{j\}, \lambda - \beta_{S,j} - m\beta_{i,S}) = 0 \tag{4.3.6}$$

for $m \geqslant M(\lambda - \beta_{j,S}, j)$. Applying the exact sequence above for this character shows that

$$H^0(S, \lambda - m\beta_{i,S} - 2\beta_{j,S}) \simeq H^0(S, \lambda - m\beta_{i,S} - \beta_{j,S}) \simeq H^0(X_S^*, \lambda - m\beta_{i,S})$$

for $m \geqslant M(\lambda - \beta_{j,S}, j)$. Continuing in this way, it is clear that we can find $M'(\lambda, j) \geqslant 1$ such that, for $m \geqslant M'(\lambda, j)$, there exists λ' with coordinates $(x_1', \dots, x_n')$ in $\mathcal{B}_S$ such that $x_j' < 0$ and $H^0(S, \lambda - m\beta_{i,S}) \simeq H^0(S, \lambda' - m\beta_{i,S})$. Hence, for large m, there exists μ with coordinates $(y_1, \dots, y_n)$ in $\mathcal{B}_S$ such that $y_j < 0$ for all $j \in S$ and $H^0(S, \lambda - m\beta_{i,S}) \simeq H^0(S, \mu)$. By assumption 2.1.6(c), this space is zero. □

LEMMA 4.3.7. *Let $d \geqslant 1$ be an integer and $R \subset \mathbf{Z}$ a subset consisting of d consecutive numbers. Let (S, j) be a normalized admissible pair for R. Write $S = \{\alpha_1, \dots, \alpha_s\}$ with $s = |S|$ and $\alpha_1 < \cdots < \alpha_s$. Then the integer $\alpha_1 + \alpha_s + s$ is odd.*

Proof. Since (S, j) is a normalized admissible pair for R, one has $j = \alpha_s$ and $\alpha_{i+1} - \alpha_i$ is odd for all $i = 1, \dots, s - 1$. Hence

$$\alpha_s - \alpha_1 = \sum_{i=1}^{s-1} (\alpha_{i+1} - \alpha_i) \tag{4.3.8}$$

has the same parity as $s - 1$. □

PROPOSITION 4.3.9. *Let (S, j) be a G-admissible pair. Let $\lambda \in \mathbf{Z}^n$ with coordinates $(x_1, \dots, x_n)$ in $\mathcal{B}_S$. If $x_j < 0$, then $H^0(S, \lambda) = 0$.*

Proof. We prove the result by induction on $|S|$. Let $m \in \{1, \dots, r\}$ such that $j \in S_m$.

(1) *Reduction to the case when $x_i < 0$ for all $i \in S \backslash S_m$.* Assume therefore $S \neq S_m$ and let $i \in S \backslash S_m$. The pair $(S - \{i\}, j)$ is again G-admissible. Let $(y_1, \dots, y_n)$ be the coordinates of λ in the basis $\mathcal{B}_{S-\{i\}}$. Since $i \notin S_m$, one has $y_j = x_j < 0$, so we deduce by induction that $H^0(S - \{i\}, \lambda) = 0$. This implies that $H^0(S, \lambda) \simeq H^0(S, \lambda - \beta_{S,i})$.

Applying the same argument successively, one eventually shows that $H^0(S,\lambda) \simeq H^0(S,\lambda')$ for some λ' whose coordinates in $\mathcal{B}_S$ are $(x'_1,\dots,x'_n)$ such that $x'_i<0$ for all $i\in S\backslash S_m$ and $x'_j<0$. Hence we may assume that this holds for λ from the start. In particular, if $S_m=\{j\}$, then one already deduces that $H^0(S,\lambda)=0$ using assumption 2.1.6(c). Therefore, we assume from now on that $s_m>1$ and $x_i<0$ for all $i\in S\backslash S_m$.

