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A finite set covering theorem III

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Let n, s, t be integers with $s > t > 2$. If a family of n different subsets of a set S, with s elements, has the properties,

(i) each member belongs to a set of $(t+1)$ members which together have union S,

(ii) no member belongs to a set of t members which together have union S,

then we prove that $n \leq (t+1)^{s-t-1}$. The result is best possible.

1. Introduction

Small letters denote non-negative integers and large letters denote sets. We denote the set $\{i, i+1, \ldots, j\}$ by $[i, j]$, with the proviso $[i, j] = \emptyset$ if ever $i > j$. If the t (disjoint) sets $X_1, X_2, \ldots, X_t$ of a given family have union S, we say that these t sets *cover* S, that the family has a *(disjoint) t-cover over* S, that X_1 is *in a (disjoint) t-cover over* S, that X_1 and X_2 are together in a *(disjoint) t-cover over* S, etc.

Given $s > t > 2$, our main problem, in this terminology, will be to find the maximum possible number of sets in a family N of n different subsets of $S = [1, s]$ chosen so that each member is in a $(t+1)$-cover, but not in a t-cover over S. In considering this problem an important example is the family

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$$F = \{X : X = P \cup Q,\ P \subset [1,\ t+1],\ |P| = 1;\ Q \subseteq S\backslash[1,\ t+1]\}$$

which contains

$$f(s,\ t) = (t+1)2^{s-t-1}$$

subsets of S. Each of the singletons $\{1\}, \{2\}, \ldots, \{t\}, \{t+1\}$ of F is in a $(t+1)$-cover over S, and since each set of F always contains just one element of $[1,\ t+1]$ it follows that every set of F is in a $(t+1)$-cover over S, but not in a t-cover. Thus in our main problem the maximum possible value of n is at least $f(s,\ t)$, and we will show not only that this is best possible, but also, when $t > 3$ in the extreme case, that N can always be obtained from F by permuting elements of S.

We observe that subsets of Q do not belong to F, which is a point of some importance, because, as we will see in the final section, if we can find a set belonging to N which has a non-empty subset that does not, then the main problem is easy to solve. For this reason we first consider a parallel but subsidiary problem, on what we call basic families, in which this does not happen.

Given $s > t > 1$, a family of different subsets of an s-set S, with properties,

(i) the family covers S,

(ii) no t members of the family cover S,

(iii) if the set X belongs to the family all subsets of X, including $\emptyset$, also belong to the family,

will be called a *basic family* $N(s,\ t)$ whose cardinality we will denote by $n(s,\ t)$. For brevity, we will always take the set S used in defining $N(s,\ t)$ to be $[1,\ s]$, unless another possibility is indicated. One further definition we need is the term *major* set, which is a member of $N(s,\ t)$ which is not a subset of any other member.

Our subsidiary problem now is to find the maximum possible number of sets in a basic family in which every non-empty set is in a $(t+1)$-cover over S. From [1] the maximum possible number of sets in any basic family is $(t+2)2^{s-t-1}$, which gives an upper bound in the subsidiary problem and also, incidentally, in the main problem. On the other hand consideration of the family

$$G = \{P : P \subseteq [1, t-1], |P| = 1\} \cup \{Q : Q \subset S \backslash [1, t-1]\}$$

shows that in our subsidiary problem $n(s, t)$ is at least

$$g(s, t) = 2^{s-t+1} + (t-2) ,$$

and the next result tells us that this is best possible.

THEOREM 1. *If every non-empty set of a basic family* $N(s, t)$ *is in a* $(t+1)$*-cover over* S *then* $n(s, t) \le g(s, t)$.

To prove this theorem, which we do in Section 3, we use a partition introduced in [1]. This partition, which we will denote by $\Delta_{\alpha\beta}$, is discussed in the next section, where we establish some properties of it that will be useful to us. It is in the final section that we return to the main problem and, using the above theorem, completely solve it. Throughout these sections we will write such expressions as $X \cup \{1\}$ in the form $X \cup 1$ so long as the meaning is clear, and all operations in expressions involving sets and families will be carried out from left to right, so that for example $X \cup Y \backslash Z$ stands for $(X \cup Y) \backslash Z$.

