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HOMOMORPHISMS FROM $S(X)$ INTO $S(Y)$

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1. Introduction. $S(X)$ is the semigroup under composition of all continuous selfmaps of the topological space X. For certain spaces X and Y we classify completely the homomorphisms from $S(X)$ into $S(Y)$. An application of the main result to $S(I)$ the semigroup of all continuous selfmaps of the closed unit interval I results in the solution of a problem which was suggested in the closing paragraph of [**6**]. We noted there the existence of at least two different types of endomorphisms of $S(I)$. To get one of the first type, simply send every element of $S(I)$ into a single idempotent. To get one of the second, choose any nonconstant idempotent v of $S(I)$ and any homeomorphism h from I onto the range of v and define $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(I)$. We concluded the paper [**6**] by remarking that we didn't know if these exhausted the possibilities for the endomorphisms of $S(I)$. We now know and they do. That is, given any endomorphism φ of $S(I)$, either φ sends every element of $S(I)$ into a single idempotent or it is injective and there exists a unique idempotent v of $S(I)$ and a unique homeomorphism h from I onto the range of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. In addition to this, we describe a moderately extensive class of spaces containing all Euclidean N-cells with the property that if X is *any* space from this class, the existence of a nonconstant homomorphism from $S(X)$ into $S(I)$ forces X to be an arc. The main results are in Section 3. Section 2 is devoted to some topological preliminaries and in Section 4, we discuss some facets of the endomorphism semigroup of $S(X)$ in general and $S(I)$ in particular. Some concluding remarks, further problems and specific conjectures are discussed in Section 5.

2. Topological preliminaries. By a *retract* of a topological space X, we mean simply the range of an idempotent continuous selfmap of the space, and the range of any function f will be denoted by $\operatorname{Ran} f$.

Definition (2.1). A topological space X is said to be *concordant* if it satisfies the following condition: Let $\mathscr{F}$ be any infinite family of distinct continuous selfmaps of X and suppose that some retract V is properly contained in $\operatorname{Ran} f$ for each f in $\mathscr{F}$. Then the ranges of at least two distinct functions in $\mathscr{F}$ also intersect outside of V.

Each finite space is trivially a concordant space and as the next result shows, these are the only possibilities for disconnected concordant spaces. Its proof

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and the proof of the proposition which follows it are both straightforward and will be omitted.

PROPOSITION (2.2). *An infinite concordant space is connected.*

PROPOSITION (2.3). *If X is concordant and Y is a retract of X, then Y is concordant.*

Recall that a *Peano continuum* is any compact connected locally connected metric space or equivalently, any Hausdorff space which is the continuous image of an arc. We characterize those Peano continua which are concordant but before we state the result, it is convenient to have the following.

Definition (2.4). A collection $\{A_\alpha : \alpha \in \Lambda\}$ of arcs in a topological space X is said to be *retractably contiguous* if there exists a retract V of the space such that $V \cap A_\alpha$ is an endpoint of A_α for each $\alpha \in \Lambda$ and for any two distinct arcs A_α and A_β of the collection, $A_\alpha \cap A_\beta$ is either empty or consists of one point and in the latter case that point also belongs to V.

PROPOSITION (2.5). *A Peano continuum is concordant if and only if each retractably contiguous collections of arcs is finite.*

Proof. Let X be any Peano continuum which is not concordant. Then there exists an infinite family $\mathscr{F}$ of distinct continuous selfmaps of X and a retract V of X such that V is properly contained in Ran f for each f in $\mathscr{F}$ and Ran $f \cap$ Ran $g \subset V$ for each pair of functions f and g from $\mathscr{F}$. Now take any $f \in \mathscr{F}$. Since Ran f is arcwise connected and V is properly contained in Ran f, there exists an arc $A_f \subset$ Ran f such that $A_f \cap V$ is one of the endpoints of the arc. Since Ran $f \cap$ Ran $g \subset V$ for distinct f and g, it readily follows that if $A_f \cap A_g \neq \emptyset$ then it consists of a point which lies in V. Consequently, $\{A_f : f \in \mathscr{F}\}$ is a retractably contiguous collection of arcs which is infinite.

