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LOCAL RINGS WITH ELEMENTARY ABELIAN UNITS

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In [2] the structure of all semiperfect rings with abelian group of units has been obtained in terms of commutative local rings. It follows easily that the structure of semiperfect rings with elementary abelian group of units is determined by commutative local rings whose unit groups are elementary abelian. In this note such local rings are completely characterized. It is shown that a local ring having an elementary abelian group of units has characteristic two, four or eight and is a homomorphic image of $Z_kG/E(Z_kG)$ where G is some elementary 2-group and $E(Z_kG)$ is the ideal of Z_kG generated by $\{1-u^2 : u \in (Z_kG)^*\}$.

Throughout, all rings will be associative with identity and all subrings shall have the same identity as the original ring. If $n \geq 1$ is an integer, Z_n will denote the ring of integers modulo n. If R is a ring, we denote the group of units by R^* and the Jacobson radical by $J(R)$. A ring R is called *local* if $R/J(R)$ is a division ring. Alternatively, R is local if $R = J(R) \cup R^*$ or if R has a unique maximal left ideal. If p is a prime integer, a group G is an elementary p-group if $a^p = 1$ for every $a \in G$. In particular, we regard the trivial group with one element as an elementary p-group for each prime p.

PROPOSITION 1. *Let R be a local ring with R^* an elementary abelian p-group for p an odd prime. Then $R \cong Z_2$ or R is a field of characteristic* 2 *such that $|R| = 2^k$ and $p = 2^k - 1$, $k \geq 2$.*

Proof. Char $R = 2$ since $(-1)^2 = 1$ and p is odd. If $a \in J(R)$, then $(1+a)^p = 1 + pa + ba = 1 + a + ba$, $b \in J(R)$. Hence $(1+b)a = 0$ and so $a = 0$. Thus $J(R) = 0$ and R is a field. Since R is a field and R^* is elementary p-group, R^* is a group of at most p elements. Since p is a prime, R^* is the group with one element or R^* is the cyclic group of order p. The result follows.

COROLLARY 2. *If F is a field such that F^* is an elementary p-group for p prime, then $F \cong Z_2$ or $F \cong Z_3$ or F has characteristic* 2 *and $|F| = 2^k$ where $p = 2^k - 1$, $k \geq 2$.*

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Proof. If $p\neq 2$, this is Proposition 1. If $p=2$, then F^* is a group of at most 2 elements, so $|F|=2$ or 3. Thus, $F\cong Z_2$ or $F\cong Z_3$.

The preceding proposition reduces the problem to considering the case where $p=2$ and R is not a field. The next result collects some basic facts.

PROPOSITION 3. *Let R be a local ring such that R^* is an elementary 2-group. Then*

(1) *R is commutative and either $R\cong Z_3$ or $R/J(R)\cong Z_2$.*
(2) *If $R\not\cong Z_3$ then* char $R=2$, 4 *or* 8.
(3) *If $a\in J(R)$ then $a^2=2a$ and $4a=0$.*

Proof. Let $a, b\in J(R)$ and $u\in R^*$. Then $1+a$, $1+b\in R^*$ and so $ab=ba$ and $au=ua$ follow since R^* is abelian. Hence R is commutative. Furthermore $(1+a)^2=1=(1-a)^2$ and so $a^2=2a=-2a$. This proves (3). Now $R/J(R)$ is a field and so is isomorphic to Z_2 or Z_3 by Corollary 2. If $R/J(R)\cong Z_2$ then $2\in J(R)$ so $8=0$ by (3). This means char $R=2,4$ or 8 and it remains to show that $R/J(R)\cong Z_3$ implies $J(R)=0$. But $R/J(R)\cong Z_3$ means $3\in J(R)$ and hence $12R=0$. Since R is local this means char $R=3$ and so if $a\in J(R)$, $a=4a=0$ by (3).

We now define elementary 2-rings and devote the remainder of this note to characterizing these rings.

DEFINITION 4. *A ring R is an elementary 2-ring if and only if R is local, R^* is an elementary 2-group and $R\not\cong Z_3$.*

COROLLARY 5. *Every subring and homomorphic image of an elementary 2-ring is again an elementary 2-ring.*

Proof. If R is an elementary 2-ring and $a\in J(R)$ then $a^3=(2a)a=2a^2=4a=0$. Hence $r^2=1$ or $r^3=0$ for each $r\in R$ and the result follows.

The next result is our first step towards a complete characterization of these elementary 2-rings.

PROPOSITION 6. *Let R be an elementary 2-ring of characteristic two (respectively four, eight). Then R is a homomorphic image of a group ring Z_2G (respectively Z_4G, Z_8G) where G is an elementary 2-group.*

Proof. Let $G=R^*$ and let L be the subring of R generated by the identity. Then $L\cong Z_k$ where $k=\text{char}\, R$. If $\sigma: Z_k\to L$ is an isomorphism, the map $\alpha: Z_kG\to R$ defined by $\alpha(\sum l_g g)=\sum \sigma(l_g)g$ is clearly a homomorphism and is onto since $1+J(R)=G$.

COROLLARY 7. *A ring R is an elementary 2-ring of characteristic 2 if and only if R is a homomorphic image of Z_2G for some elementary 2-group G.*

Proof. Let $g_i \in G$, $1 \le i \le n$. Then $g_i^2 = 1$ for each i and so $(g_1 + \cdots + g_n)^2 = \sum_{i=1}^n g_i^2 = m$ where $m = 1$ or 0 accordingly as n is odd or even. Hence Z_2G is an elementary 2-ring and the result follows.