2. The partition $\Delta_{\alpha\beta}$

Given a basic family $N(s, t)$ and elements $\alpha, \beta \in S$, let

$$B = \{X : X\cup\alpha\cup\beta \notin N(s, t), X\cup\alpha\backslash\beta \in N(s, t), X\cup\beta\backslash\alpha \in N(s, t)\} ,$$

and consider the seven families $A, B_0, B_\alpha, B_\beta, C_\alpha, C_\beta, D$, where

$$A = \{X : X\cup\alpha\cup\beta \in N(s, t)\} ,$$

$$B_0 = \{X : X \in B; \alpha, \beta \notin X\} ,$$

$$B_\alpha = \{X : X \in B, \alpha \in X\} ,$$

$$B_\beta = \{X : X \in B, \beta \in X\} ,$$

$$C_\alpha = \{X : X\cup\alpha\backslash\beta \in N(s, t), X\cup\beta\backslash\alpha \notin N(s, t)\} ,$$

$$C_\beta = \{X : X\cup\alpha\backslash\beta \notin N(s, t), X\cup\beta\backslash\alpha \in N(s, t)\} ,$$

$$D = \{X : X\cup\alpha\backslash\beta \notin N(s, t),\ X\cup\beta\backslash\alpha \notin N(s, t)\} .$$

These families are pairwise disjoint and have union $N(s, t)$, and so define a partition of $N(s, t)$ which we will denote by $\Delta_{\alpha\beta}$.

When we apply $\Delta_{\alpha\beta}$ we always ensure, among other things, that A is non-empty. In this case the sets $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha, \beta\}$ of $N(s, t)$ will belong to A , and also since $X \in A$ implies that $X \cup \alpha \cup \beta$, $X\cup\alpha\backslash\beta$, $X\backslash\alpha\cup\beta$, $X\backslash\alpha\backslash\beta \in A$, we may write $|A| = 4a$. Similarly, since $X \in C_\alpha$ implies that $X \cup \alpha$, $X\backslash\alpha \in C_\alpha$, we may write $|C_\alpha| = 2c_1$ and in exactly the same way we may write $|C_\beta| = 2c_2$. Examination of the families B_α, B_β reveals that

$$B_\alpha = \{Y : Y = X \cup \alpha,\ X \in B_0\} ,\quad B_\beta = \{Y : Y = X \cup \beta,\ X \in B_0\} ,$$

and thus $|B_0|$, $|B_\alpha|$, $|B_\beta|$ have a common value which we will denote by b . The number of sets in $N(s, t)$ is given by

(1) $$n(s, t) = 4a + 3b + 2(c_1+c_2) + d ,$$

where we have put $|D| = d$.

We now establish two useful properties of $\Delta_{\alpha\beta}$ which arise when $N(s, t)$ satisfies one or the other of two important extra conditions. Our first result is:

LEMMA 1. *If $N(s, t)$ is a basic family containing at least one doubleton in which every non-empty set is in a $(t+1)$-cover over S , then we can always choose $\alpha, \beta \in S$ so that when $\Delta_{\alpha\beta}$ is applied to $N(s, t)$ we get $a \neq 0$ and $d \geq t-2$.*

Proof. When $t = 2$ the lemma follows by letting α, β belong to the doubleton we know $N(s, 2)$ contains.

When $t > 2$ the proof is by double induction on s, t . We assume the theorem is true for $s-1$, $t-1$ and then establish it for s, t . To do this we distinguish two cases depending on whether or not $N(s, t)$ contains a singleton major set.

Case (a). $N(s, t)$ contains a singleton major set, say $\{s\}$.

In this case the family $N' = N(s, t)\backslash\{s\}$ is clearly a basic family $N(s-1, t-1)$ of $n(s, t) - 1$ different subsets of $S\backslash s$ of which at least one is a doubleton. Furthermore every non-empty set of $N(s, t)$, other than $\{s\}$, is in a $(t+1)$-cover over S with t other sets of $N(s, t)$ of which one must be $\{s\}$, so obviously every non-empty set of N' must be in a t-cover over $S\backslash s$. Hence N' satisfies the conditions of the lemma for $s-1$, $t-1$ and we can choose $\alpha, \beta \in S\backslash s$ so that when $\Delta_{\alpha\beta}$ is applied to N' we get $a \neq 0$ and $d \geq t-3$. Thus if we use the same α, β to apply $\Delta_{\alpha\beta}$ to $N(s, t)$, then because the set $\{s\}$ must belong to D we obviously get $a \neq 0$ and $d \geq t-2$, which shows that the lemma is true in this case.

Case (b). $N(s, t)$ contains no singleton major set.

