Appendix A

Conventions and notation

The conventions we use in Parts I and II are slightly different. In Part I presenting mainly a conceptual introduction to the subject of supersymmetric solitons we choose the so-called Minkowski notation. Here our notation is very close (but not identical!) to that of Bagger and Wess [40]. The main distinction is the conventional choice of the metric tensor $g_{\mu\nu} = \text{diag}(+---)$ as opposed to the $\text{diag}(-+++)$ version of Bagger and Wess. Both the spinorial and vectorial indices will be denoted by Greek letters. To differentiate between them we will use the letters from the beginning of the alphabet for the spinorial indices, e.g. α, β and so on, reserving those from the end of the alphabet (e.g. μ, ν, *etc.*) for the vectorial indices.

Those readers who venture to delve into Part II will have to switch to the so-called Euclidean notation which is more convenient for technical studies. The distinctions between these two notations are summarized in Section A.7.

A.1 Two-dimensional gamma matrices

In two dimensions we choose the gamma matrices as follows

$$\gamma^0 = \gamma^t = \sigma_2, \quad \gamma^1 = \gamma^z = i\sigma_1, \quad \gamma^5 \equiv \gamma^0\gamma^1 = \sigma_3. \tag{A.1}$$

In three dimensions

$$\gamma^t = \sigma_2, \quad \gamma^x = -i\sigma_3, \quad \gamma^z = i\sigma_1. \tag{A.2}$$

A.2 Covariant derivatives

The covariant derivative in the Minkowski space is defined as

$$D_\mu = \partial_\mu - i A_\mu^a T^a. \tag{A.3}$$

Then for the spatial derivatives we have

$$D_1 = \frac{\partial}{\partial x} + i A_x^a T^a, \tag{A.4}$$

and similar expressions for $D_{2,3}$.

A.3 Superspace and superfields

The four-dimensional space x^μ can be promoted to superspace by adding four Grassmann coordinates θ_α and $\bar\theta_{\dot\alpha}$, $(\alpha,\ \dot\alpha = 1, 2)$. The coordinate transformations

$$\{x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}\} : \quad \delta\theta_\alpha = \varepsilon_\alpha, \quad \delta\bar\theta_{\dot\alpha} = \bar\varepsilon_{\dot\alpha}, \quad \delta x_{\alpha\dot\alpha} = -2i\,\theta_\alpha\bar\varepsilon_{\dot\alpha} - 2i\,\bar\theta_{\dot\alpha}\varepsilon_\alpha \tag{A.5}$$

add SUSY to the translational and Lorentz transformations.

Here the Lorentz vectorial indices are transformed into spinorial according to the standard rule

$$A_{\beta\dot\beta} = A_\mu (\sigma^\mu)_{\beta\dot\beta}, \quad A^\mu = \frac{1}{2} A_{\alpha\dot\beta} (\bar\sigma^\mu)^{\dot\beta\alpha}, \tag{A.6}$$

where

$$(\sigma^\mu)_{\alpha\dot\beta} = \{1, \vec\tau\}_{\alpha\dot\beta}, \quad (\bar\sigma^\mu)_{\dot\beta\alpha} = (\sigma^\mu)_{\alpha\dot\beta}. \tag{A.7}$$

We use the notation $\vec\tau$ for the Pauli matrices throughout the book. The lowering and raising of the indices is performed by virtue of the $\epsilon^{\alpha\beta}$ symbol $(\varepsilon^{\alpha\beta} = i(\tau_2)_{\alpha\beta})$. For instance,

$$(\bar\sigma^\mu)^{\dot\beta\alpha} = \epsilon^{\dot\beta\dot\rho}\,\epsilon^{\alpha\gamma}\,(\bar\sigma^\mu)_{\dot\rho\gamma} = \{1, -\vec\tau\}_{\dot\beta\alpha}. \tag{A.8}$$

Note that

$$\varepsilon^{12} = -\varepsilon_{12} = 1, \tag{A.9}$$

and the same for the dotted indices.