If R denotes Z_4G or Z_8G and $\varphi: R \to S$ is a ring surjection, then S need not be an elementary 2-ring. For example, Z_4G is not an elementary 2-ring if $G = C_2 \times C_2 \times C_2$. However, we can define an ideal $E(R) \subseteq R$ such that S is an elementary 2-ring if and only if $E(R) \subseteq \ker \varphi$.

DEFINITION 8. *If R is a ring, let $E(R)$ denote the two-sided ideal of R generated by $\{1 - u^2 : u \in R^*\}$.*

LEMMA 9. *Let R be local with $1 + 1 \in J(R)$ and let $A \subseteq R$ be a two-sided ideal. Then R/A is an elementary 2-ring if and only if $E(R) \subseteq A$.*

Proof. Suppose $E(R) \subseteq A$. Then, for every $u \in R^*$, $1 - u^2 \in A$. Let $u + A \in (R/A)^*$. Then $1 - uv \in A$ for some $v \in R$. Hence $u \notin J(R)$ so, since R is local, $u \in R^*$. This means that $1 - u^2 \in A$ and so $(R/A)^*$ is an elementary 2-group. Moreover $R/A \not\cong Z_3$ (since $1 + 1 \in J(R)$) and so R/A is an elementary 2-ring. The converse is obvious.

This result together with Propositions 3 and 6 yields the following result.

THEOREM 10. *Let R be a ring. R is an elementary 2-ring if and only if* char $R = k$ *where $k = 2$, 4 or 8 and R is a homomorphic image of $Z_kG/E(Z_kG)$ for some elementary 2-group G.*

If RG is a group ring, $\delta: RG \to R$ denotes the augmentation homomorphism. That is if $r = a_1g_1 + \cdots + a_ng_n$ then $\delta(r) = a_1 + \cdots + a_n$. We now characterize the ideal $E(R)$ in the cases of interest.

THEOREM 11. *Let $R = LG$ when L is one of Z_2, Z_4 or Z_8 and when G is an elementary 2-group. Let A denote the two-sided ideal of R generated by all elements $2(1 + g + h + gh)$ when 1, g and h are distinct elements of G. (If $|G| \le 2$ take $A = 0$). Then $E(R) = A + 4J(R)$.*

Proof. If $g, h \in G$ then $(g + h)^3 = 0$ and so $1 + g + h \in R^*$. Thus $2(1 + g + h + gh) = (1 + g \times h)^2 - 1 \in E(R)$ and so $A \subseteq E(R)$. Moreover $4J(R) \subseteq E(R)$ by Proposition 3 so $A + 4J(R) \subseteq E(R)$. To obtain the reverse inclusion let $u = \sum_{i=0}^n a_ig_i \in R^*$ where $a_i \in L$, $g_i \in G$. Then $\delta(u) = \sum_{i=0}^n a_i \in L^*$ and so $1 = (\sum_{i=0}^n a_i)^2 = \sum_{i<j} a_ia_j$. Since $g_i^2 = 1$ for each i this gives

$$u^2 - 1 = \left(\sum_{i=0}^n a_i^2\right) - 1 + 2\sum_{i<j} a_ia_jg_ig_j$$

$$= 2\sum_{i<j} a_ia_j(g_ig_j - 1).$$

Now, if either a_i or a_j is even then, since $4=-4$ in L we obtain

$$2a_ia_j(g_ig_j-1)=2a_ia_j(1+g_i+g_j+g_ig_j)+2a_ia_j(g_i+g_j)\in A+4J(R).$$

Hence, if we write $v=a_0g_0+a_1g_1+\cdots+a_kg_k$ where $a_0, a_1, \ldots, a_k$ are the odd coefficients, then $(u^2-1)-(v^2-1)\in A+4J(R)$. Now it follows by [1] that R is local and $J(R)=\{r\in R \mid \delta(r)\in J(L)\}$. Thus v is a unit and it suffices to show that $v^2-1\in A+4J(R)$. We have $v=(\pm g_0\pm\cdots\pm g_h)+3(\pm g_{h+1}\pm\cdots\pm g_k)$, $g_i\in G$ and so, since $(x+3y)^2=(x-y)^2$ in R, we may assume all the coefficients are ± 1. Because $(\pm gv)^2=v^2$ for all $g\in G$ we may assume $v=1+r_1+\cdots+r_k$ where $r_i\in G$ or $-r_i\in G$ for each i. If $|G|\leq 2$, then $v=1$ and so $v^2-1=0\in A+4J(R)$. But the $k+1-v=\sum_{i=1}^k(1-r_i)$ and, if this is squared the result simplifies to

$$(v^2-1)-k(2v-k)=\sum_{i<j}2(1-r_i)(1-r_j).$$

The right side is in $A+4J(R)$ since, if 1, g and h are distinct elements of G, we have

$$\begin{aligned}2(1-g)(1-h)&=2(1+g+h+gh)-4(g+h),\\ 2(1-g)(1+h)&=2(1+g+h+gh)-4g(1+h).\\ 2(1+g)(1+h)&=2(1+g+h+gh).\end{aligned}$$

Finally k is even (since v is a unit) and so, if $k=2m$, we have $k(2v-k)=4m(v-m)$. This is zero if m is even and is in $4J(R)$ if m is odd and it follows that $v^2-1\in A+4J(R)$, as required.

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