The set $\{s\} \in N(s, t)$ is in a $(t+1)$-cover over S with t other sets of $N(s, t)$. Replace each of these t sets which is not alreadv a major set with a superset which is a major set. Thus we have a $(t+1)$-cover over S involving $\{s\}$ and t major sets of $N(s, t)$. Put $M_0 = \emptyset$ and $m_0 = 0$ and denote these t major sets by $M_1, M_2, \ldots, M_t$ in that order in such a way that, given $M_0, M_1, \ldots, M_{j-1}$,

$$m_j = \left| M_j \Big\backslash \bigcup_{k=0}^{j-1} M_k \right| , \quad (j = 1, 2, \ldots, t)$$

is a maximum with respect to the remaining sets $M_j, \ldots, M_t$. Renumber elements of $S\backslash s$ so that

$$M_j \Big\backslash \bigcup_{k=0}^{j-1} M_k = \left[1 + \sum_{k=0}^{j-1} m_k, \sum_{k=0}^{j} m_k \right] , \quad (j = 1, 2, \ldots, t) .$$

Since $\{s\}$ is not a major set of $N(s, t)$ there is at least one doubleton containing $\{s\}$ in $N(s, t)$. Let this doubleton be $\{x, s\}$ where x is as large as possible.

Let j_1 be such that $\displaystyle\sum_{k=0}^{j_1-1} m_k < x \leq \sum_{k=0}^{j_1} m_k$.

If $j_1 = t$ then put $\alpha = x$, $\beta = s$, and in the partition Δ_{xs} the

$(t-1)$ sets $M_1, M_2, \ldots, M_{t-1}$ belong to D , and we get $a \neq 0$ and $d \geq t-1$.

If $j_1 < t$ and $m_j = 1$ for $j_1 < j \leq t$ then no set of $N(s, t)$ can contain more than one of the elements $s - (t-j_1), \ldots, s$. Hence for the set $\{x\} \in N(s, t)$ to be in a $(t+1)$-cover over S there must be $(t-j_1)$ major sets containing separately the elements $s - (t-j_1), \ldots, s-1$ but not containing the elements x or s . In the partition Δ_{xs} these $(t-j_1)$ major sets together with the (j_1-1) sets $M_1, M_2, \ldots, M_{j_1-1}$ belong to D and we get $a \neq 0$ and $d \geq t-1$.

If $j_1 < t$ and $m_j \geq 2$ for $j_1 < j \leq t$ then put $\alpha = s - 2$, $\beta = s - 1$, and in the partition $\Delta_{s-2,s-1}$ the $(t-1)$ major sets $M_1, M_2, \ldots, M_{t-1}$ belong to D , and we get $a \neq 0$ and $d \geq t-1$.

Now let $j_1 < j_2 < t$ and $m_j \geq 2$ for $j_1 < j \leq j_2$ and $m_j = 1$ for $j_2 < j \leq t$. If there is a set $\{y, z\} \in N(s, t)$ such that $\sum_{k=0}^{j_2-1} m_k < y \leq \sum_{k=0}^{j_2} m_k$ and $\sum_{k=0}^{j_2} m_k < z < s$, then after interchanging s and z of S we have the second possibility above. Otherwise put $\alpha = s - (t-j_2) - 2$ and $\beta = s - (t-j_2) - 1$ and in the partition $\Delta_{\alpha\beta}$ the j_2-1 major sets $M_1, M_2, \ldots, M_{j_2-1}$ together with the $(t-j_2+1)$ major sets containing separately the elements $s - (t-j_2), \ldots, s$ but not the elements $s - (t-j_2) - 2$ or $s - (t-j_2) - 1$ belong to D and we get $a \neq 0$ and $d \geq t$.

All possibilities in this second case are now exhausted and the lemma follows.

The second result we will need is:

LEMMA 2. *If $N(s, t)$ is a basic family for which $n(s, t)$ is maximal and if on applying $\Delta_{\alpha\beta}$ the family B_β is non-empty, then each*

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set of B_β *is in one or more* t*-covers over* $S\backslash\alpha$ *, and the* $(t-1)$ *other sets in every such* t*-cover belong to* D *.*

Proof. If $T_1 \in B_\beta$ then the two sets T_1 and $\{\alpha\}$ together with some $(t-1)$ other sets $T_2, T_3, \ldots, T_t$ of $N(s, t)$ must cover S , else we can adjoin the set $T_1 \cup \alpha$ to $N(s, t)$ without creating a t-cover, which contradicts our assumption that $n(s, t)$ is maximal. If T_2 belongs to one of A, B_α, C_α then the t sets T_1, $T_2 \cup \alpha$, T_3, $\ldots$, T_t belong to $N(s, t)$ and cover S which is a contradiction. Also, if T_2 belongs to one of B_0, B_β, C_β ,then the t sets $T_1 \cup \alpha \backslash \beta$, $T_2 \cup \beta$, T_3, $\ldots$, T_t belong to $N(s, t)$ and cover S , which is again a contradiction. Thus T_2 belongs to D and the lemma follows by applying the same argument to each set other than T_1 in each such t-cover over $S\backslash\alpha$ involving T_1 .

