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GRAPH PRODUCTS OF RIGHT CANCELLATIVE MONOIDS

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Abstract

Our first main result shows that a graph product of right cancellative monoids is itself right cancellative. If each of the component monoids satisfies the condition that the intersection of two principal left ideals is either principal or empty, then so does the graph product. Our second main result gives a presentation for the inverse hull of such a graph product. We then specialize to the case of the inverse hulls of graph monoids, obtaining what we call 'polygraph monoids'. Among other properties, we observe that polygraph monoids are F^*-inverse. This follows from a general characterization of those right cancellative monoids with inverse hulls that are F^*-inverse.

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0. Introduction

Graph products of groups were introduced by Green in her thesis [14] and have since been studied by several authors, for example, [15] and [8]. In these two papers, passing reference is made to graph products of monoids, which are defined in the same way as graph products of groups and have been studied specifically by, among others, Veloso da Costa [31, 32] and Fohry and Kuske [13].

In this paper we are interested in graph products of right cancellative monoids. Free products and restricted direct products are special cases of graph products, and a free or (restricted) direct product of right cancellative monoids is again right cancellative. In Section 1, in our first main result, we generalize these observations to obtain a corresponding result for graph products.

We then concentrate on right cancellative monoids in which the intersection of two principal left ideals is either principal or empty. This property holds if and only if any two elements which have a common left multiple have a least common left multiple (see the end of Section 1), and so, following the terminology from ring theory (see, for example, [1]) we call these monoids *left LCM monoids*. A useful concept in the study of such monoids is the notion of the inverse hull of a right cancellative monoid. In Section 2, after generalities on inverse hulls, we give several (known) characterizations of inverse hulls of left LCM monoids and use them to show that a graph product of left LCM monoids is itself left LCM. We then consider presentations for inverse hulls of graph products of left LCM monoids. In Section 3 we specialize the presentation to the case where each component monoid is free on one generator, obtaining what we call *polygraph monoids*, generalizing the polycyclic monoids discussed in [18, Ch. 9].

In the final section, we concentrate on left LCM monoids with two-sided cancellation. Among these monoids we characterize those with an inverse hull that is F^*-inverse (see Section 4 for the definition), and observe that, in particular, polygraph monoids are F^*-inverse.

We assume that the reader is familiar with the basic ideas of semigroup theory (see, for example, [7, 16, 18]).

1. Graph products

For us, a *graph* $\Gamma = (V, E)$ is a set V of *vertices* together with an irreflexive, symmetric relation $E \subseteq V \times V$ whose elements are called *edges*. In particular, Γ is loop free. We say that u and v are *adjacent* in Γ if $(u, v) \in E$. For each $v \in V$, let M_v be a monoid; whenever necessary we can, without loss of generality, assume the monoids M_v are disjoint. We denote the free product of the M_v by $\prod^\star M_v$ and write $x \centerdot y$ for the product of $x, y \in \prod^\star M_v$.

We define the *graph product* $\Gamma_{v\in V} M_v$ of the M_v to be the quotient of $\prod^\star M_v$ factored by the congruence generated by the relation

$$R_\Gamma = \{(m \centerdot n, n \centerdot m) \mid m \in M_u, n \in M_v \text{ and } u, v \text{ are adjacent in } \Gamma\}.$$

Alternatively, if for each M_v we have a presentation $\langle A_v \mid R_v\rangle$, then $\Gamma_{v\in V} M_v$ is the monoid with presentation $\langle A \mid R\rangle$ where

$$A = \bigcup_{v\in V} A_v \quad \text{and} \quad R = \bigcup_{(u,v)\in E} \{ab = ba \mid a \in A_u, b \in A_v\} \cup \bigcup_{v\in V} R_v.$$

For the rest of this section we will write M for $\Gamma_{v\in V} M_v$. The M_v are called the *components* of M, and we denote multiplication in both M and its components by concatenation. It follows from Theorem 1.1 below that the latter embed naturally in the former, and so there should be no cause for confusion.

If the graph has no edges, M is the free product of the M_v, and at the other extreme, if the graph is complete, M is their restricted direct product.

A special case of interest is when all the M_v are isomorphic to the additive monoid of nonnegative integers. The graph product is then called a *graph monoid* and denoted by $M(\Gamma)$. Graph monoids are also known variously as *free partially commutative*

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monoids, *right-angled Artin monoids*, and *trace monoids*. These monoids and the corresponding groups have been extensively investigated (see, for example, [12] for monoids and [4] for groups).

Now let X be the disjoint union of the $M_v \backslash \{1\}$, and for $m \in M_v \backslash \{1\}$ write $C(m) = v$. We denote the product in the free monoid X^* by $x \circ y$ to distinguish it from the products in M and the M_v. Clearly there is a canonical surjective homomorphism $\sigma : X^* \to M$ so that each element a of M can be represented by an element of X^*, called an *expression* for a. If $x_1 \circ x_2 \circ \cdots \circ x_n \in X^*$ is an expression for $a \in M$, the x_i are the *components* of the expression, and if $C(x_i) = v$, then x_i is a v-component. If x_i and x_{i+1} are both v-components, then we may obtain a shorter expression for a by, in the terminology of [15], *amalgamating* x_i and x_{i+1}: if $x_i, x_{i+1} \in M_v$ and $x_i x_{i+1} = 1$, delete $x_i \circ x_{i+1}$; otherwise replace it by the single element y_i of M_v where $y_i = x_i x_{i+1}$ in M_v.

If $(C(x_j), C(x_{j+1})) \in E$ for some j, then we may obtain a different expression for a by replacing $x_j \circ x_{j+1}$ by $x_{j+1} \circ x_j$. Again we follow [15] and call such a move a *shuffle*. Two expressions are *shuffle equivalent* if one can be obtained from the other by a sequence of shuffles.

A *reduced expression* is an element $x_1 \circ x_2 \circ \cdots \circ x_n \in X^*$ which satisfies the following condition: whenever $i < j$ and $C(x_i) = C(x_j)$, there exists k with $i < k < j$ and $(C(x_i), C(x_k)) \notin E$. Notice that no amalgamation is possible in a reduced expression, and that a shuffle of a reduced expression is again a reduced expression. The following is the monoid version of a result of Green [14] which can also be deduced easily from [31, Theorem 6.1].

THEOREM 1.1. *Every element of M is represented by a reduced expression. Two reduced expressions represent the same element of M if and only if they are shuffle equivalent.*

The *length* of an expression is its length as an element of the free monoid X^*; it is clear that shuffle equivalent expressions have the same length, and so, in view of the theorem, all reduced expressions representing a given element of M have the same length. We shall use this observation without further comment, but we note that it also allows us to define the *length* of an element of M to be the length of any reduced expression representing it. An easy consequence of the notion of length is the following corollary which we record for later use. First, we recall that a subset U of a monoid M is *right unitary* in M if, for all elements $m \in M$ and $u \in U$, we have $m \in U$ if $mu \in U$. There is a dual notion of *left unitary*, and U is *unitary* in M if it is both right and left unitary.

COROLLARY 1.2. *Each M_v is a unitary submonoid of M.*

PROOF. If $c \in M_v$, $a \in M$ and $ac \in M_v$, then ac must have length 1 (or 0) and it follows that $a \in M_v$. Thus M_v is right unitary in M and, similarly, it is left unitary. □

It is natural to ask how properties of M are related to the corresponding properties of the M_v. Several such questions are considered in [13, 31, 32]. Our interest is in right cancellative monoids which do not seem to have been studied in this context. If M is right cancellative, then so too are the M_v since they are submonoids of M. Our first aim is to show the converse, that is, if all the M_v are right cancellative, then so is M. Towards this end we introduce the following terminology.

Let $a, a' \in M$, $v \in V$ and $c \in M_v \backslash \{1\}$. We say that a has *final v-component c* and *final v-complement a'* if a admits a reduced expression $a_1 \circ a_2 \circ \cdots \circ a_m \circ c$ such that $a_1 a_2 \ldots a_m = a'$. We say that a has *final v-component* 1 and *final v-complement a* if a has a reduced expression $a_1 \circ \cdots \circ a_m$ such that either

(i) $C(a_j) \neq v$ for all j, or
(ii) there exists k with $(C(a_k), v) \notin E$ and $C(a_j) \neq v$ for all $j \geqslant k$.

Of course, we may define the dual notions of *initial v-component* and *initial v-complement* in the obvious way.

PROPOSITION 1.3. *For each vertex v, each element of M has exactly one final v-component and exactly one final v-complement.*

PROOF. For existence, suppose that $x \in M$ and let

$$a_1 \circ \cdots \circ a_m$$

be a reduced expression for x. If condition (i) or (ii) applies, then, by definition, x has final v-component 1 and final v-complement x. Otherwise, there is a largest integer j with $C(a_j) = v$. If $(C(a_k), v) \notin E$ for some $k > j$, then condition (ii) holds. Hence $(C(a_k), v) \in E$ for all $k > j$, and it follows easily that one can shuffle a_j to the end to obtain a reduced expression

$$a = a_1 \circ \cdots \circ a_{j-1} \circ a_{j+1} \circ \cdots \circ a_m \circ a_j$$

so that x has final v-component a_j and final v-complement $a_1 \ldots a_{j-1} a_{j+1} \ldots a_m$.

For uniqueness, suppose first, towards a contradiction, that x has distinct final v-components 1 and $d \neq 1$. Then x has reduced expressions $a = a_1 \circ \cdots \circ a_m$ and $b = b_1 \circ \cdots \circ b_n \circ d$ where either

(i) $C(a_j) \neq v$ for all j, or
(ii) there exists k with $(C(a_k), v) \notin E$ and $C(a_j) \neq v$ for all $j > k$.

By Theorem 1.1, b can be obtained from a by a sequence of shuffles. But clearly in case (i) such a shuffle can never introduce a v-component, while in case (ii) no such shuffle can change the fact that there exists a_k with $(C(a_k), v) \notin E$ and $C(a_j) \neq v$ for all $j > k$. Since b does not satisfy either of the conditions (i) or (ii), this gives a contradiction.

Suppose now that x has reduced expressions

$$a = a_1 \circ \cdots \circ a_m \circ c$$

and

$$b = b_1 \circ \cdots \circ b_m \circ d$$

where $c, d \in M_v$, $c \neq 1$, $d \neq 1$. By Theorem 1.1, b can be obtained from a by a sequence of shuffles. It is clear that no such shuffle can change the value of the last v-component, so we must have $c = d$.