Using the Galois action, we may assume that $S_m=\{\alpha_1,\dots,\alpha_{s_m}\}$, $j=\alpha_{s_m}$, and $\alpha_1<\cdots<\alpha_{s_m}$. The pair $(S-\{\alpha_1\},j)$ is again admissible. Let $(y_1,\dots,y_n)$ be the coordinates of λ in the basis $\mathcal{B}_{S-\{\alpha_1\}}$. The relevant coordinates are $y_{\alpha_2},\dots,y_{\alpha_{s_m}}$. For all $i>1$, one has

$$y_{\alpha_i}=x_{\alpha_i}+(-1)^{\alpha_i+\alpha_1+i}x_{\alpha_1}\frac{2p^{n_m+\alpha_1-\alpha_i}}{p^{n_m}+(-1)^{s_m+n_m}}. \tag{4.3.10}$$

Take $i=s_m$ (so $\alpha_i=j$) in (4.3.10). Then Lemma 4.3.7 shows that the integer $\alpha_{s_m}+\alpha_1+s_m$ appearing in the formula above is always odd. Hence the formula for $i=s_m$ reads

$$y_j=x_j-x_{\alpha_1}\frac{2p^{n_m+\alpha_1-j}}{p^{n_m}+(-1)^{s_m+n_m}}. \tag{4.3.11}$$

This fact will be used in the following.

(2) *Reduction to the case when* $x_{\alpha_1}<0$. If $x_{\alpha_1}\geqslant 0$ then (4.3.11) shows that $y_j<0$. Since $(S-\{\alpha_1\},j)$ is admissible, we have by induction $H^0(S-\{\alpha_1\},\lambda)=0$. Hence, $H^0(S,\lambda)=H^0(S,\lambda-\beta_{\alpha_1,S})$.

We can apply this argument to reduce x_{α_1} by one as long as it is nonnegative, and the process stops when x_{α_1} reaches the value -1.

(3) *The case when* n_m+s_m *even.* The 'jump' from α_{s_m} to α_1 has parity $n_m+\alpha_1+\alpha_{s_m}$, which is the same as the parity of n_m+s_m+1 by Lemma 4.3.7. In this case, it is therefore odd. Hence the pair (S,α_i) is also admissible for all $i=1,\dots,s_m$.

In particular, we may apply the first step to the pair (S,α_1). It then implies that we can reduce to the case when x_{α_2} is negative. By repeating this process for all α_i, $i=1,\dots,s_m$, we obtain $H^0(S,\lambda)=H^0(S,\lambda')$ for some $\lambda'\in\mathbf{Z}^n$ with coordinates $(x'_1,\dots,x'_n)$ in the basis $\mathcal{B}_S$ and $x'_i<0$ for all $i\in\{1,\dots,n\}$. Hence $H^0(S,\lambda)=0$.

(4) *The case when* n_m+s_m *odd.* In this case, the pair $(S-\{\alpha_1\},\alpha_i)$ is G-admissible for all $i=2,\dots,s_m$ and the pair $(S-\{j\},\alpha_i)$ is G-admissible for all $i=1,\dots,s_m-1$.

Let $(z_1,\dots,z_n)$ be the change of basis of λ to $\mathcal{B}_{S-\{j\}}$. Then

$$z_{\alpha_1}=x_{\alpha_1}+x_j\frac{2p^{j-\alpha_1}}{p^{n_m}-1}. \tag{4.3.12}$$

Recall that x_j and x_{α_1} are both negative. Also, recall formula (4.3.11) above:

$$y_j=x_j-x_{\alpha_1}\frac{2p^{n_m+\alpha_1-j}}{p^{n_m}-1}, \tag{4.3.13}$$

where $(y_1,\dots,y_n)$ denote the coordinates of λ in the basis $\mathcal{B}_{S-\{\alpha_1\}}$. Note that since we assumed $p>2$, we have

$$\frac{2p^{n_m-1}}{p^{n_m}-1}<1 \tag{4.3.14}$$

because $n_m>1$. Hence one has the implications

$$|x_{\alpha_1}|\leqslant|x_j|\Longrightarrow y_j<0, \tag{4.3.15}$$

$$|x_j|\leqslant|x_{\alpha_1}|\Longrightarrow z_{\alpha_1}<0. \tag{4.3.16}$$

We reduce alternately the value of x_{α_1} and x_j. We may assume, for example, that $|x_{\alpha_1}| \leqslant |x_j|$. In this case, we have $y_j < 0$. Applying induction to the admissible subset $(S - \{\alpha_1\}, j)$ gives $H^0(S - \{s_1\}, \lambda) = 0$. This implies that $H^0(S, \lambda)$ does not change (up to isomorphism) when we replace x_{α_1} by $x_{\alpha_1} - 1$. We may repeat this argument until x_{α_1} reaches $x_j - 1$. Then we have $|x_j| \leqslant |x_{\alpha_1}|$, so $z_{\alpha_1} < 0$. Analogously we can replace x_j by $x_j - 1$. In this way, we can reduce alternately the values of x_{α_1} and x_j to arbitrarily negative integers without changing $H^0(S, \lambda)$ up to isomorphism. Applying Lemma 4.3.4, we have $H^0(S, \lambda) = 0$. □