Suppose, conversely, that X contains an infinite retractably contiguous family $\{A_\alpha : \alpha \in \Lambda\}$ of arcs and denote the associated retract by V. It follows easily that $V \cup A_\alpha$ is a Peano continuum and hence is the continuous image of X under some map f_α. Then $V \subset$ Ran f_α for each $\alpha \in \Lambda$. However the ranges of no two of the functions intersect outside of V so that X is not concordant.

Now let us recall that a Peano continuum is a *dendrite* [**9**, p. 88] if it contains no simple closed curves. By the component number of a subcontinuum V of X we mean the number (perhaps infinite) of components of $X - V$.

COROLLARY (2.6). *A dendrite is concordant if and only if the component number of every subcontinuum is finite.*

Proof. Suppose X is a concordant dendrite and let V be any subcontinuum of X. Then V itself is a dendrite and hence is a retract of X (of course it is really much more, it is an absolute retract). Let $\{B_\alpha : \alpha \in \Lambda\}$ denote the components of $X - V$. One easily verifies the existence of an arc $A_\alpha \subset B_\alpha \cup V$

such that $A_\alpha \cap V$ is an endpoint of A_α. Consequently, $\{A_\alpha : \alpha \in \Lambda\}$ is a retractably contiguous collection of arcs which, by Theorem (2.3) must be finite. Thus $X - V$ has only finitely many components.

Conversely, suppose that the component number of each subcontinuum of X is finite and let $\{A_\alpha : \alpha \in \Lambda\}$ be a retractably contiguous family of arcs with associated retract V. Let B_α be the component of $X - V$ which contains $A_\alpha - V$. Of course there are only finitely many B_α but what we really need to show is that there are only finitely many A_α. To do this, we need only show that if A_α and A_β are distinct, then so are B_α and B_β. Indeed, suppose that B_α and B_β coincide while A_α and A_β do not. Then, in fact, A_α and A_β have at most one point in common. Choose a point $a \in A_\alpha - (A_\beta \cup V)$ and a point $b \in A_\beta - (A_\alpha \cup V)$. Since $B_\alpha = B_\beta$ is arcwise connected and contains both a and b, there is an arc $H_1 \subset B_\alpha$ with a and b as endpoints. On the other hand, $A_\alpha \cup V \cup A_\beta$ is also arcwise connected so that there is an arc $H_2 \subset A_\alpha \cup V \cup A_\beta$ which also has endpoints a and b. Now $H_1 \cap V = \emptyset$ while $H_2 \cap V \neq \emptyset$. That is there are two different arcs joining a to b and this is a contradiction [**9**, p. 89]. Thus, $B_\alpha \neq B_\beta$ whenever $A_\alpha \neq A_\beta$ and the proof is complete.

Now we recall the definition of a *spray* which was introduced in [**5**, p. 150].

Definition (2.7). A topological space X is a *spray* if it is Hausdorff, connected, first countable and, in addition, satisfies the following three conditions:

(2.7.1) A discrete subspace can be at most countable.

(2.7.2) Each nondegenerate connected subset has nonempty interior.

(2.7.3) Let $\{A_\delta : \delta \in \Delta\}$ be any uncountable collection of retracts of X such that each has more than one point. Then there is at least one whose boundary intersects the interior (with respect to X) of another.

When we prove, in the next section, our main result about homomorphisms from $S(X)$ into $S(Y)$, the space Y will be a concordant spray so it is appropriate to discuss these spaces somewhat further. For any positive integer N, let I_N denote the space formed by taking N copies of the closed unit interval and identifying all the left endpoints. Let J_N denote the space formed by taking N copies of the half-open interval and identifying the endpoints and finally, form C_N by joining N copies of the unit circle together at a single point. It was shown in Proposition (3.3) of [**5**, p. 152] that all of these spaces are sprays. Since both I_N and C_N are Peano continua, it follows from Proposition (2.5) that they are concordant. Since I_N is, in fact a dendrite, it also follows from Corollary (2.6) that I_N is concordant. One shows directly from the definition that J_N is concordant. Thus, I_N, J_N and C_N are all examples of concordant sprays.