We now turn our attention to showing that final v-complements are unique. If the (unique) final v-component of x is 1 then, by definition, x is the (unique) final v-complement of itself, so there is nothing to prove. So suppose that x has final v-component $c \neq 1$, and that there are reduced expressions

$$a = a_1 \circ \cdots \circ a_m \circ c$$

and

$$b = b_1 \circ \cdots \circ b_m \circ c$$

for x. Now by Theorem 1.1, there is a sequence of shuffles which takes a to b. Clearly, just by removing those applications which involve the final v-component c of the word, we obtain a sequence of shuffles which can be applied to $a_1 \circ \cdots \circ a_m$ to yield $b_1 \circ \cdots \circ b_m$. Since these expressions are reduced, it follows by Theorem 1.1 again that $a_1 \circ \cdots \circ a_m$ and $b_1 \circ \cdots \circ b_m$ represent the same element. Thus, x has exactly one final v-complement. □

LEMMA 1.4. *Let $a \in M$ and $c \in M_v$. Suppose that a has final v-component d and final v-complement a'. Then ac has final v-component dc and final v-complement a'.*

PROOF. Suppose first that a has final v-component $d \neq 1$. Then a has a reduced expression of the form

$$a_1 \circ a_2 \circ \cdots \circ a_m \circ d \tag{1}$$

where $a_1 \circ \cdots \circ a_m$ is a reduced expression for a'. If $dc \neq 1$ then clearly

$$a_1 \circ a_2 \circ \cdots \circ a_m \circ (dc)$$

is a reduced expression for ac, from which the required result is immediate. On the other hand, if $dc = 1$ then

$$a_1 \circ a_2 \circ \cdots \circ a_m$$

is a reduced expression for $ac = a'dc = a'$. It follows easily from the fact that (1) is reduced that either this expression contains no v-components, or there exists k such that $(C(a_k), v) \notin E$ and $a_j \notin v$ for all $j \geqslant k$. Thus, ac has final v-component 1 and final v-complement a', as required.

Now consider the case in which a has final v-component $d = 1$. Then a has a reduced expression

$$a_1 \circ a_2 \circ \cdots \circ a_m$$

where $a = a' = a_1 a_2 \ldots a_m$ and either

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(i) $C(a_j) \neq v$ for all j, or
(ii) there exists k with $(C(a_k), v) \notin E$ and $C(a_j) \neq v$ for all $j \geqslant k$.

In both cases, it is easy to check that $a_1 \circ a_2 \circ \cdots \circ a_m \circ c$ is a reduced expression for ac, from which it follows that ac has final v-component $dc = c$ and final v-complement $a = a'$ as required. □

THEOREM 1.5. *A graph product of right (left, two-sided) cancellative monoids is right (left, two-sided) cancellative.*

PROOF. We prove the result for right cancellative monoids. The corresponding result for left cancellative monoids is proved similarly using initial v-components and complements, and the result for cancellative monoids is an immediate consequence of the one-sided results.

First observe that, since the graph product monoid is generated by elements from the embedded components, it suffices to show that elements of the embedded components are right cancellable, that is, that $ac = bc$ implies $a = b$ whenever c belongs M_v for some $v \in V$.

Suppose that a and b have (unique) final v-components d and e respectively, and (unique) final v-complements a' and b' respectively. Then by the preceding lemma, ac has final v-component dc and final v-complement a', while bc has final v-component ec and final v-complement b'.

Since $ac = bc$, we deduce from Proposition 1.3 that $dc = ec$ and $a' = b'$. But d, e and c lie in M_v, which by assumption is right cancellative, so we deduce that $d = e$, and hence that $a = a'd = b'e = b$ as required to complete the proof. □

We next consider the question of whether a graph product of monoids each of which is embeddable in a group is itself embeddable in a group. A positive answer is a consequence of the next proposition which gives a universal property defining the graph product. We retain the notation of this section.

PROPOSITION 1.6. *Let N be a monoid and suppose that for each $v \in V$ there is a homomorphism $\varphi_v : M_v \to N$ such that*

$$(x\varphi_v)(y\varphi_u) = (y\varphi_u)(x\varphi_v) \quad \textit{for all } (u, v) \in E \textit{ and all } x \in M_v,\, y \in M_u. \qquad (*)$$

Put $M = \Gamma_{v \in V} M_v$. Then there is a unique homomorphism $\varphi : M \to N$ such that $x\varphi = x\varphi_v$ for all $x \in M_v$ and all $v \in V$.

PROOF. For each $v \in V$, let $\langle A_v \mid R_v \rangle$ be a presentation for M_v, and let $\langle A \mid R \rangle$ be the presentation for M as at the beginning of the section. Let $\theta : A \to N$ be the function given by $a\theta = a\varphi_v$ where M_v is the unique monoid containing a. Since each φ_v is a homomorphism, θ respects the relations in each R_v, and, by hypothesis, θ also respects all the other relations in R. Hence there is a unique homomorphism $\varphi : M \to N$ which restricts to θ on A and hence to φ_v on each M_v. □

An immediate consequence is the first part of the following result.

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PROPOSITION 1.7. *Let Γ be a graph, V be its set of vertices and $\{M_v\}_{v\in V}$ and $\{N_v\}_{v\in V}$ be families of monoids. Let $M = \Gamma_{v\in V} M_v$ and $N = \Gamma_{v\in V} N_v$. Then, given homomorphisms $\varphi_v : M_v \to N_v$ for each $v \in V$, there is a unique homomorphism $\varphi : M \to N$ such that $m_v\varphi = m_v\varphi_v$ for all $v \in V$.*

Moreover, if each φ_v is injective, then so is φ.

PROOF. All that remains is to prove the final paragraph. Let $a, b \in M$ with $a\varphi = b\varphi$ and suppose that a, b have reduced expressions $a_1 \circ \cdots \circ a_m$ and $b_1 \circ \cdots \circ b_n$ respectively, where $a_i \in M_{u_i}$ and $b_j \in M_{v_j}$. Then

$$(a_1\varphi_{u_1})\ldots(a_m\varphi_{u_m}) = a\varphi = b\varphi = (b_1\varphi_{v_1})\ldots(b_n\varphi_{v_n})$$

and, since the φ_v are injective, both $(a_1\varphi_{u_1}) \circ \cdots \circ (a_m\varphi_{u_m})$ and $(b_1\varphi_{v_1}) \circ \cdots \circ (b_n\varphi_{v_n})$ are reduced expressions for $a\varphi$. Hence they are shuffle equivalent so that $m = n$ and, for some permutation σ, $a_i\varphi_{u_i} = b_{i\sigma}\varphi_{v_{i\sigma}}$ for all i. Since $\operatorname{im}\varphi_v \subseteq N_v$ for all v, we see that $u_i = v_{i\sigma}$ for each i, and so $a_i = b_{i\sigma}$ since φ_{u_i} is injective. It is now clear that $a_1 \circ \cdots \circ a_m$ and $b_1 \circ \cdots \circ b_n$ are shuffle equivalent so that $a = b$ and hence φ is injective. □

The following corollary, which can also be easily proved directly, is now immediate.

COROLLARY 1.8. *Let Γ be a graph with vertex set V. If, for each $v \in V$, the monoid M_v is embeddable in a group G_v, then the graph product ΓM_v is embeddable in the group ΓG_v.*

In the next section we use ideas about inverse hulls to demonstrate another result about the closure of a class of right cancellative monoids under graph products. Specifically, we consider right cancellative monoids which satisfy the condition that the intersection of two principal left ideals is either principal or empty. A right cancellative monoid satisfying this condition is called a *left LCM monoid*. We show that a graph product of left LCM monoids is again a left LCM monoid.

The reason for the terminology, which is borrowed from ring theory, is that the defining condition may also be expressed in terms of divisibility. For a right cancellative monoid C and $a, b \in C$, we say that a is a *left multiple* of b (and that b is a *right factor or divisor* of a) if $a = cb$ for some $c \in C$. If m is is a left multiple of both b and d, we say it is a *common left multiple* of these elements, and such a common left multiple m is a *least common left multiple* of b and d if every common left multiple of b and d is a left multiple of m. Equivalently, m is a least common left multiple of b and d if and only if

$$Cb \cap Cd = Cm.$$

Least common left multiples are sometimes known as left least common multiples. We note that a left LCM monoid is a right cancellative monoid in which any two elements having a common left multiple have a least common left multiple.

In ring theory (see [1]), an integral domain (not necessarily commutative) is called a *left LCM domain* if the intersection of any two principal left ideals is principal. Thus an integral domain R is a left LCM domain if and only if the cancellative monoid of its nonzero elements is a left LCM monoid.

Similarly, one defines *common right factors* and *highest common right factors*. An element d of C is a highest common right factor of a and b in C if and only if Cd is the least upper bound of Ca and Cb in the partially ordered set of principal left ideals of C.

We remark that least common left multiples and highest common right factors are not uniquely determined in general, being defined only up to left multiplication by a unit.

If C is actually cancellative, *common right multiple*, *common left factor*, *least common right multiple* and *highest common left factor* are defined symmetrically.

Examples of right cancellative LCM monoids abound: the right locally Garside monoids of Dehornoy [10] which, as he points out, include all Artin monoids and all Garside monoids; from ring theory, we have already mentioned the multiplicative monoid of nonzero elements of any LCM domain. Examples of LCM monoids which are right cancellative but not left cancellative are provided by principal left ideal right cancellative monoids; specific examples are the monoids of ordinal numbers less than ω^α (where α is any ordinal number greater than 1) under the dual of the usual operation of ordinal addition.

2. Inverse hulls

To any right cancellative monoid C, one can associate an inverse monoid called the inverse hull of C. Before giving the definition we recall some of the basic concepts of inverse monoids. For more on the general theory of inverse monoids see [16, Ch. 5] and [18].

An *inverse monoid* is a monoid M such that, for all $a \in M$, there is a unique $b \in M$ such that $aba = a$ and $bab = b$. The element b is the *inverse* of a and is denoted by a^{-1}. It is worth noting that $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in M$. The set of idempotents $E(M)$ of M forms a commutative submonoid, referred to as the *semilattice of idempotents* of M. In fact, a monoid M is an inverse monoid if and only if $E(M)$ is a commutative submonoid and, for every $a \in M$, there is an element $b \in M$ such that $aba = a$ (that is, M is regular).

