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PRIMARY DECOMPOSITION FOR Σ-GROUPS

BY
DON BRUNKER AND DENIS HIGGS*

ABSTRACT. A Σ-group is an abelian group on which is given a collection of infinite sums having properties suggested by those of absolutely convergent series in **R** or **C**. It is shown that the usual decomposition of a torsion abelian group into its p-components carries over to the case of Σ-groups when the property of being torsion is replaced by an appropriate uniform version. For a certain class of Σ-groups, it turns out that being torsion is already sufficient for primary decomposition to hold.

1. **Introduction.** A Σ-group is an abelian group on which is given a collection of infinite sums having properties abstracted from certain of the properties of absolutely convergent series in **R** and **C**. The notion is originally due to Wylie [5]. Later, in [2] and [3], a somewhat narrower notion was considered (the Σ-groups of [2] and [3] coincide with those Σ-groups in the present sense which, in terms of concepts defined later in this note, are complete and regular). The present definition was suggested to us by Fleischer (this was prior to our learning of Wylie's work). Happily, it turns out to be equivalent to Wylie's (the only difference being quite inessential: Wylie's formulation is in terms of the series which sum to 0).

After introducing the relevant notions in Section 2, we show in Section 3 that the usual decomposition of a torsion abelian group into its p-components carries over to the case of Σ-groups when the property of being torsion is replaced by a uniform version, which we call 'Σ-torsion'. For a certain class of Σ-groups (the regular adic Σ-groups) it turns out that torsion and Σ-torsion are equivalent.

2. **Σ-groups.** A *series* on an abelian group A is a family $\mathbf{x} = (x_i: i \in I)$ of elements of A. Let $\mathbf{x} = (x_i: i \in I)$ be a series on A. Then $-\mathbf{x}$ is the series $(-x_i: i \in I)$, a *subseries* of $\mathbf{x}$ is a series of the form $(x_i: i \in I')$ where $I' \subseteq I$, a *contraction* of $\mathbf{x}$ is a series of the form $(\sum_{i \in I_j} x_i: j \in J)$ where $(I_j: j \in J)$ is a partition of I into finite sets I_j, and an *addiction* of $\mathbf{x}$ is a series of the form

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$(x_{\theta(j)}: j \in J)$ where $\theta: J \to I$ is a function such that $\theta^{-1}(i)$ is finite for all i in I. If $\mathbf{y} = (y_j: j \in J)$ is also a series on A then $\mathbf{x} + \mathbf{y}$ is the series $(z_k: k \in K)$ where K is the disjoint union of I and J and $z_i = x_i$ for i in I, $z_j = y_j$ for j in J.

A *Σ-group* (A, S, Σ) is an abelian group A together with a class S of series on A and a function $\Sigma: S \to A$ such that:

(S) S contains every series on A consisting of a single term and S is closed under the operations $\mathbf{x} + \mathbf{y}$, $-\mathbf{x}$, contraction, and the insertion/deletion in a series of arbitrarily many 0's;

(Σ) If $\mathbf{x}$ consists of a single term a in A then $\Sigma(\mathbf{x}) = a$, if $\mathbf{x}$ and $\mathbf{y}$ are in S then $\Sigma(\mathbf{x} + \mathbf{y}) = \Sigma(\mathbf{x}) + \Sigma(\mathbf{y})$ and $\Sigma(-\mathbf{x}) = -\Sigma(\mathbf{x})$, and if $\mathbf{x}$ is in S and $\mathbf{z}$ is a contraction of S then $\Sigma(\mathbf{z}) = \Sigma(\mathbf{x})$.

Every Hausdorff topological abelian group A becomes a Σ-group when a series $\mathbf{x}$ on A is taken to be in S with $\Sigma(\mathbf{x}) = a$ if and only if the net of finite subsums of $\mathbf{x}$ converges to a. (This is unconditional summability; see Bourbaki [1, Ch. III, Section 5] for a survey.)