Two invariant subspaces $\{x_L^\mu, \theta_\alpha\}$ and $\{x_R^\mu, \bar\theta_{\dot\alpha}\}$ are spanned on 1/2 of the Grassmann coordinates,

$$\begin{aligned} \{x_L^\mu, \theta_\alpha\} : &\quad \delta\theta_\alpha = \varepsilon_\alpha, \quad \delta(x_L)_{\alpha\dot\alpha} = -4i\,\theta_\alpha\bar\varepsilon_{\dot\alpha}; \\ \{x_R^\mu, \bar\theta_{\dot\alpha}\} : &\quad \delta\bar\theta_{\dot\alpha} = \bar\varepsilon_{\dot\alpha}, \quad \delta(x_R)_{\alpha\dot\alpha} = -4i\,\bar\theta_{\dot\alpha}\varepsilon_\alpha, \end{aligned} \tag{A.10}$$

where

$$(x_{L,R})_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} \mp 2i\,\theta_\alpha\bar\theta_{\dot\alpha}. \tag{A.11}$$

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The minimal supermultiplet of fields includes one complex scalar field $\phi(x)$ (two bosonic states) and one complex Weyl spinor $\psi^\alpha(x)$, $\alpha = 1, 2$ (two fermionic states). Both fields are united in one *chiral superfield*,

$$\Phi(x_L, \theta) = \phi(x_L) + \sqrt{2}\theta^\alpha \psi_\alpha(x_L) + \theta^2 F(x_L), \tag{A.12}$$

where F is an auxiliary component. The field F appears in the Lagrangian without the kinetic term.

In the gauge theories one also uses a *vector superfield*,

$$\begin{aligned} V(x, \theta, \bar\theta) = C &+ i\theta\chi - i\bar\theta\bar\chi + \frac{i}{\sqrt{2}}\theta^2 M - \frac{i}{\sqrt{2}}\bar\theta^2 \bar M \\ &- 2\theta_\alpha \bar\theta_{\dot\alpha} v^{\dot\alpha\alpha} + \left\{ 2i\theta^2 \bar\theta_{\dot\alpha}\left[\bar\lambda^{\dot\alpha} - \frac{i}{4}\partial^{\alpha\dot\alpha}\chi\right] + \text{H.c.}\right\} \\ &+ \theta^2\bar\theta^2\left[D - \frac{1}{4}\partial^2 C\right]. \end{aligned} \tag{A.13}$$

The superfield V is real, $V = V^\dagger$, implying that the bosonic fields C, D and $v^\mu = \sigma^\mu_{\alpha\dot\alpha} v^{\dot\alpha\alpha}/2$ are real. Other fields are complex, and the bar denotes the complex conjugation. The field strength superfield has the form

$$W_\alpha = i\left(\lambda_\alpha + i\theta_\alpha D - \theta^\beta F_{\alpha\beta} - i\theta^2 D_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}\right). \tag{A.14}$$

The gauge field strength tensor is denoted by $F^a_{\mu\nu}$. Sometimes we use the abbreviation F^2 for

$$F^2 \equiv F^a_{\mu\nu} F^{\mu\nu\, a}, \tag{A.15}$$

while

$$F F^* \equiv F^a_{\mu\nu} F^{*\mu\nu\, a} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{A.16}$$

The transformations (A.10) generate the SUSY transformations of the fields which can be written as

$$\delta V = i\left(Q\varepsilon + \bar Q\bar\varepsilon\right) V \tag{A.17}$$

where V is a generic superfield (which could be chiral as well). The differential operators Q and $\bar Q$ can be written as

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} + \partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \quad \bar Q_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \theta^\alpha\partial_{\alpha\dot\alpha}, \quad \left\{Q_\alpha, \bar Q_{\dot\alpha}\right\} = 2i\partial_{\alpha\dot\alpha}. \tag{A.18}$$

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These differential operators give the explicit realization of the SUSY algebra,

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2P_{\alpha\dot\alpha}, \quad \left\{Q_\alpha, Q_\beta\right\} = 0, \quad \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0, \quad \left[Q_\alpha, P_{\beta\dot\beta}\right] = 0, \tag{A.19}$$

where Q_α and $\bar{Q}_{\dot\alpha}$ are the supercharges while $P_{\alpha\dot\alpha} = i\partial_{\alpha\dot\alpha}$ is the energy-momentum operator. The *superderivatives* are defined as the differential operators $\bar{D}_\alpha$, $D_{\dot\alpha}$ anticommuting with Q_α and $\bar{Q}_{\dot\alpha}$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \quad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\partial_{\alpha\dot\alpha}, \quad \left\{D_\alpha, \bar{D}_{\dot\alpha}\right\} = 2i\partial_{\alpha\dot\alpha}. \tag{A.20}$$

A.4 The Grassmann integration conventions

$$\int d^2\theta\,\theta^2 = 1, \quad \int d^2\theta d^2\,\bar\theta\;\theta^2\,\bar\theta^2 = 1. \tag{A.21}$$