An *inverse submonoid* of an inverse monoid M is simply a submonoid N closed under taking inverses.

For a nonempty set X, a *partial permutation* is a bijection $\sigma : Y \to Z$ for some subsets Y, Z of X. We allow Y and Z to be empty so that the empty function is regarded as a partial permutation. The set of all partial permutations of X is made into a monoid by using the usual rule for composition of partial functions; it is called the *symmetric inverse monoid* on X and denoted by $\mathscr{I}_X$. That it is an inverse monoid follows from the fact that if σ is a partial permutation of X, then so is its inverse (as a

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function) σ^{-1}, and this is the inverse of σ in $\mathscr{I}_X$ in the sense above. The idempotents of $\mathscr{I}_X$ are the *partial identities* ε_Y for all subsets Y of X where ε_Y is the identity map on the subset Y. It is clear that, for $Y, Z \subseteq X$, $\varepsilon_Y \varepsilon_Z = \varepsilon_{Y \cap Z}$ and hence that $E(\mathscr{I}_X)$ is isomorphic to the Boolean algebra of all subsets of X.

The concept of an inverse hull was introduced by Rees [28] to give an alternative proof of Ore's theorem about the existence of a group of fractions of a left (or right) Ore cancellative monoid C. The name was introduced in [7], where the inverse hull of a right cancellative semigroup C is defined. A detailed study of the inverse hull is carried out in [5] where the authors use a definition slightly different from that in [7]. However, the two definitions coincide in the case of inverse hulls of right cancellative monoids, the only case that we consider.

After defining what we mean by an inverse hull and recalling some general results, we show that a graph product of left LCM monoids is also a left LCM monoid, and continue by finding a presentation for the inverse hull of a such a graph product in terms of presentations for its constituent monoids. As a special case we obtain a presentation of the inverse hull of a graph monoid.

2.1. Generalities about inverse hulls As well as being significant in the question of embeddability in a group, the inverse hull of a right cancellative semigroup is also important in describing the structure of bisimple, 0-bisimple, simple and 0-simple inverse semigroups.

Let C be a right cancellative monoid. For an element a of C, the mapping ρ_a with domain C defined by

$$x\rho_a = xa$$

is the *inner right translation* of C determined by a. It is injective since C is right cancellative, and so it can be regarded as a member of $\mathscr{I}_C$. The inverse submonoid of $\mathscr{I}_C$ generated by all the inner right translations of C is the *inverse hull* $IH(C)$ of C. The inverse of ρ_a is, of course, the partial map $\rho_a^{-1} : Ca \to C$, so if C is not a group, then $IH(C)$ contains maps which are not total.

The mapping $\eta : C \to IH(C)$ given by $a\eta = \rho_a$ is an embedding of C into $IH(C)$. Moreover, $C\eta$ is the right unit subsemigroup of $IH(C)$, that is, it consists of those elements $\rho \in IH(C)$ for which there is an element τ with $\rho\tau = 1_C$. The group of units of $IH(C)$ is $G\eta$, where G is the group of units of C. The left unit submonoid L of $IH(C)$ consists of the elements ρ_c^{-1} for $c \in C$. For notational convenience, we introduce a left cancellative monoid C^{-1} containing G as its group of units and such that there is an anti-isomorphism $c \mapsto c^{-1}$ from C to C^{-1}. Here, if $c \in G$, then c^{-1} is its inverse in G, and if $c \notin G$, then c^{-1} is a new symbol. We can now extend η from G to an isomorphism, also denoted by η, from C^{-1} to L given by $c^{-1}\eta = \rho_c^{-1}$.

We remark that if C is a group, then every inner right translation is a permutation of C and η is just the Cayley representation of C.

The empty mapping $\emptyset$ is sometimes a member of $IH(C)$. When it is, it is the zero of $IH(C)$. For ease of expression of some results, we often state them in terms of $IH^0(C)$, where we define $IH^0(C)$ to be the submonoid $IH(C) \cup \{\emptyset\}$ of $\mathscr{I}_C$.

Clearly, if $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of C, then $\rho = \rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$ is a member of $IH(C)$. It is easy to verify that every element of $IH(C)$ can be expressed in this way (see [5, Lemma 2.5]) using the fact that if $a, b \in C$, then $\rho_a\rho_b = \rho_{ab}$ and $\rho_a^{-1}\rho_b^{-1} = \rho_{ba}^{-1}$. Thus every element can be written in the form $(a_1\eta)(b_1^{-1}\eta) \ldots (a_n\eta)(b_n^{-1}\eta)$.

It is noted in [7] that the inverse hull of an infinite cyclic monoid $\{x\}^*$ is the bicyclic monoid. This example was generalized by Nivat and Perrot in [26] where they introduced polycyclic monoids as the inverse hulls of free monoids. They give several characterizations of polycyclic monoids and, in particular, show that the polycyclic monoid P_X on a set X with more than one element has the following presentation as a monoid with zero:

$$\langle X \cup X^{-1} \mid xx^{-1} = 1,\ xy^{-1} = 0 \text{ for } x \neq y\ (x, y \in X)\rangle.$$

More information on polycyclic monoids can be found in [18, Ch. 9] and [25].

An independent study of the inverse hull of the free monoid on an arbitrary nonempty set X was carried out in [17] where Knox describes it as a Rees quotient of a semidirect product of a semilattice by the free group on X.

Further examples of inverse hulls are calculated in [23].

We recall that a compatible partial order called the natural partial order is defined on any inverse semigroup S by the rule that $a \leqslant b$ if $a = eb$ for some idempotent e. For later use, we characterize this relation between certain elements of an inverse hull in the following well-known lemma. See [19] for a version of this and its corollary.

LEMMA 2.1. *Let C be a right cancellative monoid and let $a, b, c, d \in C$. Then in $IH(C)$,*

$$\rho_a^{-1}\rho_b \leqslant \rho_c^{-1}\rho_d \quad \textit{if and only if } a = xc \textit{ and } b = xd \textit{ for some } x \in C.$$

PROOF. If $\rho_a^{-1}\rho_b \leqslant \rho_c^{-1}\rho_d$, then $a \in \operatorname{dom} \rho_a^{-1}\rho_b$, so $a \in \operatorname{dom} \rho_c^{-1}\rho_d$, that is, $a \in Cc$, say $a = xc$. Then

$$b = a\rho_a^{-1}\rho_b = a\rho_c^{-1}\rho_d = xd.$$

Conversely,

$$\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_x^{-1}\rho_x\rho_d \leqslant \rho_c^{-1}\rho_d.$$

COROLLARY 2.2. *Let C be a right cancellative monoid and let $a, b, c, d \in C$. Then in $IH(C)$,*

$$\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d \quad \textit{if and only if } a = uc \textit{ and } b = ud \textit{ for some unit } u \in C.$$

PROOF. By Lemma 2.1, there are elements $x, y \in C$ such that $a = xc$, $b = xd$, $c = ya$ and $d = yb$. Hence $a = xya$ and, by right cancellation, $1 = xy$. It follows that x and y are units. □

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Recall that in any monoid M, Green's relation $\mathscr{R}$ is defined by the rule that $a\mathscr{R}b$ if and only if $aM = bM$. The relation $\mathscr{L}$ is the left-right dual of $\mathscr{R}$; we define $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$ and $\mathscr{D} = \mathscr{R} \vee \mathscr{L}$. In fact, by [16, Proposition 2.1.3], $\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$. Finally, $a \mathscr{J} b$ if and only if $MaM = MbM$. In an inverse monoid, $a\mathscr{R}b$ if and only if $aa^{-1} = bb^{-1}$ and, similarly, $a\mathscr{L}b$ if and only if $a^{-1}a = b^{-1}b$. In $\mathscr{I}_X$, we have $\rho\mathscr{R}\sigma$ if and only if $\operatorname{dom}\rho = \operatorname{dom}\sigma$, and $\rho\mathscr{L}\sigma$ if and only if $\operatorname{im}\rho = \operatorname{im}\sigma$ [16, Exercise 5.11.2]. The following lemma thus follows immediately from [18, Proposition 3.2.11].

LEMMA 2.3. *Let C be a right cancellative monoid. Then, for elements ρ, σ of $IH^0(C)$:*

(1) $\rho\mathscr{R}\sigma$ *in $IH^0(C)$ if and only if* $\operatorname{dom}\rho = \operatorname{dom}\sigma$;
(2) $\rho\mathscr{L}\sigma$ *in $IH^0(C)$ if and only if* $\operatorname{im}\rho = \operatorname{im}\sigma$.

We mention that $\mathscr{L}$ is a right congruence and $\mathscr{R}$ is a left congruence. More information on Green's relations can be found in [16, 18]. Finally, an inverse monoid (or semigroup) is 0-*bisimple* if all its nonzero elements are $\mathscr{D}$-related; it is *bisimple* if all its elements are $\mathscr{D}$-related. Thus if a, b are nonzero elements of a 0-bisimple inverse monoid M, then there are elements $c, d \in M$ such that $a\mathscr{L}c\mathscr{R}b$ and $a\mathscr{R}d\mathscr{L}b$.

It is pointed out in [26] that the equivalence of (1) and (3) in the next proposition can be obtained by slightly modifying the theory of Clifford [6]. A proof of the whole result can be extracted from [21], but for the convenience of the reader and completeness we give an elementary proof.

PROPOSITION 2.4. *The following are equivalent for a right cancellative monoid C:*

(1) *$IH^0(C)$ is* 0*-bisimple;*
(2) *the domain of each nonzero element of $IH^0(C)$ is a principal left ideal;*
(3) *C is a left LCM monoid;*
(4) *every nonzero element of $IH^0(C)$ can be written in the form $\rho_c^{-1}\rho_d$ for some $c, d \in C$.*

PROOF. Suppose that (1) holds, and let ρ be a nonzero element of $IH^0(C)$. Then ρ is $\mathscr{D}$-related to the identity, and so $\mathscr{R}$-related to an element σ of the left unit submonoid. Hence $\operatorname{dom}\rho = \operatorname{dom}\sigma$ and, since $\sigma = \rho_a^{-1}$ for some $a \in C$, $\operatorname{dom}\rho = Ca$ so that (2) holds.