Just one more definition and we will be able to proceed to the next section and prove our main result.

Definition (2.8). A topological space X is said to be *clonable* if it is a first countable compact Hausdorff space and, in addition, satisfies the following two conditions:

(2.8.1) Each continuous map from a closed subset of X into X can be extended to a continuous function which maps X into X.

(2.8.2) Every nonempty open subset of X contains a copy of X.

One easily verifies that all Euclidean N-cells are clonable and that the Cantor discontinuum is also clonable.

3. The homomorphism theorem. We are now in a position to prove the main result of the paper. In it, we completely determine the homomorphisms from $S(X)$ the semi-group of all continuous selfmaps of X into the semigroup $S(Y)$ whenever X is clonable and Y is a concordant spray.

THEOREM (3.1). *Let X be any clonable space, let Y be any concordant spray and let φ be a homomorphism from $S(X)$ into $S(Y)$. Then either φ maps everything into one single idempotent of $S(Y)$ or φ is injective. If φ is injective and, in addition, X is not totally disconnected then there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from X onto the range V of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$.*

Proof. Suppose φ does not map everything into one single element of $S(Y)$. We prove that φ is injective by assuming the contrary and deriving a contradiction. Let p be any point of X and let $\langle p \rangle$ denote the constant function which maps everything into the point p. Then $\varphi(\langle p \rangle)$ is some idempotent element v of $S(Y)$. We shall verify that

(3.1.1) $\varphi(\langle x \rangle) = v$ for each $x \in X$.

Since φ is not injective, there exist distinct elements f and g of $S(X)$ such that $\varphi(f) = \varphi(g)$. Now $f \neq g$ implies $f(a) \neq g(a)$ for some $a \in X$. By condition (2.8.1), there exists a continuous selfmap h of X such that $h(f(a)) = x$ and $h(g(a)) = p$. Thus $h \circ f \circ \langle a \rangle = \langle x \rangle$ and $h \circ g \circ \langle a \rangle = \langle p \rangle$ and we get

$$\begin{aligned}\varphi(\langle x \rangle) &= \varphi(h \circ f \circ \langle a \rangle) = \varphi(h) \circ \varphi(f) \circ \varphi(\langle a \rangle) \\ &= \varphi(h) \circ \varphi(g) \circ \varphi(\langle a \rangle) = \varphi(h \circ g \circ \langle a \rangle) = \varphi(\langle p \rangle) = v.\end{aligned}$$

This verifies (3.1.1). Now suppose f is any element of $S(X)$. We use (3.1.1) and get

$$v \circ \varphi(f) = \varphi(\langle p \rangle) \circ \varphi(f) = \varphi(\langle p \rangle \circ f) = \varphi(\langle p \rangle) = v$$

and

$$\varphi(f) \circ v = \varphi(f) \circ \varphi(\langle p \rangle) = \varphi(f \circ \langle p \rangle) = \varphi(\langle f(p) \rangle) = v.$$

That is,

(3.1.2) v is a two-sided zero for $\varphi[S(X)]$.

Hereafter, we will denote Ran v by V and we next show that

(3.1.3) $V \subset \text{Ran } g$ for each $g \in \varphi[S(X)]$, and

$V = \text{Ran } g$ if and only if $v = g$.

Since $g \circ v = v$, it follows that $V \subset \text{Ran } g$. Now suppose that $V = \text{Ran } g$. Since v is idempotent, it is the identity on its range. We use this fact and (3.1.2) to get

$$g(y) = v(g(y)) = v \circ g(y) = v(y)$$

for any $y \in Y$. This verifies (3.1.3).