This completes our discussion of the partition $\Delta_{\alpha\beta}$.

3. Proof of Theorem 1

In proving this theorem there is obviously no loss of generality in assuming that $n(s, t)$ is maximal. Thus throughout this section we will assume that $N(s, t)$ is a basic family in which every non-empty set is in a $(t+1)$-cover over S , and that it contains the maximum possible number of sets.

In the simple case when $s = t + 1$ only $\emptyset$ and singletons can belong to $N(s, t)$, and so $n(s, s-1) \leq s + 1 = g(s, s-1)$, which shows the theorem is true for values of s, t satisfying this relationship. Proof in the general case is by induction on s . We assume the theorem is true for s', t when $1 < t < s' < s$ and establish it for s, t *by showing that the further assumption* $n(s, t) > g(s, t)$ *always leads to a contradiction.*

Since we are assuming $n(s, t)$ is maximal, when $s > t+1$ the family $N(s, t)$ must contain at least one doubleton. Thus Lemma 1 applies to $N(s, t)$ in the general case and we can always choose $\alpha, \beta \in S$ so as to get $a \neq 0$ and $d \geq t-2$ in the partition $\Delta_{\alpha\beta}$. With no loss of generality we assume these values of α, β are 1, 2 respectively, and we then distinguish two cases depending on the relative values of b and d arising when Δ_{12} is applied to $N(s, t)$.

Case (a). $t-2 \leq d < b+t-2$.

The above inequality shows that in this case B_2 is non-empty, and so it contains at least one set, say T_1 , whose cardinality, which by definition must be at least two, is greater than or equal to the cardinality of every other set in B_2 . Lemma 2 now tells us that there are $(t-1)$ other sets $T_2, T_3, \ldots, T_t$ in $N(s, t)$, which must belong to D , and which together with T_1 form a t-cover of $S\backslash 1$. We lose no generality in assuming this is a disjoint t-cover because, remembering that all subsets of each set T_i $(i = 2, \ldots, t)$ also belong to $N(s, t)$, we can obviously replace each T_i $(i = 2, \ldots, t)$ by a set $T_i' \subseteq T_i$ chosen so that the t sets $T_1, T_2', \ldots, T_t'$ are disjoint and cover $S\backslash 1$, and it then follows from Lemma 2 that each of these new sets T_i' $(i = 2, \ldots, t)$ also belongs to D .

Using these facts we will now obtain our contradiction for this case by showing that from the family V_1 of different subsets of $S\backslash 1$ defined by

$$V_1 = A' \cup B_0 \cup B_2' \cup C_1' \cup C_2' \cup H ,$$

$$A' = \{X : X \in A, 1 \notin X\} ,$$

$$B_2' = B_2 \backslash T_1 ,$$

$$C_1' = \{X : X \in C_1, 1 \notin X\} ,$$

$$C_2' = \{X : X \in C_2, 2 \notin X\} ,$$

$$H = \{X : X \subseteq T_i \ (i = 2, \ldots, t), X \in D\} ,$$

we can always derive a family satisfying the conditions of the theorem at $s-1$, t which contains, however, more than $g(s-1, t)$ members.

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If $X \in V_1$, clearly all subsets of X belong to V_1 and also the sets $\{2\}$, $T_1 \backslash 2, T_2, \ldots, T_t$ belong to V_1 and cover $S \backslash 1$. Thus V_1 satisfies the first two conditions for a basic family. Now suppose the t sets $X_1, X_2, \ldots, X_t$ belong to V_1 and cover $S \backslash 1$. Since the element 2 must be covered by at least one of these sets, say X_1, belongs either to A' or B_2'. If $X_1 \in A'$ the t sets $X_1 \cup 1, X_2, \ldots, X_t$ belong to $N(s, t)$ and cover S, which is a contradiction. On the other hand if $X_1 \in B_2'$ then by Lemma 2 the remaining sets $X_2, \ldots, X_t \in H \subseteq D$, and so $X_2 \cup X_3 \cup \ldots \cup X_t \subseteq T_2 \cup T_3 \cup \ldots \cup T_t$. This means that for the t sets $X_1, X_2, \ldots, X_t$ to cover $S \backslash 1$ we must have $T_1 \subset X_1$, which contradicts the definition of T_1. Thus V_1 contains no t-cover over $S \backslash 1$, which is the third condition for a basic family. Noting that