If (2) holds, and $a, b \in C$, then since $Ca \cap Cb$ is the domain of $\rho_a^{-1}\rho_a\rho_b^{-1}\rho_b$, we see that $Ca \cap Cb$ is either principal or empty. Thus (3) holds.

Now suppose that (3) holds and let ρ be a nonzero element of $IH^0(C)$. We have noted that $\rho = \rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$ for some $a_i, b_i \in C$, and so it is enough to show that if $c, d \in C$ and $\rho_c\rho_d^{-1}$ is nonzero, then $\rho_c\rho_d^{-1} = \rho_a^{-1}\rho_b$ for some $a, b \in C$. Now the domain of $\rho_c\rho_d^{-1}$ is $(Cc \cap Cd)\rho_c^{-1}$, and, by assumption, $Cc \cap Cd = Cs$ for some $s \in C$. Thus $s = rc = td$ for some $r, t \in C$ and an easy calculation shows that $\rho_c\rho_d^{-1} = \rho_r^{-1}\rho_t$.

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Finally, if (4) holds, let $\rho = \rho_a^{-1}\rho_b$ be a nonzero element of $IH^0(C)$. Now ρ_a^{-1} is $\mathscr{L}$-related to the identity, and since $\mathscr{L}$ is a right congruence, we get $\rho \mathscr{L} \rho_b$. But $\rho_b \mathscr{R} 1$, so ρ is $\mathscr{D}$-related to the identity, and (1) follows. □

It is worth noting that if C is a left LCM monoid, then the product of two nonzero elements in $IH^0(C)$ is given by

$$(\rho_a^{-1}\rho_b)(\rho_c^{-1}\rho_d) = \begin{cases} 0 & \text{if } Cb \cap Cc = \emptyset, \\ \rho_{sa}^{-1}\rho_{td} & \text{if } Cb \cap Cc = Csb = Ctc. \end{cases}$$

Although it is not relevant to the present paper, it is worth noting that every 0-bisimple inverse monoid M is isomorphic to $IH^0(C)$ where C is the right unit submonoid of M [26], so that the preceding proposition applies to all such monoids. We make use of the proposition to prove the next theorem, for which we also need the following lemma.

LEMMA 2.5. *Let $\Gamma = (V, E)$ be a graph and, for each $v \in V$, let C_v be a right cancellative monoid and $C = \Gamma_{v \in V} C_v$. Let c, d be nonunits in C_v, C_u respectively, where $(u, v) \in E$. Then*

$$Cc \cap Cd = Ccd.$$

PROOF. Since $(u, v) \in E$, we have $cd = dc$ so that $Ccd \subseteq Cc \cap Cd$. Now suppose that $a \in Cc \cap Cd$ so that $a = sc = td$ for some $s, t \in C$. By Lemma 1.4, a has final v-component $c'c$ and final u component $d'd$, where c' is the final v-component of s and d' is the final u-component of t. Neither $c'c$ nor $d'd$ can be 1 since c, d are not units. Thus a has reduced expressions $x_1 \circ \cdots \circ x_n \circ (c'c)$ and $y_1 \circ \cdots \circ y_n \circ (d'd)$ which, by Theorem 1.1, must be shuffle equivalent. Hence one of the x_i, say x_j, must be $d'd$ and one can shuffle it to the end to obtain a reduced expression

$$x_1 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_n \circ (c'c) \circ (d'd)$$

for a. Hence $a = x_1 \ldots x_{j-1}x_{j+1} \ldots x_n(c'c)(d'd)$ and, since $c \in C_v$, $d' \in C_u$ so that $cd' = d'c$ (as $(u, v) \in E$),

$$a = x_1 \ldots x_{j-1}x_{j+1} \ldots x_n c'd'cd \in Ccd,$$

completing the proof. □

THEOREM 2.6. *Let $\Gamma = (V, E)$ be a graph and, for each $v \in V$, let C_v be a left LCM monoid. Then the graph product $C = \Gamma_{v \in V} C_v$ is also a left LCM monoid.*

PROOF. C is right cancellative by Theorem 1.5. To prove that C is a left LCM monoid, we show that every nonzero element of $IH^0(C)$ can be written in the form $\rho_a^{-1}\rho_b$ for some $a, b \in C$, and appeal to Proposition 2.4.

We claim that if $c, d \in C$ and $\tau = \rho_c\rho_d^{-1}$ is nonzero, then $\tau = \rho_a^{-1}\rho_b$ for some $a, b \in C$. The result follows from this claim and our earlier observation that every nonzero element of $IH^0(C)$ can be written in the form $\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$.

We note that the claim is true if one of c, d is a unit: if $r = c^{-1}$ exists, then

$$\tau = \rho_{r^{-1}}\rho_d^{-1} = \rho_r^{-1}\rho_d^{-1} = \rho_{dr}^{-1} = \rho_{dr}^{-1}\rho_1,$$

and if d is a unit, then

$$\rho_c\rho_d^{-1} = \rho_c\rho_{d^{-1}} = \rho_{cd^{-1}} = \rho_1^{-1}\rho_{cd^{-1}}.$$

We now assume that c, d are both nonunits and continue by proving the claim in the case where they both have length 1, so that $c \in C_v$ for some $v \in V$. If $d \in C_v$, then $\tau = \rho_a^{-1}\rho_b$ since C_v is a left LCM monoid. Let $d \in C_u$ with $u \neq v$. If $(u, v) \notin E$, then no reduced expression ending in c is shuffle equivalent to one ending in d, and it follows that $Cc \cap Cd = \emptyset$. Thus $\tau = \emptyset$, a contradiction. Hence $(u, v) \in E$ so that $cd = dc$. By Lemma 2.5, $Cc \cap Cd = Ccd$. It follows that dom $\rho_c\rho_d^{-1} = Cd =$ dom $\rho_d^{-1}\rho_c$, and it is easily verified that $\rho_c\rho_d^{-1} = \rho_d^{-1}\rho_c$. Hence the claim holds for all c and d of length 1; in fact $\rho_c\rho_d^{-1} = \rho_a^{-1}\rho_b$, where a and b also have length 1.

To complete the proof, let $c, d \in C$ have reduced expressions $c_1 \circ \cdots \circ c_h$ and $d_1 \circ \cdots \circ d_k$ so that $\rho_c\rho_d^{-1} = \rho_{c_1} \ldots \rho_{c_h}\rho_{d_1}^{-1} \ldots \rho_{d_k}^{-1}$. Now apply the length 1 case repeatedly. □

In the next lemma we compare intersections of principal left ideals in the graph product and in its component monoids.

LEMMA 2.7. *Let $\Gamma = (V, E)$ be a graph and, for each $v \in V$, let C_v be a left LCM monoid and let $C = \Gamma_{v \in V} C_v$. If $x, y \in C_v$ for some $v \in V$, then*

$$C_v x \cap C_v y = \emptyset \quad \textit{if and only if } Cx \cap Cy = \emptyset.$$

Moreover, if $C_v x \cap C_v y = C_v z$, then $Cx \cap Cy = Cz$.

PROOF. Clearly, if $Cx \cap Cy = \emptyset$, then $C_v x \cap C_v y = \emptyset$. Conversely, suppose that $ax = by$ for some $a, b \in C$. Let a and b have final v-components c and d, respectively. Then by Lemma 1.4, ax has final v-component cx and by has final v-component dy. But $ax = by$ so, by Proposition 1.3, $cx = by \in C_v x \cap C_v y$.

Suppose that $C_v x \cap C_v y = C_v z$; then certainly $Cz \subseteq Cx \cap Cy$. If $r = ax = by$ for some $a, b \in C$ then, applying Lemma 1.4 and Proposition 1.3 again, we see that r has final v-component $cx = dy$ where c and d are the final v-components of a and b, respectively. Thus $cx \in C_v x \cap C_v y$ so $cx = mz$ for some $m \in C_v$, and if r' is the final v-complement of r, then $r = r'mz \in Cz$ as required. □

We are now in a position to prove the following result which will be important in the next subsection.

PROPOSITION 2.8. *If C is the graph product $\Gamma_{v \in V} C_v$ of left LCM monoids C_v, then, for each $v \in V$, the inverse hull $IH^0(C_v)$ is embedded in $IH^0(C)$.*

PROOF. For $x \in C_v$ denote the inner right translations of C_v and C determined by x by ρ_x and δ_x, respectively. Nonzero elements of $IH^0(C_v)$ have the form $\rho_x^{-1}\rho_y$ and so we can define $\theta : IH^0(C_v) \to IH^0(C)$ by $0\theta = 0$ and $(\rho_x^{-1}\rho_y)\theta = \delta_x^{-1}\delta_y$.

To see that θ is well defined, suppose that $\rho_x^{-1}\rho_y = \rho_z^{-1}\rho_t$. Then by Corollary 2.2, $x = uz$ and $y = ut$ for some unit u of C_v. Certainly u is a unit of C, so $\delta_x^{-1}\delta_y = \delta_z^{-1}\delta_t$ as required.

To see that θ is injective, suppose that $\delta_x^{-1}\delta_y = \delta_z^{-1}\delta_t$, where $x, y, z, t \in C_v$. Then by Corollary 2.2, $x = qz$ and $y = qt$ for some unit q of C. By Corollary 1.2, C_v is unitary in C, and since $qt, t \in C_v$, we have $q \in C_v$. It is easy to see that q^{-1} is also in C_v, so that q is a unit of C_v and so $\rho_x^{-1}\rho_y = \rho_z^{-1}\rho_t$ as required.

Finally, we show that θ is a homomorphism. Let $\rho_x^{-1}\rho_y, \rho_z^{-1}\rho_t$ be elements of $IH^0(C_v)$.

If $C_v y \cap C_v z = \emptyset$ then, by Lemma 2.7, $Cy \cap Cz = \emptyset$. From the rule for multiplication following Proposition 2.4, $(\rho_x^{-1}\rho_y)(\rho_z^{-1}\rho_t) = 0$ and, since, by Theorem 2.6, C is left LCM, also $(\delta_x^{-1}\delta_y)(\delta_z^{-1}\delta_t) = 0$.