Proof of Theorem 4.2.3. Fix a G-admissible subset $S \subset \{1, \ldots, n\}$. We prove that $C_{\mathcal{Y},S} = C_{Y,S}$. The inclusion $C_{\mathcal{Y},S} \subset C_{Y,S}$ is clear. Conversely, let $\lambda \notin C_{\mathcal{Y},S}$. Then there exists $i \in S$ such that $x_i < 0$, where $(x_1, \ldots, x_n)$ denote the coordinates of λ in the basis $\mathcal{B}_S$. By definition, the pair (S, i) is G-admissible. It follows from Proposition 4.3.9 that $H^0(S, N\lambda) = 0$ for all $N \geqslant 1$. Hence, $\lambda \notin C_{Y,S}$. □

5. Further examples and counterexamples: Types A_2, C_2 and C_3

5.1 The results

THEOREM 5.1.1. *Suppose $\mathcal{Z}$ is a proper zip datum of connected Hodge type whose underlying group G is either of type C_2 or $\mathbf{F}_p$-split of type A_2. Then Conjecture 2.1.6 holds for $\mathcal{Z}$. More precisely, $C_{Y,w} = C_{\mathcal{Y},w}$ for all $w \in W$, except possibly in each case when w' is the unique lower neighbor of w_0 satisfying $w' \notin {}^IW$.*

For all $w \in W$, the $C_{\mathcal{Y},w}$ are given explicitly in Figures 1 and 2. By the theorem, this gives an explicit description of $C_Y = C_X$. In terms of §§ 5.2–5.3, the exceptional strata in Theorem 5.1.1 are given respectively by $w = (14)$ and $w = (123)$. The results recalled in § 2.2 imply the following corollary.

COROLLARY 5.1.2. *Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum with $\mathbf{G} = \mathrm{GSp}(4)$ or $\mathbf{G}$ a unitary group associated to an imaginary quadratic field in which p splits and $\mathbf{G}_{\mathbf{R}} = \mathrm{GU}(2,1)$. Let X be the special fiber of the associated Shimura variety at a level hyperspecial at p. If X satisfies assumption 2.1.6(c), then Conjecture 2.1.6 holds for X.*

By contrast, the following gives a counterexample to Conjecture 2.1.6 when the zip datum is not of connected Hodge type.

PROPOSITION 5.1.3. *Let X be the special fiber of the Siegel Shimura variety of type $\mathrm{GSp}(6)$ at a level which is hyperspecial at p. The inclusion $C_{\mathcal{X}_B} \subset C_Y$ is strict for the pair $(Y, \Psi \circ \zeta_Y)$ (§ 1.3).*

5.2 Groups of type C_n

For $n \geqslant 1$, let G be an $\mathbf{F}_p$-group of type C_n and $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ a zip datum of connected Hodge type. Identify the root system of (T, G) with $(\mathbf{Q}^n, \Phi)$, where

$$\Phi = \{\pm \mathbf{e}_i \pm \mathbf{e}_j | 1 \leqslant i \neq j \leqslant n\} \cup \{\pm 2\mathbf{e}_i | 1 \leqslant i \leqslant n\}. \tag{5.2.1}$$

Then $W \cong \{\sigma \in S_{2n} | \sigma(a) + \sigma(2n + 1 - a) = 2n + 1 \text{ for all } 1 \leqslant a \leqslant 2n\}$. Fix $\Delta = \{\mathbf{e}_i - \mathbf{e}_{i+1} | 1 \leqslant i \leqslant n - 1\} \cup \{2\mathbf{e}_n\}$.

Since $\mathcal{Z}_\mu$ is of connected Hodge type, $\mu^{\mathrm{ad}} \in X_*(G^{\mathrm{ad}})$ is minuscule. The unique Δ-dominant, minuscule cocharacter of G^{ad} is the fundamental coweight corresponding to $2\mathbf{e}_n$. Since $L = \mathrm{Cent}_G(\mu)$, the type of L is $I = \Delta \backslash \{2\mathbf{e}_n\}$.

For $G = \mathrm{Sp}(2n)$, we identify $X^*(T) \cong \mathbf{Z}^n$ compatibly with (5.2.1).