$$|A'| = 2a\ , \quad |B_0| = b\ , \quad |B_2'| = b - 1\ , \quad |C_1'| = c_1\ ,$$
$$|C_2'| = c_2\ , \quad |H| = h \geq t-1\ ,$$

the number of sets in V_1 is given by

$$|V_1| = 2a + 2b + (c_1+c_2) + (h-1)\ .$$

Thus

$$2|V_1| = 4a + 3b + 2(c_1+c_2) + d + [b-d+2(h-1)]\ ,$$

which upon using (1) gives

$$2|V_1| = n(s, t) + [b-d+2(h-1)]\ .$$

Remembering that in this case $d < b+t-2$ while $h \geq t-1$, our assumption that $n(s, t) > g(s, t)$ now yields

$$2|V_1| > g(s, t) + t - 2 = 2g(s-1, t)\ ,$$

that is

$$|V_1| > g(s-1, t)\ .$$

Thus V_1 is a basic family $N(s-1, t)$ of more than $g(s-1, t)$

different subsets of $S\backslash 1$.

From V_1 we will now derive a new family that will be denoted by V_k , which is also a basic family $N(s-1, t)$ of more than $g(s-1, t)$ different subsets of $S\backslash 1$, but in which every singleton is in a $(t+1)$-cover over $S\backslash 1$. This will immediately give the contradiction we mentioned above, because if every singleton in this new family V_k is in a $(t+1)$-cover over $S\backslash 1$, then obviously so is every other non-empty set of V_k .

Noting that if $x \in T_1$ the $(t+1)$ sets $\{x\}$, $T_1\backslash x, T_2, \ldots, T_t$ belong to V_1 and cover $S\backslash 1$, it is clear that all singletons $\{x\} \subset T_1$ and all sets T_i $(i = 2, \ldots, t)$ which are themselves singletons, are already in $(t+1)$-covers over $S\backslash 1$ in V_1 . Thus we suppose, with no loss of generality, that each set T_i , other than the sets $T_1, T_2, \ldots, T_j$, is a singleton, and we then define a sequence of families $V_1, V_2, \ldots, V_k$, where $k \leq j \leq t$, as follows. Assuming V_{l-1} , where $l \geq 2$, has been defined, if every singleton subset of $T_l \cup \ldots \cup T_j$ is in a $(t+1)$-cover over $S\backslash 1$ in V_{l-1} we set $l - 1 = k$ and end the sequence. On the other hand if some singleton subset of $T_l \cup \ldots \cup T_j$ is not in such a $(t+1)$-cover we assume, with no loss of generality, that it is $\{x_l\} \subset T_l$ and then define the next family in the sequence to be

$$V_l = V_{l-1} \cup V_l' \cup T_{l-1}\backslash T_l ,$$

where

$$V_l' = \{Y : Y = X \cup x_l, X \subset T_{l-1}\} .$$

Clearly V_k covers $S\backslash 1$, contains at least as many members as V_1 , and if $X \in V_k$ then so do all subsets of X . Thus to complete this case we only have to show that when $k > 1$ no t sets of V_k cover $S\backslash 1$, and that every singleton of V_k is in a $(t+1)$-cover over $S\backslash 1$.

So suppose that, whereas each family V_i $(1 \leq i < l)$ contains no t-cover, the t sets $X_1, X_2, \ldots, X_t$ of V_l do cover $S\backslash 1$. Clearly some of the X_i $(i = 1, \ldots, t)$ must belong to $V'_l \cup T_{l-1}$, because this is the essential difference between V_{l-1} and V_l in respect to t-covers over $S\backslash 1$. Now if no X_i is T_{l-1} the $(t+1)$ sets $\{x_l\}$, $X_1\backslash x_l, \ldots, X_t\backslash x_l$ belong to V_{l-1} and cover $S\backslash 1$, while if $X_{l-1} = T_{l-1}$ and $X_1 \in V'_l$ the $(t+1)$ sets $\{x_l\}$

$$X_1\backslash x_l, \ldots, X_{l-2}\backslash x_l, X_{l-1}\backslash X_1, X_l\backslash x_l, \ldots, X_t\backslash x_l$$