On the other hand, if $C_v y \cap C_v z \neq \emptyset$, then since C_v is a left LCM monoid, $C_v y \cap C_v z = C_v a$ for some $a \in C_v$, say $a = ry = sz$, where $r, s \in C_v$. Also, by Lemma 2.7, $Cy \cap Cz = Ca$, and so by the rule for multiplication we see that

$$(\rho_x^{-1}\rho_y)(\rho_z^{-1}\rho_t) = \rho_{rx}^{-1}\rho_{st}^{-1}$$

and

$$(\delta_x^{-1}\delta_y)(\delta_z^{-1}\delta_t) = \delta_{rx}^{-1}\delta_{st}^{-1}.$$

It follows that θ is a homomorphism as required. □

2.2. Inverse hulls of graph products of left LCM monoids Let $\Gamma = (V, E)$ be a graph and $\{C_v\}_{v\in V}$ be a family of left LCM monoids. Let $C = \Gamma_{v\in V} C_v$ be the graph product of the C_v; we have just proved that C is also a left LCM monoid. In this subsection our first goal is to find a presentation (as a monoid with zero) for $IH^0(C)$ in terms of given presentations for the inverse monoids $IH^0(C_v)$.

We begin by establishing some notation. Let D be any right cancellative monoid with group of units G and let Y be a symmetric set of monoid generators for G (that is, $y \in Y$ if and only if $y^{-1} \in Y$). We assume that $1 \notin Y$ and take Y to be empty if $G = \{1\}$. Let X be a set of nonunits in D such that $X \cup Y$ generates D. Let $X^{-1} = \{x^{-1} \mid x \in X\}$ be a set disjoint from X such that $x \mapsto x^{-1}$ is a bijection, and $X^{-1} \cup Y$ generates the left cancellative monoid D^{-1} anti-isomorphic to D. Since any element of $IH(D)$ can be written in the form $\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$, it follows that there is a homomorphism from the free monoid $(X \cup X^{-1} \cup Y)^*$ onto $IH(D)$ sending x to ρ_x, y to ρ_y and x^{-1} to ρ_x^{-1}. Thus $IH(D)$ has a presentation of the form $\langle X \cup X^{-1} \cup Y \mid R\rangle$ for some set of relations R. We can also regard $\langle X \cup X^{-1} \cup Y \mid R\rangle$ as a presentation for $IH^0(D)$ in the class of monoids with zero. Since $\rho_x\rho_x^{-1} = 1$ for all $x \in X$, we can assume that $xx^{-1} = 1$ is a relation in R for every $x \in X$. Similarly, since ρ_y is a unit for all $y \in Y$, we can assume that we have relations $yy^{-1} = 1 = y^{-1}y$ in R for all $y \in Y$.

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Turning to the graph product $C = \Gamma_{v \in V} C_v$, we note that we have a corresponding graph product $C^{-1} = \Gamma_{v \in V} C_v^{-1}$ of the left cancellative monoids C_v^{-1}. Writing G_v for the common group of units of C_v and C_v^{-1}, we remark that, by [31, Proposition 7.1], the common group of units of C and C^{-1} is $G = \Gamma_{v \in V} G_v$. We also observe that the anti-isomorphisms between the C_v and the C_v^{-1} extend, by a slight variation of Proposition 1.7, to an anti-isomorphism between C and C^{-1}. Now put $S_v = IH^0(C_v)$ for each $v \in V$, and let $\langle X_v \cup X_v^{-1} \cup Y_v \mid R_v \rangle$ be a presentation for S_v of the type described in the previous paragraph. It will be convenient to adopt the following notational convention: x_v, y_v denote elements of X_v, Y_v, respectively; t_v denotes an element of $X_v \cup Y_v$; and z_v denotes any element of $Z_v = X_v \cup X_v^{-1} \cup Y_v$.

We now put $X = \bigcup_{v \in V} X_v$, $X^{-1} = \bigcup_{v \in V} X_v^{-1}$, $Y = \bigcup_{v \in V} Y_v$, and $Z = X \cup X^{-1} \cup Y$. As in Section 1, we will want to consider the free monoid on $\bigcup_{v \in V} C_v$ as well as the free monoid Z^*. To avoid confusion about the various products, we write $\circ$, as before, for the product in the former free monoid, and $\diamond$ for that in Z^*.

Next, we introduce several sets of relations amongst words over $X \cup X^{-1} \cup Y$ (and zero) as follows:

(1) $R = \bigcup_{v \in V} R_v$;

(2) $N = \{x_v \diamond y_{u_1} \diamond \cdots \diamond y_{u_m} \diamond x_w^{-1} = 0 \mid m \geqslant 0, \forall x_v \in X_v, x_w \in X_w, y_{u_i} \in Y_{u_i} \text{ with } (v, w) \notin E \text{ and } v \neq w\}$;

(3) $\mathrm{Com} = \{z_u \diamond z_v = z_v \diamond z_u \mid \forall z_u \in Z_u, z_v \in Z_v \text{ with } (u, v) \in E\}$.

The *polygraph product* of the S_v is defined to be the monoid $\mathrm{PG} = \mathrm{PG}_{v \in V}(S_v)$ given by the presentation

$$\langle Z \mid R \cup N \cup \mathrm{Com} \rangle.$$

There is thus a surjective homomorphism $\zeta : Z^* \to \mathrm{PG}$. For each $v \in V$, the generators and relations of $IH^0(C_v)$ are among those for PG and so there is a monoid homomorphism ψ_v from $IH^0(C_v)$ into PG determined by $\rho_{t_v} \psi_v = t_v \zeta$ and $\rho_{x_v}^{-1} \psi_v = x_v^{-1} \zeta$ for $t_v \in X_v \cup Y_v$ and $x_v \in X_v$.

The map $\eta_v : C_v \to IH_0(C_v)$ given by $c\eta_v = \rho_c$ is an isomorphism of C_v with the right unit submonoid of $IH_0(C_v)$. As noted in the preceding subsection, we can also extend η_v from G_v (the group of units of C_v) to the left cancellative monoid C_v^{-1} to give an isomorphism onto the left unit submonoid of $IH^0(C_v)$. Composing η_v with the restriction of ψ_v first to the right unit submonoid of $IH^0(C_v)$, then to the left unit submonoid, we obtain monoid homomorphisms from C_v and C_v^{-1} into PG, both of which we denote by θ_v. There is no ambiguity here since these homomorphisms agree on the common group of units of C_v and C_v^{-1}. We observe that if $c_v = t_1 \ldots t_n$, where $t_i \in X \cup Y$, then

$$c_v \theta_v = (t_1 \eta_v \psi_v) \ldots (t_n \eta_v \psi_v) = \rho_{t_1} \psi_v \ldots \rho_{t_n} \psi_v = t_1 \zeta \ldots t_n \zeta = (t_1 \diamond \cdots \diamond t_n) \zeta$$

and

$$c_v^{-1} \theta_v = (t_n^{-1} \ldots t_1^{-1}) \theta_v = \rho_{t_n}^{-1} \psi_v \ldots \rho_{t_1}^{-1} \psi_v = t_n^{-1} \zeta \ldots t_1^{-1} \zeta = (t_n^{-1} \diamond \cdots \diamond t_1^{-1}) \zeta.$$

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Now by Proposition 1.6 and its dual, there are unique homomorphisms from C into the right unit submonoid of PG, and from C^{-1} into the left unit submonoid of PG which restrict to θ_v on each C_v and C_v^{-1}, respectively. We have noted that the common group of units of C and C^{-1} is $G = \Gamma_{v\in V} G_v$, where G_v is the common group of units of C_v and C_v^{-1}. As no nonunits are in both C and C^{-1}, there is no ambiguity in denoting both homomorphisms by θ.

From the above we see that the squares

$$\begin{tikzcd} (X \cup Y)^* \arrow[r] \arrow[d, "\iota"'] & C \arrow[d, "\theta"] \\ (X \cup X^{-1} \cup Y)^* \arrow[r, "\zeta"] & \mathrm{PG} \end{tikzcd} \qquad \begin{tikzcd} (X^{-1} \cup Y)^* \arrow[r] \arrow[d, "\iota"'] & C^{-1} \arrow[d, "\theta"] \\ (X \cup X^{-1} \cup Y)^* \arrow[r, "\zeta"] & \mathrm{PG} \end{tikzcd}$$

are commutative where ι is the inclusion map. It follows that every nonzero element of PG can be written in the form $(a_1\theta)(b_1^{-1}\theta)\dots(a_k\theta)(b_k^{-1}\theta)$, where $a_i, b_i \in C$. In fact, we can do better than this, as we see in the next lemma.

LEMMA 2.9. *Every nonzero element of* $\mathrm{PG} = \mathrm{PG}_{v\in V}(S_v)$ *can be written in the form* $(a^{-1}\theta)(b\theta)$*, where* $a, b \in C$*.*

PROOF. In view of the remark preceding the lemma, it is enough to show that if $c, d \in C$, then either $(c\theta)(d^{-1}\theta) = 0$ or $(c\theta)(d^{-1}\theta) = (a^{-1}\theta)(b\theta)$ for some $a, b \in C$. This is clearly true if c or d is a unit of C, so we may assume that neither is a unit.

We use induction on the length, as defined in Section 1, of c and d. We start by considering d of length 1, and proving by induction on the length of c that for any $c \in C$, either $(c\theta)(d^{-1}\theta) = 0$ or $(c\theta)(d^{-1}\theta) = (a^{-1}\theta)(b\theta)$ for some $a, b \in C$ with a of length 1. First, suppose that c has length 1. Then $c \in C_u$, $d \in C_v$ for some u, v. If $u = v$, then

$$(c\theta)(d^{-1}\theta) = (c\theta_u)(d^{-1}\theta_u) = (\rho_c\psi_u)(\rho_d^{-1}\psi_u) = (\rho_c\rho_d^{-1})\psi_u.$$

Since C_u is left LCM, we have, by Proposition 2.4, that $\rho_c\rho_d^{-1}$ is either zero or equal to $\rho_a^{-1}\rho_b$ for some $a, b \in C_u$. Hence, if nonzero,

$$(c\theta)(d^{-1}\theta) = (\rho_c\rho_d^{-1})\psi_u = (\rho_a^{-1}\rho_b)\psi_u = (\rho_a^{-1}\psi_u)(\rho_b\psi_u) = (a^{-1}\theta)(b\theta).$$

If $u \neq v$, let $c = t'_1 \dots t'_m$ and $d = t_1 \dots t_n$, where $t'_i \in X_u \cup Y_u$ and $t_j \in X_v \cup Y_v$. If $(u, v) \in E$, then $t'_i \diamond t_j = t_j \diamond t'_i$ is a relation in Com for all i, j and it follows that $(c\theta)(d^{-1}\theta) = (d^{-1}\theta)(c\theta)$.