Next we construct two families of functions in $S(X)$. Since φ is nonconstant and X is a clonable space, it cannot be finite. In addition, X is Hausdorff, so it contains a countably infinite family $\{G_n\}_{n=1}^{\infty}$ of mutually disjoint nonempty open subsets. Since each nonempty open subset of X contains a copy of X, there exists a homeomorphism f_n from X into G_n for each positive integer n. For each n, define a function g_n with domain $\text{Ran } f_n \cup [X - G_n]$ by

$$g_n(x) = f_n^{-1}(x) \quad \text{for } x \in \text{Ran } f_n$$

$$g_n(x) = p \quad \text{for } x \in X - G_n.$$

Then by condition (2.8.1) g_n can be continuously extended to a function $k_n \in S(X)$. Note that

(3.1.4) $k_n \circ f_n = i_X$, the identity on X, and

$k_m \circ f_n = \langle p \rangle$ whenever $m \neq n$.

Next we want to show that

(3.1.5) $\varphi(f_n) \neq \varphi(f_m)$ whenever $n \neq m$.

Let σ be the congruence on $S(X)$ which is induced by φ. That is, $(f, g) \in \sigma$ if and only if $\varphi(f) = \varphi(g)$. Now σ is a proper congruence since φ does not map everything into a single element. According to Theorem (1.6) in Chapter 6 of [**7**, p. 262], there is a largest proper congruence Ω on $S(X)$ and it is defined by $(f, g) \in \Omega$ if and only if anytime one of the functions is injective on a subspace which is homeomorphic to X, the other function is also injective there and, in fact, the two functions coincide there. It is immediate that $(f_n, f_m) \notin \Omega$ whenever $n \neq m$ and since $\sigma \subset \Omega$, this means that $(f_n, f_m) \notin \sigma$ whenever $n \neq m$. In other words (3.1.5) is valid. Furthermore, it is also immediate that $(\langle p \rangle, f_n) \notin \Omega$ for each n and hence that $(\langle p \rangle, f_n) \notin \sigma$. Thus

(3.1.6) $\varphi(f_n) \neq v$ for each n.

In view of (3.1.3), (3.1.5) and (3.1.6), $\{\varphi(f_n)\}_{u=1}^{\infty}$ is an infinite family of distinct functions in $S(Y)$ whose ranges all properly contain the retract V. Thus, since Y is concordant, the ranges of at least two of those functions intersect outside of V. We can take these two functions to be $\varphi(f_1)$ and $\varphi(f_2)$

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and we have

$$q \in \text{Ran } \varphi(f_1) \cap \text{Ran } \varphi(f_2) \cap [X - V].$$

Then $\varphi(f_1)(r_1) = q$ for some $r_1 \in Y$ and $\varphi(f_2)(r_2) = q$ for some $r_2 \in Y$. We use (3.1.4) and we get

$$v(r_1) = \varphi\langle p\rangle(r_1) = \varphi(g_2 \circ f_1)(r_1) = \varphi(g_2)(\varphi(f_1)(r_1)) = \varphi(g_2)(q)$$
$$= \varphi(g_2)(\varphi(f_2)(r_2) = \varphi(g_2 \circ f_2)(r_2) = \varphi(i_X)(r_2).$$

That is,

(3.1.7) $\quad v(r_1) = \varphi(i_X)(r_2)$

and we will soon see that here is where the contradiction arises. Specifically, $v(r_1)$ belongs to V but we will show that $\varphi(i_X)(r_2)$ does not. We observe that

$$\varphi(f_2)(\varphi(i_X)(r_2)) = \varphi(f_2 \circ i_X)(r_2) = \varphi(f_2)(r_2) = q$$

which does not belong to V. To show that (3.1.7) is contradictory we need only show that $\varphi(f_2)$ maps points of V into V. Suppose $z \in V$. Then we use (3.1.2) and the fact that v is the identity on V to get

$$\varphi(f_2)(z) = \varphi(f_2)(v(z)) = \varphi(f_2) \circ v(z) = v(z) = z.$$

Thus, (3.1.7) is, indeed, a contradiction and we conclude that φ is injective.

Next, we assume that X is not totally disconnected and we verify that there exists a unique idempotent continuous selfmap v of Y and a unique homeomorphism h from X onto the range of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. But all this follows from the Main Theorem of [**5**, p. 149]. One easily verifies that condition (2.8.1) implies that X is strongly conformable (Definition (2.4), [**5**, p. 149]) and that Conditions (2.8.1) and (2.8.2) together imply that X is quasi-homogeneous (Definition (2.6), [**5**, p. 149]). Consequently, the Main Theorem of [**5**, p. 149] applies and the proof of the present result is complete.