Let (A, S, Σ) be a Σ-group. For $\mathbf{x} = (x_i: i \in I)$ in S, it is convenient to write $\Sigma(\mathbf{x})$ as $\sum_i x_i$. If $(x_i: i \in I)$ and $(y_i: i \in I)$ are in S then $(x_i + y_i: i \in I)$ is in S with $\sum_i (x_i + y_i) = \sum_i x_i + \sum_i y_i$, and similarly for $(x_i - y_i: i \in I)$ and $(nx_i: i \in I)$ where n is any integer (in which case $\sum_i nx_i = n \sum_i x_i$). Also, any series on A which consists entirely of 0's is in S and has $\Sigma = 0$.

A Σ-group (A, S, Σ) is *discrete* if S consists precisely of the series on A with only finite many non-zero terms, (A, S, Σ) is *complete* if every subseries of a series in S is in S, and (A, S, Σ) is *adic* if every addiction of a series in S is in S. Finally, (A, S, Σ) is *regular* if, for every series $(x_i: i \in I)$ in S and partition $(I_j: j \in J)$ of I such that each of the series $(x_i: i \in I_j)$ is in S, it is the case that the series $(\sum_{i \in I_j} x_i: j \in J)$ is in S and $\sum_{j \in J} (\sum_{i \in I_j} x_i) = \sum_{i \in I} x_i$. It is easy to see that finite $\Rightarrow$ discrete $\Rightarrow$ adic $\Rightarrow$ complete, also that discrete $\Rightarrow$ regular, and simple examples show that none of these implications can be reversed. For any property P of abelian groups, say that a Σ-group (A, S, Σ) has P if A has P.

If (A, S, Σ) is a Σ-group and B is a subgroup of A then B becomes a Σ-group (B, T, Σ'), called a *Σ-subgroup* of (A, S, Σ), if we take T to consist of the series $\mathbf{x}$ on B such that $\mathbf{x}$ is in S and $\Sigma(\mathbf{x})$ is in B, putting $\Sigma'(\mathbf{x}) = \Sigma(\mathbf{x})$ for such $\mathbf{x}$. B is *Σ-closed* in A if, for every series $\mathbf{x}$ on B such that $\mathbf{x}$ is in S, $\Sigma(\mathbf{x})$ is in B.

A *morphism* from a Σ-group (A, S, Σ) to a Σ-group (B, T, Σ) is a function $f: A \to B$ such that, for each $(x_i: i \in I)$ in S, $(f(x_i): i \in I)$ is in T and $f(\sum_i x_i) = \sum_i f(x_i)$. The kernel of such a morphism f is evidently a Σ-closed subgroup of A.

The *product* $\prod_j (A_j, S_j, \Sigma)$ of a family of Σ-groups is the Σ-group (A, S, Σ) with $A = \prod_j A_j$, a series $(x_i: i \in I)$ on A being in S if and only if $(x_i(j): i \in I)$ is in S_j for each j, with $(\sum_i x_i)(j) = \sum_i x_i(j)$ for each j.

A Σ-group (A, S, Σ) is the *direct sum* of subgroups A_j, j in J, of A if

(1) $A = \oplus_j A_j$ as abelian groups,

(2) each A_j is a Σ-closed subgroup of A,

and (3) for each $(x_i: i \in I)$ in S, there exists a finite subset J_1 of J such that

(a) $\pi_j(x_i) = 0$ for all i in I and j in $J \backslash J_1$, and (b) $(\pi_j(x_i): i \in I)$ is in S for all j in J_1 (π_j is the projection of A onto A_j relative to the decomposition $A = \oplus_j A_j$).

This definition can be seen to give an internal description of (A, S, Σ) as the coproduct of its Σ-subgroups (A_j, S_j, Σ); see [2, 4.2.3]. It is easily verified that if each (A_j, S_j, Σ) is discrete then so is (A, S, Σ), and likewise for completeness, adicity, and regularity. Note that if infinitely many of the A_j's are non-zero then the 'direct sum' Σ-structure on $A = \oplus_j A_j$ is strictly smaller than the 'product' Σ-structure on $\oplus_j A_j$ induced by regarding $\oplus_j A_j$ as a Σ-subgroup of $\prod_j (A_j, S_j, \Sigma)$ via the embedding $\oplus_j A_j \to \prod_j A_j$: if $a_0, a_1, a_2, \ldots$ are non-zero elements of $A_{j_0}, A_{j_1}, A_{j_2}, \ldots$, where $j_0, j_1, j_2, \ldots$ are distinct, then the series $(a_0, a_1 - a_0, a_2 - a_1, \ldots)$, for example, is in the latter S but not in the former.