Suppose that $(u, v) \notin E$. Since c, d are nonunits, not all the t'_i are units and not all the t_j are units. Let h and k be the largest integers such t'_h and t_k are nonunits. Then we can write x'_h for t'_h and x_k for t_k, and, similarly, we can write y'_i for t'_i when $i > h$ and y_j for t_j when $j > k$. Consider

$$(x'_h \diamond y'_{h+1} \diamond \dots \diamond y'_m \diamond y_n^{-1} \diamond \dots \diamond y_{k+1}^{-1} \diamond x_{k+1}^{-1})\zeta.$$

This element is zero (by virtue of the relations in N) and so $(c\theta)(d^{-1}\theta) = 0$.

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Thus our claim is true for all c and d of length 1. Now suppose that, for any $c, d \in C$ with c of length less than m and d of length 1, $(c\theta)(d^{-1}\theta) = 0$ or $(c\theta)(d^{-1}\theta) = (a^{-1}\theta)(b\theta)$ for some $a, b \in C$ with a of length 1.

Next, let $c \in C$ have length m, let $c_1 \circ \cdots c_m$ be a reduced expression for c, and let $d \in C_v$. By the current induction assumption, $(c_2 \ldots c_m\theta)(d^{-1}\theta)$ is either zero or can be written in the form $(a^{-1}\theta)(b\theta)$ with a of length 1. In the former case, it is clear that $(c\theta)(d^{-1}\theta) = 0$. In the latter case, if $(c\theta)(d^{-1}\theta)$ is nonzero, then

$$\begin{aligned}
(c\theta)(d^{-1}\theta) &= ((c_1 \ldots c_m)\theta)(d^{-1}\theta) = (c_1\theta)((c_2 \ldots c_m)\theta)(d^{-1}\theta) \\
&= (c_1\theta)(a^{-1}\theta)(b\theta) \\
&= (a_1^{-1}\theta)(b_1\theta)(b\theta) = (a_1^{-1}\theta)((b_1 b)\theta)
\end{aligned}$$

where a_1 has length 1, using the fact that c_1 and a both have length 1.

Thus we have proved our claim that, for any $c, d \in C$ with d of length 1, either $(c\theta)(d^{-1}\theta) = 0$ or $(c\theta)(d^{-1}\theta) = (a^{-1}\theta)(b\theta)$ for some $a, b \in C$ with a of length 1.

Now assume inductively that for any $c \in C$ and any $d \in C$ of length $n-1$, if $(c\theta)(d^{-1}\theta) \neq 0$, then $(c\theta)(d^{-1}\theta) = (a^{-1}\theta)(b\theta)$ for some $a, b \in C$. Let $d \in C$ have a reduced expression $d_1 \circ \cdots \circ d_n$ so that

$$\begin{aligned}
(c\theta)(d^{-1}\theta) &= (c\theta)(d_n^{-1}\theta)((d_{n-1}^{-1} \ldots d_1^{-1})\theta) \\
&= (a_1^{-1}\theta)(b_1\theta)((d_{n-1}^{-1} \ldots d_1^{-1})\theta) \quad \text{for some } a_1, b_1 \in C \text{ (by the case for } n = 1) \\
&= (a_1^{-1}\theta)((b_1\theta)(d_{n-1}^{-1} \ldots d_1^{-1})\theta) \\
&= (a_1^{-1}\theta)(a_2^{-1}\theta)(b_2\theta) \quad \text{for some } a_2, b_2 \in C \text{ (by the induction assumption)} \\
&= (a_1^{-1}a_2^{-1})\theta(b_2\theta) \\
&= (a^{-1}\theta)(b\theta) \quad \text{where } a = a_2 a_1 \text{ and } b = b_2.
\end{aligned}$$

This completes the proof of the lemma. □

We now consider $IH^0(C)$. We remind the reader that (as a monoid with zero) each $IH^0(C_v)$ is generated by $\{\rho_{x_v}, \rho_{x_v}^{-1}, \rho_{y_v} : x_v \in X_v, y_v \in Y_v\}$ and that $IH^0(C)$ is generated by $Q = \{\rho_x, \rho_x^{-1}, \rho_y \mid x \in X, y \in Y\}$, where $X = \bigcup_{v \in V} X_v$, $X^{-1} = \bigcup_{v \in V} X_v^{-1}$ and $Y = \bigcup_{v \in V} Y_v$. As before, we also assume that R_v is a set of defining relations for $IH^0(C_v)$ and put $R = \bigcup_{v \in V} R_v$.

LEMMA 2.10. *With respect to the generating set Q, the relations in R are satisfied by $IH^0(C)$.*

PROOF. By Proposition 2.8, $IH^0(C_v)$ is embedded in $IH^0(C)$ for all $v \in V$. The relations in R are relations in R_v for some v, so hold in $IH^0(C_v)$ and hence in $IH^0(C)$. □

LEMMA 2.11. *With respect to the generating set Q, the relations in N are satisfied by $IH^0(C)$.*

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PROOF. Suppose that $x_v \diamond y_{u_1} \diamond \cdots \diamond y_{u_m} \diamond x_w^{-1} = 0$ is a relation in N so that $(v, w) \notin E$ and $v \neq w$. Then in $IH^0(C)$,

$$\operatorname{dom} \rho_{x_v}\rho_{y_{u_1}} \ldots \rho_{y_{u_m}}\rho_{x_w}^{-1} = (Cx_v y_{u_1} \ldots y_{u_m} \cap Cx_w)(\rho_{x_v}\rho_{y_{u_1}} \ldots \rho_{y_{u_m}})^{-1}.$$

Since x_v is not a unit and $(v, w) \notin E$, in an expression for an element a of $Cx_v y_{u_1} \ldots y_{u_m}$, any amalgamation involving x_v produces a nonunit of C_v, so a nonunit of C_w cannot be shuffled to the end of the expression. Hence the final w-component of a is a unit. But the final w-component of an element of Cx_w must be a left multiple of x_w and hence be a nonunit. It follows from Proposition 1.3 that $Cx_v y_{u_1} \ldots y_{u_m} \cap Cx_w = \emptyset$ and so $\rho_{x_v}\rho_{y_{u_1}} \ldots \rho_{y_{u_m}}\rho_{x_w}^{-1} = 0$. □

LEMMA 2.12. *With respect to the generating set Q, the relations in* Com *are satisfied by $IH^0(C)$.*

PROOF. Following our convention that t_u, x_u denote arbitrary elements of $X_u \cup Y_u$ and X_u respectively, relations in Com have one of the forms

(i) $t_u \diamond t_v = t_v \diamond t_u$,
(ii) $x_u \diamond x_v^{-1} = x_v^{-1} \diamond x_u$, or
(iii) $x_u^{-1} \diamond x_v^{-1} = x_v^{-1} \diamond x_u^{-1}$,

where $(u, v) \in E$. Relations of the form (i) are satisfied in $IH^0(C)$ since

$$\rho_{t_u}\rho_{t_v} = \rho_{t_u t_v} = \rho_{t_v t_u} = \rho_{t_v}\rho_{t_u}.$$

Consider a relation as in (ii). By Lemma 2.5, $Cx_u \cap Cx_v = Cx_u x_v$, and since $x_u x_v = x_v x_u$ in C,

$$\operatorname{dom} \rho_{x_u}\rho_{x_v}^{-1} = (\operatorname{im} \rho_{x_u} \cap \operatorname{dom} \rho_{x_v}^{-1})\rho_{x_u}^{-1} = (Cx_v x_u)\rho_{x_u}^{-1} = Cx_v.$$

Similarly, we calculate $\operatorname{im} \rho_{x_u}\rho_{x_v}^{-1} = Cx_u$.

Since $\operatorname{im} \rho_{x_v}^{-1} = C = \operatorname{dom} \rho_{x_u}$, it is easy to see that we also have $\operatorname{dom} \rho_{x_v}^{-1}\rho_{x_u} = Cx_v$ and $\operatorname{im} \rho_{x_v}^{-1}\rho_{x_u} = Cx_u$, and it follows that $\rho_{x_u}\rho_{x_v}^{-1} = \rho_{x_v}^{-1}\rho_{x_u}$.

Finally consider a relation of the form (iii). In this case, since $(u, v) \in E$, we also have that $x_u \diamond x_v = x_v \diamond x_u$ is a relation in Com. Hence $\rho_{x_v}\rho_{x_u} = \rho_{x_u}\rho_{x_v}$ follows by (i), and since $IH^0(C)$ is an inverse monoid,

$$\rho_{x_u}^{-1}\rho_{x_v}^{-1} = (\rho_{x_v}\rho_{x_u})^{-1} = (\rho_{x_u}\rho_{x_v})^{-1} = \rho_{x_v}^{-1}\rho_{x_u}^{-1}. \qquad \square$$

We now use the lemmas together to obtain the following theorem, in which we retain the notation of this section.

THEOREM 2.13. *The monoids $\mathrm{PG}_{v \in V}(S_v)$ and $IH^0(C)$ are isomorphic.*

PROOF. Consider the function $\beta : X \cup X^{-1} \cup Y \to IH^0(C)$ given by $x\beta = \rho_x$, $x^{-1}\beta = \rho_x^{-1}$ and $y\beta = \rho_y$. It follows from Lemmas 2.10–2.12 that β extends to a homomorphism, again denoted by β, from PG to $IH^0(C)$. Since the latter is generated by Q, the homomorphism is surjective.