will belong to V_{l-1} and cover $S\backslash 1$. Thus, because the singleton $\{x_l\}$ in V_{l-1} is not in a $(t+1)$-cover over $S\backslash 1$, it follows that one of the X_i, say X_{l-1}, is T_{l-1} and of the rest none belong to V'_l. Now suppose there exists m satisfying $1 \leq m < l-1$, such that the set T_m does not appear among the X_i whereas the sets $T_{m+1}, \ldots, T_{l-1}$ do appear among the X_i. In this case it is easy to see that among the t sets $X_1\backslash x_{m+1}, \ldots, X_m\backslash x_{m+1}, T_{m+1}, \ldots, T_{l-1}, X_l\backslash x_{m+1}, \ldots, X_t\backslash x_{m+1}$ there are at most t sets belonging to V_m and covering $S\backslash 1$, which is a contradiction. Thus every one of the $(l-1)$ sets $T_1, \ldots, T_{l-1}$ must appear among the X_i and we suppose, with no loss of generality, that the t-cover over $S\backslash 1$ in V_l actually consists of the sets $T_1, \ldots, T_{l-1}, X_l, \ldots, X_t$. Now every one of these sets also belongs to $N(s, t)$, because if some X_i, where $l \leq i \leq t$, does not belong to $N(s, t)$ it must be a subset of $T_1 \cup \ldots \cup T_{l-1}$, which implies $N(s, t)$ has a t-cover over S, a contradiction. Thus Lemma 2 applies and tells us that the sets $X_l, \ldots, X_t$ belong to $H \subseteq D$. But T_l does not belong to V_l, which means that $X_1 \cup \ldots \cup X_t \subset T_1 \cup \ldots \cup T_t$, and then because the sets $T_1, \ldots, T_t$ are disjoint it follows that the t sets $X_1, \ldots, X_t$ of V_l cannot possibly cover $S\backslash 1$. Thus if each family V_i

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$(i = 1, \ldots, l-1)$ contains no t-cover over $S\backslash 1$ then neither does V_l, and it follows by induction on l that V_k, like V_1, contains no t-cover over $S\backslash 1$.

Our final requirement that every singleton subset of V_k is in a $(t+1)$-cover over $S\backslash 1$, is easy to check. If $x \in T_k$ the $(t+1)$ sets $T_1, \ldots, T_{k-1}, \{x\}, T_k\backslash x, T_{k+1}, \ldots, T_t$ belong to V_k and cover $S\backslash 1$, and if $x \in T_m$ $(1 \leq m \leq k-1)$ the $(t+1)$ sets $\{x\}, T_1, \ldots, T_{m-1}$, $T_m\backslash x \cup x_{m+1}, T_{m+1}\backslash x_{m+1} \cup x_{m+2}, \ldots, T_{k-1}\backslash x_{k-1} \cup x_k, T_k\backslash x_k, T_{k+1}, \ldots, T_t$ also belong to V_k and cover $S\backslash 1$. Thus, remembering that by definition of V_k every singleton subset of $T_{k+1} \cup \ldots \cup T_j$ is in a $(t+1)$-cover, it follows that all singletons in V_k are in $(t+1)$-covers over $S\backslash 1$.

This completes the case.

Case (b). $b+t-2 \leq d$.

In this case we obtain our contradiction by showing that from $N(s, t)$ we can always derive another family, satisfying the conditions of the theorem for s', t, where $1 < t < s' < s$, but which contains more than $g(s', t)$ members.

With no loss of generality, we suppose that the common intersection of the major sets of the family A defined by applying Δ_{12} to $N(s, t)$ is $[1, j+1]$, where $j \geq 1$, and putting $W_0 = N(s, t)$ we define a sequence of families $W_0, W_1, \ldots, W_j$ as follows. Assuming W_{i-1} is defined, we apply $\Delta_{i,i+1}$ to W_{i-1} and (using generic notation) define W_i by

$$W_i = A' \cup B_0 \cup C_i' \cup C_{i+1}' \cup D,$$

$$A' = \{X : X \in A, i \notin X\},$$

$$C_i' = \{X : X \in C_i, i \notin X\},$$

$$C_{i+1}' = \{X : X \in C_{i+1}, i+1 \notin X\}.$$

We claim that W_j is a basic family $N(s-j, t)$ of more than

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$g(s-j, t)$ different subsets of $S\backslash[1, j]$ in which every singleton is in a $(t+1)$-cover over $S\backslash[1, j]$. If this is so, the contradiction mentioned above arises for W_j and the case is complete.