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5.3 Groups of type A_2

Let G be an $\mathbf{F}_p$-split group of type A_2 and and $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ a zip datum of connected Hodge type. Identify the root system of (T, G) with $(\{(a_1, a_2, a_3) \in \mathbf{Q}^3 \mid a_1 + a_2 + a_3 = 0\}, \Phi)$, where $\Phi = \{\pm(\mathbf{e}_i - \mathbf{e}_j) | 1 \leqslant i < j \leqslant 3\}$. Then $W \cong S_3$. The two choices $I = \{\mathbf{e}_1 - \mathbf{e}_2\}$ and $I = \{\mathbf{e}_2 - \mathbf{e}_3\}$ correspond to isomorphic zip data. Choose $I = \{\mathbf{e}_1 - \mathbf{e}_2\}$. For $G = \mathrm{GL}(3)$, identify $X^*(T) \cong \mathbf{Z}^3$ compatibly with Φ. It suffices to consider characters of the form $(a_1, a_2, 0) \in \mathbf{Z}^3$; denote such by (a_1, a_2).

5.4 Cone diagrams

Recall the sub-cone $C_{\mathrm{Sbt},w} \subset C_{\mathcal{Y},w}$ for all $w \in W$ (Definition 3.4.1). In Figures 1 and 2, the equations for $C_{\mathrm{Sbt},w}$ appear beside each $w \in W$. In Figure 1, (a_1, a_2) stands for $(a_1, a_2, 0)$. A line connecting w to a lower element w' means that w' is a lower neighbor of w. Furthermore, the line joining w and w' is labeled by $\lambda \in \mathbf{Z}^2$ to indicate that there exists $h \in H^0(\mathcal{Y}_w^*, \mathscr{L}_{\mathcal{Y}}(\lambda))$ whose vanishing locus is exactly $\mathcal{Y}_{w'}$.

In the A_2-split case, every stratum admits a separating system of partial Hasse invariants (Definition 3.4.2). In the C_2 case, every stratum except $\mathcal{Y}_{(14)}$ admits such a system; $\mathcal{Y}^*_{(14)}$ has sections h_1 and h_2 such that $\mathrm{div}(h_1) = 2\mathcal{Y}^*_{(1324)}$ and $\mathrm{div}(h_2) = 2\mathcal{Y}^*_{(1243)}$.

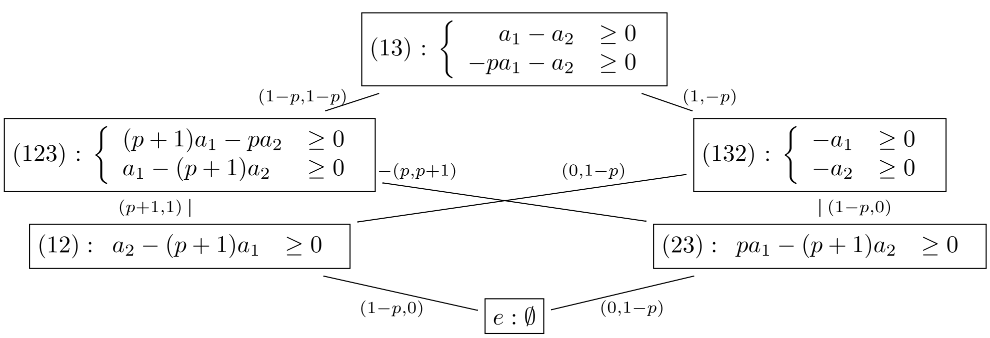

Figure 1. Strata cones and partial Hasse invariants for type A_2-split.