A.5 (1,0) and (1,0) sigma matrices

To convert the two-index spinorial symmetric representation in the vectorial representation we will need the following sigma matrices:

$$\begin{aligned}
(\vec\sigma)^{\alpha\beta} &= \{\tau^3,\ i,\ -\tau^1\}_{\alpha\beta}, \quad (\vec\sigma)_{\alpha\beta} = \{-\tau^3,\ i,\ \tau^1\}_{\alpha\beta},\\
(\vec\sigma)^{\dot\alpha\dot\beta} &= \{\tau^3,\ -i,\ -\tau^1\}_{\dot\alpha\dot\beta}, \quad (\vec\sigma)_{\dot\alpha\dot\beta} = \{-\tau^3,\ -i,\ \tau^1\}_{\dot\alpha\dot\beta}.
\end{aligned} \tag{A.22}$$

A.6 The Weyl and Dirac spinors

If we have two Weyl (right-handed) spinors ξ^α and η_β, transforming in the representations R and $\bar{R}$ of the gauge group, respectively, then the Dirac spinor Ψ can be formed as

$$\Psi = \begin{pmatrix} \xi^\alpha \\ \bar\eta_{\dot\alpha} \end{pmatrix}. \tag{A.23}$$

The Dirac spinor Ψ has four components, while ξ^α and η_β have two components each.

A.7 Euclidean notation

As was mentioned, in Part II we switch to a formally Euclidean notation e.g.

$$F_{\mu\nu}^2 = 2F_{0i}^2 + F_{ij}^2, \tag{A.24}$$

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and

$$(\partial_\mu a)^2 = (\partial_0 a)^2 + (\partial_i a)^2, \tag{A.25}$$

etc. This is appropriate, since we mostly consider static (time-independent) field configurations, and $A_0 = 0$. Then the Euclidean action is nothing but the energy functional.

Then, in the fermion sector we have to define the Euclidean matrices

$$(\sigma_\mu)^{\alpha\dot{\alpha}} = (1, -i\vec{\tau})_{\alpha\dot{\alpha}}, \tag{A.26}$$

and

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha} = (1, i\vec{\tau})_{\dot{\alpha}\alpha}. \tag{A.27}$$

Lowering and raising of the spinor indices is performed by virtue of the antisymmetric tensor defined as

$$\begin{aligned} \varepsilon_{12} &= \varepsilon_{\dot{1}\dot{2}} = 1, \\ \varepsilon^{12} &= \varepsilon^{\dot{1}\dot{2}} = -1. \end{aligned} \tag{A.28}$$

The same raising and lowering convention applies to the flavor $\mathrm{SU}(2)_R$ indices f, g, etc.

When the contraction of the spinor indices is assumed inside the parentheses we use the following notation:

$$(\lambda\psi) \equiv \lambda_\alpha \psi^\alpha, \quad (\bar{\lambda}\bar{\psi}) \equiv \bar{\lambda}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}. \tag{A.29}$$

A.8 Group-theory coefficients

As was mentioned, the gauge group is assumed to be SU(N). For a given representation R of SU(N), the definitions of the Casimir operators to be used below are

$$\mathrm{Tr}(T^a T^b)_R = T(R)\delta^{ab}, \quad (T^a T^a)_R = C(R)\, I, \tag{A.30}$$

where I is the unit matrix in this representation. It is quite obvious that

$$C(R) = T(R)\, \frac{\dim(G)}{\dim(R)}, \tag{A.31}$$

where $\dim(G)$ is the dimension of the group (= the dimension of the adjoint representation). Note that $T(R)$ is also known as (one half of) the Dynkin index, or the dual Coxeter number. For the adjoint representation, $T(R)$ is denoted by $T(G)$. Moreover, $T(\mathrm{SU}(N)) = N$.

A.9 Renormalization-group conventions

We use the following definition of the β function (also known as the Gell–Mann–Low function) and anomalous dimensions:

$$\mu \frac{\partial \alpha}{\partial \mu} \equiv \beta(\alpha) = -\frac{\beta_0}{2\pi}\alpha^2 - \frac{\beta_1}{4\pi^2}\alpha^3 + \cdots \tag{A.32}$$

while

$$\gamma = -d \ln Z(\mu)/d \ln \mu. \tag{A.33}$$

In supersymmetric theories

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left[3\,T(G) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{T(G)\,\alpha}{2\pi}\right)^{-1}, \tag{A.34}$$

where the sum runs over all matter supermultiplets. This is the so-called Novikov–Shifman–Vainshtein–Zakharov (NSVZ) beta function [236]. The anomalous dimension of the ith matter superfield is

$$\gamma_i = -2C(R_i)\,\frac{\alpha}{2\pi} + \cdots \tag{A.35}$$

Sometimes, when one-loop anomalous dimensions are discussed, the coefficient in front of $-\alpha/(2\pi)$ in (A.33) is also referred to as an "anomalous dimension."