Clearly W_j is a family of different subsets of $S\backslash[1, j]$ and if X belongs to W_j so do all subsets of X . Roughly speaking sets of W_{i-1} and of W_i differ only with respect to elements i and $i+1$, and so if W_{i-1} covers $S\backslash[1, i-1]$ then W_i covers $S\backslash[1, i]$ because when $\Delta_{i,i+1}$ is applied to W_{i-1} $(i = 1, \ldots, j)$ the corresponding family A' , which must contain the set $[i+1, j+1]$, is non-empty. Also if W_{i-1} contains no t-cover over $S\backslash[1, i-1]$ whereas the t-sets $X_1, X_2, \ldots, X_t$ of W_i do cover $S\backslash[1, i]$, then since the element $(i+1)$ is covered by W_i , one of these t sets, say X_1 , must belong to A' . This implies that the t sets $X_1 \cup i, X_2, \ldots, X_t$ belong to W_{i-1} and cover $S\backslash[1, i-1]$, which is a contradiction that shows W_i contains no t-cover over $S\backslash[1, i]$. Thus by induction on i the cover of $S\backslash[1, j]$ in W_j contains more than t sets and W_j is therefore a basic family.

Now consider what happens when $\Delta_{i,i+1}$ is applied to W_{i-1} , $(i = 2, \ldots, j)$ from the point of view of the numbers of sets involved. The singleton $\{i+1\}$ together with some t other sets $X_1, \ldots, X_t$ of $N(s, t)$ cover S , and so after truncation each X_k $(k = 1, \ldots, t)$ gives rise to a corresponding set Y_k in W_{i-1} . These sets $Y_1, \ldots, Y_t$ must be different, for if say $Y_1 = Y_2$, then $X_1\backslash[1, i-1] = X_2\backslash[1, i-1]$ and the t sets $[1, i+1]$, $X_2, \ldots, X_t$ belong to $N(s, t)$ and cover S , which is a contradiction. Using generic notation, the element i only occurs in sets of W_{i-1} that belong to A , which incidentally means that $B_i = \emptyset$ and $b = 0$. Thus if $i \in Y_1$ say, the set $Y_1 \cup \{i+1\}$ belongs to W_{i-1} , the set $X_1 \cup [1, i+1]$ belongs to $N(s, t)$ and the t sets $X_1 \cup [1, i+1], X_2, \ldots, X_t$ of $N(s, t)$ cover S , which is a contradiction.

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Also if $i+1 \in Y_1$ say, then $i+1 \in X_1$ and the t sets $X_1, X_2, \ldots, X_t$ belong to $N(s, t)$ and cover S, which is again a contradiction. Thus none of the t sets $Y_1, \ldots, Y_t$ can have either i or $i+1$ adjoined and so when $\Delta_{i,i+1}$ is applied to W_{i-1} they belong to D. Hence $b = 0$ and $d \geq t$ and the inequality $b+t-2 \leq d$, which in this case holds when Δ_{12} is applied to W_0, also holds when $\Delta_{i,i+1}$ is applied to W_{i-1} $(i = 2, \ldots, j)$.

Now suppose the number of sets in W_{i-1} $(i = 1, \ldots, j)$ is greater than $g(s-i+1, t)$. Then using generic notation, since

$$W_i = A' \cup B_0 \cup C'_i \cup C'_{i+1} \cup D$$

and clearly

$$|A'| = 2a , \quad |B_0| = b , \quad |C'_i| = c_1 , \quad |C'_{i+1}| = c_2 , \quad |D| = d ,$$

the number of sets in W_i is given by

$$|W_i| = 2a + b + c_1 + c_2 + d .$$

Hence

$$2|W_i| = 4a + 3b + 2(c_1+c_2) + d + (d-b) ,$$

which, on using (1) and the inequality $b+t-2 \leq d$, gives

$$2|W_i| \geq |W_{i-1}| + (t-2) > g(s-i+1, t) + (t-2) = 2g(s-i, t) ,$$

that is

$$|W_i| > g(s-i, t) .$$

Thus by induction on i our assumption that $n(s, t) > g(s, t)$ for $W_0 = N(s, t)$ gives $|W_j| > g(s-j, t)$.

So far we have shown that W_j is a basic family $N(s-j, t)$ of more than $g(s-j, t)$ different subsets of $S\setminus[1, j]$, and hence to finish this case and the theorem we now only have to justify the claim that every singleton set of W_j is a $(t+1)$-cover over $S\setminus[1, j]$.