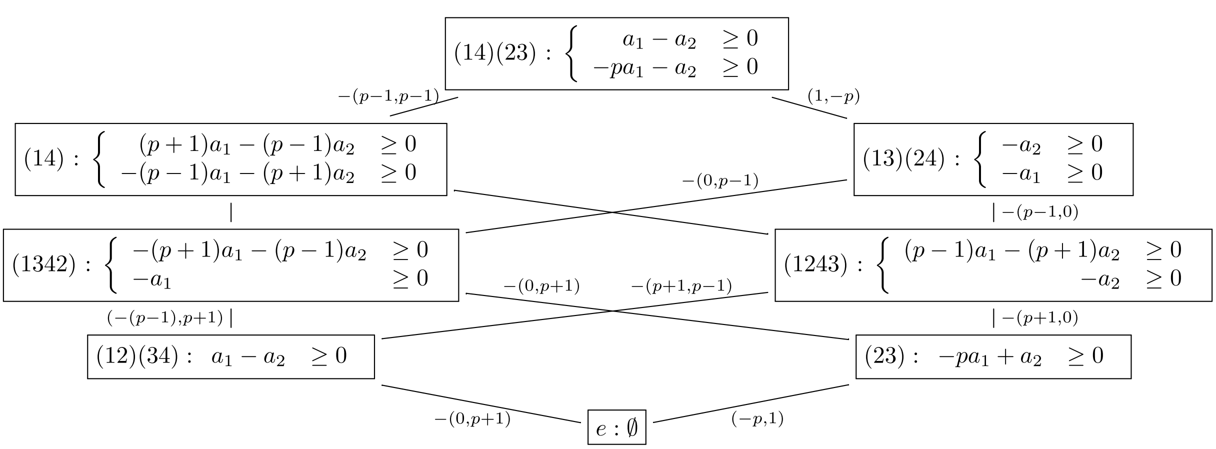

Figure 2. Strata cones and partial Hasse invariants for type C_2.

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5.5 Proof of Theorem 5.1.1

We prove the case C_2; the case A_2 is completely analogous but one step shorter, so it is left to the reader. The inclusions between cones used in the proof below are checked by consulting Figure 2; in case A_2 use Figure 1. Note that both $G = \mathrm{Sp}(2n)$ and $G = \mathrm{GL}(n)$ satisfy $\mathrm{Pic}(G) = 0$.

Proof of Theorem 5.1.1. By Lemma 3.3.1, it suffices to treat $G = \mathrm{Sp}(4)$. We proceed in four steps. First, since (12)(34) and (23) have length one, they satisfy the equality of cones by Proposition 3.2.1. Second, since $C_{(12)(34)} \cap C_{(23)} \subset C_{(1342)} \cap C_{(1243)}$, both (1342) and (1243) satisfy the equality of cones by Proposition 3.4.4. Third, as $C_{(1342)} \cap C_{(1243)} \subset C_{(13)(24)}$, the equality of cones also holds for $C_{(13)(24)}$, again by Proposition 3.4.4. Finally, one has $X^*_{+,I}(T) \cap C_{(13)(24)} \subset C_{\mathcal{X}}$, so the result for (14)(23) follows from the third step and Proposition 3.4.7. □

5.6 The counterexample

Let $G = \mathrm{GSp}(6)$. Then $\tilde{G} = \mathrm{Sp}(6)$ and $\iota : \mathrm{Sp}(6) \to \mathrm{GSp}(6)$ is the inclusion. Let $X = \mathcal{A}_{3,K}$ be the moduli of prime-to-p polarized abelian 3-folds in characteristic p with level K, assumed hyperspecial at p. It is endowed with a smooth morphism $\zeta : X \to G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ (§ 2.2). Let ω be the Hodge line bundle of $\mathcal{A}_{3,K}$. With our identifications, the Hodge character η_ω (such that $\mathscr{V}(\eta_\omega) = \omega$) satisfies $\iota^*(\eta_\omega) = (-1, -1, -1)$.

Recall the stack $\mathcal{X}_B$ (§ 1.3). Proposition 5.1.3 is an immediate consequence of the following lemma.

LEMMA 5.6.1. *Let $(\lambda_n)_{n \geqslant 0} \subset X^*(T)$ be a sequence of characters satisfying*

$$\iota^*(\lambda_n) = -(p + 1 + n, p^2 + n, p^2 + 1 + n).$$

Then one has:

(a) *$H^0(\mathcal{X}_B, \mathscr{V}(m\lambda_n)) = 0$ for all $n, m \geqslant 1$. In other words, $\lambda_n \notin C_{\mathcal{X}_B}$ for all $n \geqslant 1$;*

(b) *for sufficiently large n, the line bundle $\mathscr{L}_Y(\lambda_n)$ is ample on Y;*

(c) *in particular, for sufficiently large $n \geqslant 1$, $\lambda_n \in C_Y$.*