Let $r, s \in \mathrm{PG}$ and suppose that $r\beta = s\beta$. By Lemma 2.9, $r = (a^{-1}\theta)(b\theta)$ and $s = (c^{-1}\theta)(d\theta)$ for some $a, b, c, d \in C$. Hence $((a^{-1}\theta)(b\theta))\beta = ((c^{-1}\theta)(d\theta))\beta$ so that $\rho_a^{-1}\rho_b = \rho_c^{-1}\rho_d$, and hence, by Corollary 2.2, there is a unit e of C such that $c = ea$ and $d = eb$. If $m, n \in C$, then there are correponding elements m^{-1}, n^{-1} in C^{-1} and $(mn)^{-1} = n^{-1}m^{-1}$. Thus, using the fact that e is a unit in C,

$$\begin{aligned} s &= (c^{-1}\theta)(d\theta) = ((ea)^{-1}\theta)((eb)\theta) \\ &= (a^{-1}e^{-1})\theta(eb)\theta = (a^{-1}\theta)(e^{-1}\theta)(e\theta)(b\theta) \\ &= (a^{-1}\theta)((e^{-1}e)\theta)(b\theta) = (a^{-1}\theta)(b\theta) \\ &= r. \end{aligned}$$

Thus β is an isomorphism and the proof is complete. □

3. Polygraph monoids

Theorem 2.13 gives us a presentation for $IH^0(C)$ and also allows us to write the elements of PG in the form $a^{-1}b$ with $a, b \in C$ where $a^{-1}b = c^{-1}d$ if and only if $c = ea$ and $d = eb$ for some unit e of C. The presentation simplifies considerably in the case when each C_v (and hence also C) has a trivial group of units, in that $Y = \emptyset$ and consequently

$$N = \{x_u \diamond x_v^{-1} = 0 \mid \forall x_u \in X_u,\, x_v \in X_v \text{ with } (u, v) \notin E \text{ and } u \neq v\}.$$

Thus we have the presentation

$$\langle X \cup X^{-1} \mid R \cup N \cup \mathrm{Com}\rangle$$

for $IH^0(C)$.

A particular instance of this is when each C_v is a free monogenic monoid. Then $S_v = IH^0(C_v)$ is the bicyclic monoid with zero adjoined, and as a monoid with zero it has the presentation with two generators: $\langle x_v, x_v^{-1} \mid x_v x_v^{-1} = 1\rangle$. In this case, the graph product of the C_v is a graph monoid $M(\Gamma)$ with presentation

$$\langle x_v \ (v \in V) \mid x_u x_v = x_v x_u \text{ if } (u, v) \in E\rangle.$$

The monoid $IH^0(M(\Gamma))$ is called a *polygraph monoid* and we denote it by $P(\Gamma)$. Put $X = \{x_v \mid v \in V\}$ and, for $x \in C_u$, $y \in C_v$, write $x \sim y$ if $(u, v) \in E$, and, in an abuse of notation, write $x \nsim y$ to mean $u \neq v$ and $(u, v) \notin E$. Then our polygraph monoid has a presentation

$$\begin{aligned} \langle X \cup X^{-1} \mid xx^{-1} = 1;\ xy^{-1} = 0 \text{ if } x \nsim y;& \\ xy = yx,\ xy^{-1} = y^{-1}x,\ x^{-1}y^{-1} = y^{-1}x^{-1} \text{ if } x \sim y\rangle.& \end{aligned}$$

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If Γ has no edges, then $M(\Gamma) = X^*$ is the free monoid on X and the polygraph monoid $IH^0(M(\Gamma))$ is the monoid with presentation

$$\langle X \cup X^{-1} \mid xx^{-1} = 1;\ xy^{-1} = 0 \text{ if } x \neq y\rangle,$$

that is, it is the polycyclic monoid introduced in [26] and studied in, among others, [17, 18, 25].

Let $P(\Gamma)$ be the polygraph monoid determined by the graph $\Gamma = (V, E)$. Since $P(\Gamma)$ is the inverse hull (with zero adjoined if necessary) of the graph monoid $M(\Gamma)$, it follows from the remarks following Theorem 2.13 that every nonzero element of $P(\Gamma)$ can be written as $a^{-1}b$ for some $a, b \in M(\Gamma)$. Since the identity is the only unit in $M(\Gamma)$ it follows that if $a, b, c, d \in M(\Gamma)$, then $a^{-1}b = c^{-1}d$ if and only if $a = c$ and $b = d$. Thus we may regard the nonzero elements of $P(\Gamma)$ as pairs (a, b) where $a, b \in M(\Gamma)$. With this notation, the product in $P(\Gamma)$ is given by

$$(a, b)(c, d) = \begin{cases} 0 & \text{if } M(\Gamma)b \cap M(\Gamma)c = \emptyset, \\ (sa, td) & \text{if } M(\Gamma)b \cap M(\Gamma)c = M(\Gamma)sb = M(\Gamma)tc. \end{cases}$$

PROPOSITION 3.1. *The monoid $P(\Gamma)$ is a 0-bisimple (bisimple if it has no zero) inverse monoid with*

$$E(P(\Gamma)) = \{(a, a) \mid a \in M(\Gamma)\} \cup \{0\}$$

as its set of idempotents.

PROOF. Since graph monoids are left LCM, Proposition 2.4 gives that $P(\Gamma)$ is a 0-bisimple (bisimple if it has no zero) inverse monoid.

It is easy to verify that any element of the form (a, a) is idempotent. Suppose that $(a, b)(a, b) = (a, b)$. Then $(ta, sb) = (a, b)$ where $M(\Gamma)a \cap M(\Gamma)b = M(\Gamma)sb = M(\Gamma)ta$. Hence, by the criterion for equality, $ta = a$ and $sb = b$ in $M(\Gamma)$ so that $t = s = 1$. Thus $M(\Gamma)a = M(\Gamma)b$ and hence $a = b$. □

Since $P(\Gamma)$ is 0-bisimple, $\mathscr{D} = \mathscr{J}$ and two elements are $\mathscr{D}$-related if and only if they are both nonzero or both equal to zero. In the next proposition we characterize the other Green relations on $P(\Gamma)$.

PROPOSITION 3.2. *For elements (a, b), (c, d) of $P(\Gamma)$:*

(1) $(a, b)^{-1} = (b, a)$;
(2) $(a, b)\mathscr{L}(c, d)$ *if and only if* $b = d$;
(3) $(a, b)\mathscr{R}(c, d)$ *if and only if* $a = c$;
(4) $\mathscr{H}$ *is trivial.*

PROOF. (1) is an easy calculation. In an inverse monoid, elements s, t are $\mathscr{L}$-related if and only if $s^{-1}s = t^{-1}t$. Using this and (1) we see that in $P(\Gamma)$ we have $(a, b)\mathscr{L}(c, d)$ if and only if $b = d$.

The result for $\mathscr{R}$ is similar, and then it follows that $\mathscr{H}$ is trivial. □

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We next consider the properties of being E^*-unitary or strongly E^*-unitary. For any inverse monoid S, the semilattice of idempotents of S is denoted by $E(S)$, and if S has a zero, then $E^*(S)$ denotes the set of nonzero idempotents. Recall from Section 1 that a subset U of S is *right unitary* in S if for $u \in U$, $s \in S$ we have $su \in U$ if and only if $s \in U$. There is a dual notion of *left unitary*, and if U is both left and right unitary, it is said to be *unitary* in S. If U is either $E(S)$ or $E^*(S)$, then it is left unitary if and only if it is right unitary. We say that S is E*-unitary* if $E(S)$ is a unitary subset of S, and that it is E^**-unitary* [30] (or 0-E-unitary [18, 25]) if $E^*(S)$ is a unitary subset of S. E^*-unitary inverse semigroups are discussed in detail in [18, Ch. 9].

A special class of E^*-unitary inverse semigroups was introduced independently in [3] and [19]. In general, if we adjoin a zero to a semigroup S, we denote the semigroup obtained by S^0. An inverse semigroup S with zero is *strongly* E^**-unitary* if there is a group G and a function $\theta : S \to G^0$ satisfying:

(1) $a\theta = 0$ if and only if $a = 0$;
(2) $a\theta = 1$ if and only if $a \in E^*(S)$;
(3) if $ab \neq 0$, then $(ab)\theta = (a\theta)(b\theta)$.

Condition (1) says that θ is 0-*restricted*; conditions (1) and (2) together say that θ is *idempotent pure*, that is, the only elements which map to idempotents are idempotents; and condition (3) says that θ is a *prehomomorphism*. In general, prehomomorphisms between inverse monoids are defined in terms of the natural order on the monoids, but the general definition is equivalent to condition (3) when the codomain is a group with zero adjoined. Implicit in [3] is the result that an inverse semigroup with zero is strongly E^*-unitary if and only if it is a Rees quotient of an E-unitary inverse semigroup. This was made explicit with an easy proof in [29]. As well as [3] and [29], further information about strongly E^*-unitary inverse semigroups, including many examples, can be found in the surveys [20] and [22].

We are interested in the connection between strongly E^*-unitary inverse monoids and embeddability of cancellative monoids in groups. The following result is due to Margolis [24]; we include a proof for completeness.

PROPOSITION 3.3. *Let S be a cancellative monoid. Then S is embeddable in a group if and only if $IH^0(S)$ is strongly E^*-unitary.*

PROOF. Suppose first that S is embedded in a group G. As noted in Section 2.1, every (nonzero) element ρ of $IH^0(S)$ can be expressed as $\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$ for some elements $a_1, b_1, \ldots, a_n, b_n$ of S. Define a mapping $\theta : IH^0(S) \to G^0$ by putting $0\theta = 0$ and $(\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1})\theta = a_1 b_1^{-1} \ldots a_n b_n^{-1}$ if $\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1}$ is nonzero.

If $\rho = \rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1} = \rho_{c_1}\rho_{d_1}^{-1} \ldots \rho_{c_m}\rho_{d_m}^{-1}$ is nonzero, then, for every element x in dom ρ,

$$x\rho = x\rho_{a_1}\rho_{b_1}^{-1} \ldots \rho_{a_n}\rho_{b_n}^{-1} = x\rho_{c_1}\rho_{d_1}^{-1} \ldots \rho_{c_m}\rho_{d_m}^{-1},$$

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so that, in G,

$$x\rho = xa_1b_1^{-1}\ldots a_nb_n^{-1} = xc_1d_1^{-1}\ldots c_md_m^{-1}$$

and hence $a_1b_1^{-1}\ldots a_nb_n^{-1} = c_1d_1^{-1}\ldots c_md_m^{-1}$. Thus θ is well defined.