The singleton $\{j+1\}$ of W_j is obviously in a $(t+1)$-cover over $S\backslash[1, j]$ in W_j, because the set $[1, j+1]$ of $N(s, t)$ was in a $(t+1)$-cover over S in $N(s, t)$. For each singleton $\{x\} \subset S\backslash[1, j+1]$ there must be at least one set X_1 say, which belongs to the family A defined by applying Δ_{12} to $N(s, t)$, such that $\{x\}$ and X_1 are together in a $(t+1)$-cover over S in $N(s, t)$. Otherwise, either $X_1 \cup x$ could be adjointed to $N(s, t)$, which contradicts the assumption that $n(s, t)$ is maximal, or x belongs to every major set of A implying $x \in [1, j+1]$, which is again a contradiction. Thus, if the $(t-1)$ other sets in this t-cover are $X_2, \ldots, X_t$, then clearly the $(t+1)$ sets

$$\{x\}, X_1\backslash[1, j], X_2\backslash[1, j], \ldots, X_t\backslash[1, j]$$

belong to W_j and cover $S\backslash[1, j]$. Hence every singleton of W_j is in a $(t+1)$-cover over $S\backslash[1, j]$.

This completes the proof of Theorem 1.

It is interesting to note that the extreme value $g(s, t)$ can be attained in ways other than G. For instance, given $q \geq 1$ when $t > 2$ and $s = 2q + t - 1$ the family

$$G' = \{P : P \subseteq [1, t-2], |P| = 1\} \cup \{Q : Q \subset S\backslash[1, t-2], |Q| \leq q\}$$

satisfies the conditions of Theorem 1 and contains $g(s, t)$ members. However, both G and G' do contain $(t-2)$ singleton major sets and, although we do not prove it, it is not very difficult to show that this property characterizes $N(s, t)$ in the extreme case.

4. The main theorem

With Theorem 1 proved in the last section it is now easy to establish our main result:

THEOREM 2. *Given $s > t > 2$, if N is a family of n different subsets of $S = [1, s]$ chosen so that each member is in a $(t+1)$-cover but not in a t-cover over S, then*

(i) $n \leq f(s, t)$, *and*

(ii) when $t > 3$ *in the extreme case* N *is obtainable from* F *by permuting elements of* S .

The result is best possible.

Proof of *(i)*. When $s = t + 1$ only singletons can belong to N and so *(i)* is true for this simple case. Proof in the general case $s > t + 1$, where from our example F we know that the maximum value of n is at least $f(s, t)$, is by induction on s . We assume the theorem is true for $s-1, t$ and then establish it for s, t by showing that if we assume n is maximal and greater than $f(s, t)$, then we get a contradiction.

These assumptions yield the inequality $n > f(s, t) \geq g(s, t) - 1$ when $t > 2$ and $s > t + 1$, and so by comparing the conditions of Theorems 1 and 2 (and remembering that the empty set always belongs to basic families) there must clearly be a set belonging to N which has a non-empty subset Z that does not belong to N . Obviously the only reason why Z cannot be adjoined to N is because it would not be in a $(t+1)$-cover over S . Thus no non-empty subset of Z belongs to N and we may assume, with no loss of generality, that Z is the singleton $\{s\}$. In these circumstances we claim that for any X belonging to N the sets $X \cup s$ and $X \backslash s$ also belong to N . If $s \notin X$ the set $X \cup s$ cannot be in a t-cover over S because it would imply that $\{s\}$ is already in a $(t+1)$-cover over S which is a contradiction, and if X is in a $(t+1)$-cover over S so obviously is $X \cup s$. On the other hand if $s \in X$ then the set $X \backslash s$ cannot be in a t-cover over S , and since there are t sets $X_1, \ldots, X_t$ belonging to N which together with X cover S (while we have shown that $X_1 \cup s$ belongs to N), clearly the $(t+1)$ sets $X \backslash s$, $X_1 \cup s, X_2, \ldots, X_t$ cover S . Hence whichever possibility arises, since n is maximal, both $X \cup s$ and $X \backslash s$ must belong to N . Thus N contains exactly twice as many sets as the family $N = \{X : X \in N, s \notin X\}$ of different subsets of $S \backslash s$, which is obviously a family that satisfies the conditions of the theorem for $s-1, t$. Hence $n \leq 2f(s-1, t) = f(s, t)$ which contradicts our assumption that $n > f(s, t)$ and establishes *(i)*.

Proof of *(ii)*. If $n = f(s, t)$, then for $t > 3$ and $s > t + 1$ we

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find $n = f(s, t) > g(s, t) - 1$. This inequality means that we can apply the reasoning used in the proof of *(i)*, and so when $t > 3$ in the extreme case we can always derive from N another family N' containing $f(s-1, t)$ members, which satisfies the conditions of the theorem for $s-1, t$. Hence, because this result is obviously true when $s = t + 1$, *(ii)* follows easily by induction on s .

This completes the proof of our main theorem.

Reference

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