If, in the previous theorem, we take Y to be either the closed unit interval or the space of real numbers, we can relax one of the conditions on X.

THEOREM (3.2). *Let X be any clonable space, let Y be either the closed unit interval I or the space R of real numbers and let φ be a homomorphism from $S(X)$ into $S(Y)$. Then either φ maps everything into one single idempotent or φ is injective in which case there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from X onto the range of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$.*

Proof. Suppose that φ does not map everything into a single idempotent. Then X must have more than one point and since it is clonable, it must then be infinite. Assume X is totally disconnected. Then $X = A \cup B \cup C$ where A, B and C are mutually disjoint, nonempty subsets which are simultaneously both open and closed. Choose $a \in A$, $b \in B$, $c \in C$ and define f in $S(X)$ by

$$f(x) = b \quad \text{for } x \in A$$
$$f(x) = c \quad \text{for } x \in B$$
$$f(x) = a \quad \text{for } x \in C.$$

Then $\{f, f^2, f^3\}$ is a subgroup of $S(X)$ of order three. Since φ is nonconstant, Theorem (3.1) assures us that it is then injective. But then $S(Y)$ would have to have a subgroup of order three and this would contradict Theorem (5.6) of [**3**, p. 145]. Consequently, X is not totally disconnected and the conclusion now follows from Theorem (3.1).

COROLLARY (3.3). *Suppose that X is a clonable space and that there exists a nonconstant homomorphism φ from $S(X)$ into $S(I)$. Then φ is injective and X is an arc.*

Proof. The conclusion follows from Theorem (3.2) and the fact that the range of any nonconstant idempotent of $S(I)$ is an arc.

COROLLARY (3.4). *Suppose that X is a clonable space and that there exists a nonconstant homomorphism φ from $S(X)$ into $S(R)$. Then φ is injective and X is an arc.*

Proof. It follows immediately from Theorem (3.2) that φ is injective and X is homeomorphic to a subinterval of R. Since X is clonable and every clonable space is compact, this means that X must be an arc.

4. The endomorphism semigroup of $S(X)$. In this section we associate with each semigroup S with identity, another semigroup $EC(S)$ which we call the *endocore* of S. We show that there is a natural homomorphism η from $EC(S)$ into End S the endomorphism semigroup of S. For a large number of semigroups, the homomorphism η is injective and in the case of $S(I)$, it is actually an isomorphism onto the endomorphism semigroup of $S(I)$.

We define the endocore of S. Let $S_A = \{(a, b) \in S \times S : ba = e\}$ where e is the identity of S and let S_E denote the collection of all idempotents of S. Let $EC(S) = S_A \cup S_E$ and define a binary operation on $EC(S)$ by

$$(a, b)(c, d) = (ac, db) \quad \text{for } (a, b), (c, d) \in S_A;$$
$$(a, b)v = avb \quad \text{for } (a, b) \in S_A \text{ and } v \in S_E;$$
$$v(a, b) = v \quad \text{for } (a, b) \in S_A \text{ and } v \in S_E;$$
$$vw = v \quad \text{for } v, w \in S_E.$$

$EC(S)$ with the binary operation just defined is a semigroup and is the semigroup we have already referred to as the endocore of S. Note that $S_A S_E \subset S_E$. In fact, S_E is precisely the set of left zeros of $EC(S)$.

Now we define a map η from $EC(S)$ into End S by

$$\eta(a, b)(x) = axb \quad \text{for all } (a, b) \in S_A \text{ and } x \in S;$$
$$\eta(v)(x) = v \quad \text{for all } v \in S_E \text{ and } x \in S.$$

It is a routine matter to verify that η is a homomorphism from $EC(S)$ into End S and we omit the proof. This is the map we mean when we speak of the *natural* or the *canonical* homomorphism from $EC(S)$ into End S.

Definition (4.1). We say that the collection of left zeros of a semigroup S *distinguishes elements* if $a, b \in S$ and $a \neq b$ implies $az \neq bz$ for some left zero z of S.