Proof. By Lemma 3.3.1, part (a) is equivalent to $\iota^*\lambda_n \notin \tilde{C}_{\mathcal{X}_B}$ for all n, where $\tilde{C}_{\mathcal{X}_B}$ is the cone attached to the group $\tilde{G}$. Using the methods of [GK17, § 5], we find that the cone $\tilde{C}_{\mathcal{X}_B}$ is generated by the three vectors $v_1 = (1, 0, -p)$, $v_2 = (1, -(p-1), -p)$, and $v_3 = -(p-1, p-1, p-1)$, as these are the pullbacks of the three fundamental weights for $\tilde{G}$ relative to Δ along the map Ψ (§ 1.3). By inverting the 3×3 matrix whose columns are v_1, v_2, v_3, we find that $\tilde{C}_{\mathcal{X}_B}$ is the set of tuples (k_1, k_2, k_3) satisfying:

$$\begin{array}{rrrl} k_1 & -(p+1)k_2 & +k_3 & \geqslant 0, \\ -pk_1 & +(p+1)k_2 & -k_3 & \geqslant 0, \\ pk_1 & & +k_3 & \geqslant 0. \end{array} \tag{5.6.2}$$

For all $n \geqslant 1$, $\iota^*\lambda_n$ fails to satisfy the second inequality in (5.6.2); hence $\iota^*\lambda_n \notin \tilde{C}_{\mathcal{X}_B}$.

Consider 5.6.1(b). By Moret-Bailly [Mor85], ω is ample on $\mathcal{A}_{3,K}$. Since λ_0 is I-dominant and regular, it follows from the discussion surrounding Kempf's vanishing theorem (cf. [Jan03, II, Proposition 4.4]) that the line bundle $\mathscr{L}_Y(\lambda_0)$ is relatively ample for $Y \to \mathcal{A}_{3,K}$. The result now follows from the general lemma which says that if $f : S_1 \to S_2$ is a morphism of schemes, $\mathcal{M}$ is an ample line bundle on S_2, and $\mathcal{N}$ is an f-ample line bundle on S_1, then $f^*\mathcal{M}^a \otimes \mathcal{N}$ is ample on S_1 for sufficiently large a (take $S_1 = Y$, $S_2 = \mathcal{A}_{3,K}$, $\mathcal{M} = \omega$, and $\mathcal{N} = \mathscr{V}(\lambda_0)$).

Finally, (c) follows from (b) because every ample line bundle has a positive power which is very ample, whence it admits a nonzero global section. □

5.7 Concluding remarks

Conjecture 2.1.6 concerns equality of the cones C_Y and $C_{\mathcal{Y}}$. But we also have the cones $C_{\mathrm{Sbt}} \subset C_{\mathcal{Y}}$ of sections pulled back from Sbt (§ 1.3, Definition 3.4.1). The cone C_{Sbt} is easily determined for all G.

The cases of Conjecture 2.1.6 proved here all satisfy $C_Y = C_{\mathcal{Y}} = C_{\mathrm{Sbt}}$. Proposition 5.1.3 shows that $C_Y \neq C_{\mathrm{Sbt}}$ when $X = \mathcal{A}_{3,K}$. In fact, $C_{\mathrm{Sbt}} \neq C_{\mathcal{Y}}$ in this case; the cone $C_{\mathcal{Y}}$ is much more complicated. This leaves hope that $C_{\mathcal{Y}} = C_Y$ holds even though $C_Y \neq C_{\mathrm{Sbt}}$.

In conclusion, a first step to proving Conjecture 2.1.6 for more general groups (e.g., $G = \mathrm{Sp}(2n)$) is to determine the cone $C_{\mathcal{Y}}$. This is the object of our forthcoming work.

Acknowledgements

We thank Fred Diamond for sharing his conjecture with us and both Diamond and Payman Kassaei for helpful discussions and correspondence. We are grateful to Yohan Brunebarbe and Benoît Stroh for collaboration on related questions concerning ampleness of automorphic bundles and period maps in the setting of G-zips. We thank Torsten Wedhorn for helpful comments on an earlier version of this paper.

W.G. thanks the University of Zurich for providing excellent working conditions and the opportunity to present some of the results of this paper during a visit in the Fall of 2016.

Finally, we thank the referees for their helpful comments.

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Wushi Goldring wushijig@gmail.com
Department of Mathematics, Stockholm University, Stockholm SE-10691, Sweden

Jean-Stefan Koskivirta jeanstefan.koskivirta@gmail.com
Department of Mathematics, South Kensington Campus, Imperial College London,
London SW7 2AZ, UK

https://doi.org/10.1112/S0010437X18007467 Published online by Cambridge University Press