By definition, θ is 0-restricted. If ρ is as defined above and $\rho\theta = 1$, then $a_1b_1^{-1}\ldots a_nb_n^{-1} = 1$ and it follows that $x\rho = x$ for all $x \in \operatorname{dom}\rho$ so that $\rho = I_{\operatorname{dom}\rho}$ and θ is idempotent pure. Finally, it is clear from the definition that if $\rho, \sigma \in IH^0(S)$ and $\rho\sigma \neq 0$, then $(\rho\sigma)\theta = (\rho\theta)(\sigma\theta)$ so that θ is a prehomomorphism. Thus $IH^0(S)$ is strongly E^*-unitary.

For the converse, we suppose that $IH^0(S)$ is strongly E^*-unitary and consider a 0-restricted idempotent pure prehomomorphism $\theta : IH^0(S) \to G^0$ from $IH^0(S)$ to a group G with zero adjoined. For each $a \in S$, we have the element ρ_a of $IH(S)$, and since $\operatorname{dom}\rho_a = S$, it follows that $\rho_a\rho_b = \rho_{ab}$ for any $a, b \in S$. Since θ is 0-restricted, $\rho_a\theta \in G$ and

$$(\rho_a\theta)(\rho_b\theta) = (\rho_a\rho_b)\theta = \rho_{ab}\theta.$$

Hence we can define $\psi : S \to G$ by $a\psi = \rho_a\theta$, and $(a\psi)(b\psi) = (ab)\psi$, that is, ψ is a homomorphism. It is also injective, for if $a\psi = b\psi$, then $\rho_a\theta = \rho_b\theta$. Now $\rho_a^{-1}\rho_b$ is a nonzero element of $IH^0(S)$, and so

$$(\rho_a^{-1}\rho_b)\theta = (\rho_a^{-1}\theta)(\rho_b\theta) = (\rho_a^{-1}\theta)(\rho_a\theta) = (\rho_a^{-1}\rho_a)\theta = 1$$

since $\rho_a^{-1}\rho_a$ is a nonzero idempotent. But θ is idempotent pure, so $\rho_a^{-1}\rho_b$ is an idempotent, that is, it is the identity map on its domain. Hence, for $x \in \operatorname{dom}(\rho_a^{-1}\rho_b)$, $x\rho_a^{-1}\rho_b = x$. Now $x\rho_a^{-1} = u$ where $x = ua$ and also $x = u\rho_b = ub$ so that $ua = ub$ and $a = b$ by cancellation.

Thus S is embedded in G. $\square$

It is well known (and a consequence of Corollary 1.8) that there is an embedding $\theta : M(\Gamma) \to G(\Gamma)$ of the graph monoid $M(\Gamma)$ into the graph group $G(\Gamma)$, and so we have the following corollary.

COROLLARY 3.4. *For any graph Γ, the polygraph monoid $P(\Gamma)$ is strongly E^*-unitary.*

In the next section, we see that $P(\Gamma)$ has another special property, namely that it is F^*-inverse.

4. F^*-inverse 0-bisimple inverse monoids

Recall that an inverse monoid S is F^*-*inverse* if every nonzero element of S is under a unique maximal element in the natural partial order. If S does not have a zero, it is said to be F-inverse, and in this case the definition is equivalent to every σ-class containing a maximum element. (Here σ is the minimum group congruence on S.) However, we shall use the term F^*-inverse to include both cases. It is easy to

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verify that every F^*-inverse monoid is E^*-unitary. An F^*-inverse monoid which is also strongly E^*-unitary is called *strongly F^*-inverse*. It follows from Corollary 3.4 and the results of this section that a polygraph monoid is strongly F^*-inverse.

We find a criterion for a 0-bisimple inverse monoid with cancellative right unit submonoid to be F^*-inverse in terms of a property of its right unit submonoid. We remark that by a result of Lawson [19], for a 0-bisimple inverse monoid, having a cancellative right unit submonoid is equivalent to being E^*-unitary.

LEMMA 4.1. *Let C be a right cancellative monoid and suppose that $IH^0(C)$ is 0-bisimple. If $a, b \in C$ have only units as common left factors, then $\rho_a^{-1}\rho_b$ is maximal in $IH^0(C)$.*

PROOF. Since $IH^0(C)$ is 0-bisimple, every element has the form $\rho_a^{-1}\rho_b$ for some $a, b \in C$. The result is now immediate from Lemma 2.1 and its corollary. $\square$

If C is a cancellative monoid, we denote the partially ordered set of principal right (left) ideals by $P_r(C)$ ($P_\ell(C)$). From the remarks at the end of Section 1, we see that $P_r(C)$ is a join semilattice if and only if every pair of elements has a highest common left factor, and it is a meet semilattice if and only if every pair of elements has a least common right multiple. Corresponding remarks apply to $P_\ell(C)$.

PROPOSITION 4.2. *Let C be a cancellative monoid and suppose that $IH^0(C)$ is 0-bisimple. Then $IH^0(C)$ is F^*-inverse if and only if $P_r(C)$ is a join semilattice.*

PROOF. Suppose that every pair of elements of C has a highest common left factor and let α be a nonzero element of $IH^0(C)$. Then $\alpha = \rho_a^{-1}\rho_b$ for some $a, b \in C$. Let x be a highest common left factor of a and b, say $a = xc$ and $b = xd$. Then the only common left factors of c and d are units, so, by Lemma 4.1, $\rho_c^{-1}\rho_d$ is maximal. But $\rho_a^{-1}\rho_b \leqslant \rho_c^{-1}\rho_d$ by Lemma 2.1, so α lies beneath a maximal element.

If $\rho_a^{-1}\rho_b \leqslant \rho_p^{-1}\rho_q$ for some $p, q \in C$ then, by Lemma 2.1, $a = yp$ and $b = yq$ for some $q \in C$. Hence $x = yz$ for some $z \in C$ so that $a = yp = yzc$ and $b = yq = yzd$. By left cancellation, $p = zc$ and $q = zd$ so that $\rho_p^{-1}\rho_q \leqslant \rho_c^{-1}\rho_d$ by Lemma 2.1. Thus $\rho_c^{-1}\rho_d$ is the unique maximal element above $\rho_a^{-1}\rho_b$, and $IH^0(C)$ is F^*-inverse.

Conversely, suppose that $IH^0(C)$ is F^*-inverse, and let $a, b \in C$. Then there is a unique maximal element $\rho_c^{-1}\rho_d$ above $\rho_a^{-1}\rho_b$. By Lemma 2.1, $a = xc$ and $b = xd$ for some $x \in C$. If y is a common left factor of a and b, then $a = yp$ and $b = yq$ for some $p, q \in C$ so that $\rho_a^{-1}\rho_b \leqslant \rho_p^{-1}\rho_q$. Now $\rho_p^{-1}\rho_q \leqslant \alpha$ for some maximal α, and, by uniqueness, $\alpha = \rho_c^{-1}\rho_d$. It follows that $p = zc$ and $q = zd$ for some z so that $xc = a = yzc$, whence $x = yz$ and y is a left factor of x. Thus x is a highest common left factor of a and b. $\square$

An abstract version of this proposition is given in the following result.

PROPOSITION 4.3. *Let S be an E^*-unitary 0-bisimple (E-unitary bisimple) inverse monoid, and let C be its right unit submonoid. Then S is F^*-inverse (F-inverse) if and only if $P_r(C)$ is a join semilattice.*

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PROOF. Since S is 0-bisimple, the right unit submonoid C of S is a left LCM monoid by [26, Proposition 1], and from the same proposition we have that S is isomorphic to $IH^0(C)$. By [19, Theorem 5], C is cancellative so that the result is now immediate by Proposition 4.2. □

A *Garside monoid* is defined to be a cancellative monoid whose only unit is the identity, that is, a lattice with respect to both left and right divisibility, and that satisfies additional finiteness conditions (see, for example, [9]). Such monoids have proved to be important in the study of algebraic and algorithmic properties of braid groups and, more generally, Artin groups of finite type. We note that if C is a Garside monoid, then since the identity is the only unit, regarded as a partially ordered set under left divisibility, C is order-isomorphic to $P_r(C)$ under reverse inclusion. Thus $P_r(C)$ is a lattice so that $IH(C)$ does not have a zero, and hence the next corollary follows immediately from Propositions 2.4 and 4.2.

COROLLARY 4.4. *The inverse hull of a Garside monoid C is a bisimple F-inverse monoid.*

We now turn to Artin monoids. Recall that an Artin monoid is a monoid generated by a nonempty set X that is subject to relations of the form $xyx \ldots = yxy \ldots$, where $x, y \in X$, both sides of a given relation have the same length, and at most one such relation holds for each pair $x, y \in X$. Thus graph monoids are Artin monoids where both sides of each defining relation have length 2. The associated *Artin group* of a given Artin monoid A is the group given by the presentation of A regarded as a group presentation. Rather than the definition, we use some of the properties of Artin monoids which we now recall. The first three in the list below can be found in [2], the third is also given in [11], and the fourth is from [27]. Let A be an Artin monoid. Then we have the following properties:

(1) A is cancellative;
(2) the intersection of two principal left (right) ideals of A is either empty or principal;
(3) A is left (and right) Ore if and only if it is of finite type;
(4) A embeds in its associated Artin group.

PROPOSITION 4.5. *The inverse hull $IH(A)$ of an Artin monoid A is strongly F^*-inverse.*

PROOF. It follows from Proposition 3.3 and item (4) above that $IH(A)$ is strongly E^*-unitary (E-unitary if A is of finite type). Moreover, we have already noted that (4) of Proposition 2.4 is satisfied. Hence $IH(A)$ is 0-bisimple (bisimple if A is of finite type).

Thus by Proposition 4.2, it is enough to show that any two elements of A have a highest common left factor. This is noted in [2]. The argument is as follows. Since the defining relations of A are homogeneous (that is, the two words in each relation have the same length), it follows that any factor (left or right) of an element w of A has length at most $|w|$. Hence any element of A has only finitely many left factors.

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Let $x_1, \ldots, x_k$ be the common left factors of two elements v and w of A. Then by the right-handed version of item (2),

$$x_1 A \cap \cdots \cap x_k A = x A$$

for some x. (That is, x is the least common left multiple of $x_1, \ldots, x_k$.) Now x is a common left factor of v and w (so must be one of the x_i), and is clearly the highest common left factor of v and w. □

Since a graph monoid is a special type of Artin monoid, we immediately have the following corollary.

COROLLARY 4.6. *For a graph Γ, the polygraph monoid is a strongly F^*-inverse monoid.*

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