PROPOSITION (4.2). *Let S be any semigroup with identity whose collection of left zeros distinguishes elements. Then the canonical homomorphism from $EC(S)$ into* End S *is injective.*

Proof. The conclusion is immediate if S has only one element so we assume card $S > 1$. The map η is certainly injective on S_E. To show that $\eta(a, b) \neq \eta(v)$ for any $(a, b) \in S_A$ and $v \in S_E$, we need only show that $\eta(a, b)$ is nonconstant. In fact, $\eta(a, b)$ is injective, for suppose $\eta(a, b)(c) = \eta(a, b)(d)$. Then $acb = adb$ and since $ba = e$, this implies that $c = d$. It remains for us to show that η is injective on S_A. Suppose $(a, b) \neq (c, d)$. Assume first that $a = c$. Then $b \neq d$ and we have

$$\eta(a, b)(e) = aeb = ab$$
$$\eta(c, d)(e) = ced = ad.$$

If $ab = ad$, then $b = bab = bad = d$ which is a contradiction. Thus $\eta(a, b) \neq \eta(c, d)$ in this particular case. Now we consider the case where $a \neq c$. Since the left zeros of S distinguish elements, we have $az \neq cz$ for some left zero z of S. Then

$$\eta(a, b)(z) = azb = az \neq cz = czd = \eta(c, d)(z)$$

and we see that $\eta(a, b) \neq \eta(c, d)$ in this case also.

COROLLARY (4.3). *Let X be any topological space whatsoever. Then the canonical homomorphism from $EC(S(X))$ into* End $S(X)$ *is injective.*

Proof. Suppose $f, g \in S(X)$ and $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$ which implies $f \circ \langle x \rangle \neq g \circ \langle x \rangle$ where $\langle x \rangle$ is the constant function which maps everything into the point x. Thus, the left zeros of $S(X)$ distinguish the elements of $S(X)$ and the previous result applies.

The latter corollary tells us that in some sense the endomorphism semigroup of $S(X)$ is at least as large as the endocore of $S(X)$. The next result tells us that for $S(I)$ the two are in fact isomorphic.

PROPOSITION (4.4). *The canonical homomorphism from $EC(S(I))$ into* End $S(I)$ *is actually an isomorphism onto* End $S(I)$.

Proof. In view of the Corollary (4.3), we need only show that η is surjective. Let φ be any endomorphism of $S(I)$. According to Theorem (3.2), there are

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two possibilities:

(4.4.1) $\varphi(f) = v$ for all $f \in S(I)$ where v is some idempotent of $S(I)$.

(4.4.2) $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for all $f \in S(I)$ where v is some idempotent of $S(I)$ and h is a homeomorphism from I onto the range of v.

In the first case, η maps v onto φ while in the second it maps $(h, h^{-1} \circ v)$ onto φ. This completes the proof.

For any semigroup S with identity and more than one element, $EC(S)$ is the disjoint union of the two subsemigroups S_A and S_E. The algebraic structure of S_E is not interesting as it is just a left zero semigroup regardless of the semigroup S. The semigroup S_A however can be interesting and we look at it more closely in the case $S = S(I)$. Now $S(I)_A$ is just the collection of all pairs (h, k) of continuous selfmaps h and k of I such that $k \circ h$ is the identity map on I, with multiplication defined by

$$(k, h)(r, t) = (k \circ r, t \circ h).$$

Suppose we let $\mathscr{I}$ denote the subsemigroup of $S(I)$ consisting of all injective elements and $\mathscr{S}$ the subsemigroup of surjective elements which map some subinterval onto I. One easily sees that the mapping which sends (k, h) into k is a homomorphism from $S(I)_A$ onto $\mathscr{I}$ while the mapping which sends (k, h) into h is an antihomomorphism from $S(I)_A$ onto $\mathscr{S}$. The semigroup $\mathscr{I}$ has been studied in some detail by L. M. Gluskin in [**1**] and [**2**] and some of his results can be applied in conjunction with some other results to immediately yield results about $S(I)_A$ and hence also about $EC(S(I))$. Item [**2**] is an English translation of [**1**] and specific references will be to [**2**]. Now we verify the following

Proposition (4.5). *The semigroup $\mathscr{I}$ can be embedded into $S(I)_A$ and hence into* End $S(I)$, *the endomorphism semigroup of* $S(I)$.

Proof. Let h be any element of $\mathscr{I}$. Then h maps I injectively onto some subinterval $[a, v]$. Define $k(x) = h^{-1}(x)$ for $x \in [a, b]$. Then either $k(a) = 0$ or $k(a) = 1$. In the former case, extend k by defining $k(x) = 0$ for $0 \leqq x \leqq a$ and $k(x) = 1$ for $b \leqq x \leqq 1$ and in the latter case, extend k by defining $k(x) = 1$ for $0 \leqq x \leqq a$ and $k(x) = 0$ for $b \leqq x \leqq 1$. Then $k \circ h = \mathrm{id}$ and hence (h, k) belongs to $S(I)_A$. One can then verify that the mapping which sends h into (h, k) is an isomorphism from $\mathscr{I}$ into $S(I)_A$.

Corollary (4.6). End $S(I)$ *contains a subsemigroup with a continuum of elements that has no nontrivial homomorphic images and in addition has the property that each element of the subsemigroup generates a subsemigroup which is isomorphic to the semigroup of natural numbers under addition.*

Proof. This is an immediate consequence of Proposition (4.5) and Theorem (3.1) of [**2**, p. 280].

5. A few closing remarks. The endomorphism semigroup of $S(I)$ is completely determined in Proposition (4.4). The result follows quite easily from Theorem (3.2) and the crucial point is that I happens to be a clonable space for Theorem (3.2) does not otherwise apply. Note that R is not a clonable space and indeed the canonical homomorphism (which by Corollary (4.3) in injective) from $EC(S(R))$ into End $S(R)$ turns out not to be surjective. In fact a stronger statement is true. The endocore of $S(R)$ and the endomorphism semigroup of $S(R)$ are *definitely not isomorphic*. Let us first look at $S(R)_A$. A pair (h, k) belongs to $S(R)_A$ if and only if $k \circ h = i_R$ the identity map on R. But this means that both h and k are homeomorphisms mapping R onto R and that $k = h^{-1}$. One readily verifies that $S(R)_A$ is isomorphic to the homeomorphism group of R. Therefore, $S(R)_A$ is a group and $S(R)_E$ is, of course, a left zero semigroup. It follows that the only idempotent element of $EC(S(R))$ which is not a left zero is the identity of that semigroup. So in order to show that $EC(S(R))$ is *not* isomorphic to End $S(R)$, it is quite sufficient to produce an idempotent in End $S(R)$ which is neither the identity nor a left zero. This is not at all difficult to do. Choose any constant function $\langle x \rangle$ in $S(R)$ and define $\varphi(f) = i_R$ if f is a homeomorphism mapping R onto R and $\varphi(f) = \langle x \rangle$ otherwise. Then $\varphi \in$ End $S(R)$ is certainly idempotent but it is not the identity. Nor is it a left zero since $\varphi \circ \alpha \neq \varphi$ for any α which maps everything into a single idempotent. One may consult [**6**, pp. 350-352] for other examples of nonconstant endomorphisms of $S(R)$ whose ranges are nevertheless finite.

Further problems immediately suggest themselves. One is to determine, in some sense, the endomorphisms of $S(R)$ and more generally of $S(R^N)$ where R^N denotes the Euclidean N-space. Although we have completely determined the endomorphisms of $S(I)$, there still remains the problem of determining the endomorphisms of $S(I^N)$ where I^N is the Euclidean N-cell. We conclude with the following.

CONJECTURE. *Let φ be a nonconstant endomorphism of $S(I^N)$. Then $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each $f \in S(I^N)$ where v is an idempotent of $S(I^N)$ and h is a homeomorphism from I^N onto the range of v.*

If the conjecture is true, it would then follow that $EC(S(I^N))$ and End $S(I^N)$ are isomorphic.

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