3. **Primary decomposition.** Let (A, S, Σ) be a torsion Σ-group. If (A, S, Σ) is the direct sum of the p-components A_p of A then we say that primary decomposition holds for (A, S, Σ). It does not hold for all torsion Σ-groups, for take $\mathbf{Q}/\mathbf{Z}$ with the Σ-structure of unconditional summability with respect to the quotient topology on $\mathbf{Q}/\mathbf{Z}$ ($\mathbf{Q}$ with the usual topology). Then the p-components of $\mathbf{Q}/\mathbf{Z}$ are not even Σ-closed since $1/2 - 1/4 + 1/8 \ldots = 1/3$ for instance. An example of the failure of primary decomposition in which the p-components are Σ-closed is obtained by taking $\oplus_p Z(p)$ with the product Σ-structure as defined in Section 2 ($Z(n)$ denotes the cyclic group of order n); in this case primary decomposition fails since, although each $(Z(p), S_p, \Sigma)$ is necessarily a discrete Σ-group, the whole Σ-group isn't.

A condition which does ensure that primary decomposition holds is the following. Say that a Σ-group (A, S, Σ) is Σ-*torsion* if for every series $(x_i: i \in I)$ in S there exists a positive integer n such that $nx_i = 0$ for all i in I. Clearly every Σ-torsion Σ-group is torsion and every discrete torsion Σ-group is Σ-torsion, likewise every Σ-group of bounded exponent. It is also clear that the property of being Σ-torsion is inherited by Σ-subgroups and direct sums.

THEOREM 1. *Primary decomposition holds for all Σ-torsion Σ-groups.*

PROOF. Let (A, S, Σ) be a Σ-torsion Σ-group. Then $A = \oplus_p A_p$. To see that each A_p is Σ-closed in A, let $(x_i: i \in I)$ be a series in S with each term x_i in A_p. Since (A, S, Σ) is Σ-torsion, there exists a positive integer n such that $nx_i = 0$ for all i in I. The smallest such n will be of the form p^k for some k and so $p^k \sum_i x_i = \sum_i p^k x_i = 0$, whence $\sum_i x_i$ is in A_p. To verify condition (3) in the definition of direct sum, let $(x_i: i \in I)$ be in S and let n be such that $nx_i = 0$ for all i in I, where $n = p_1^{m_1} \ldots p_k^{m_k}$ say. Then certainly $\pi_p(x_i) = 0$ for all i in I and all $p \notin \{p_1, \ldots, p_k\}$. Put $q_j = n/p_j^{m_j}$ for $j = 1, \ldots, k$ and let $r_1, \ldots, r_k$ be integers such that $q_1 r_1 + \ldots + q_k r_k = 1$. Then for any x in A for which

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$nx = 0$ we have $\pi_{p_j}(x) = q_j r_j x$, $j = 1, \ldots, k$. Hence from $(x_i: i \in I)$ in S it follows that $(\pi_{p_j}(x_i): i \in I)$ is in S for all $j = 1, \ldots, k$. Thus condition (3) holds and Theorem 1 is proved.

It is not true that if primary decomposition holds for a Σ-group then that Σ-group is necessarily Σ-torsion. For example, consider $\oplus_n Z(2^n)$ with the product Σ-structure; as we have seen in Section 2, this Σ-group has some genuinely infinite sums and it is easy to find such a sum in which the terms are of unbounded order.

We wish to show that Σ-torsion holds for all regular adic torsion Σ-groups, and need:

LEMMA. *In a metric group A with the Σ-structure of unconditional summability, a subgroup B is topologically closed if and only if it is Σ-closed.*

PROOF. B topologically closed clearly implies B Σ-closed without the assumption of metrizability. For the converse, let B be Σ-closed, let x be a point in the closure of B, and let $(x_n: n \in \omega)$ be a sequence of points in B such that $x_n \to x$. We may assume without loss of generality that $d(x, x_n) < 1/2^n$ for all n, from which it follows that if we put $y_0 = x_0$, $y_{n+1} = x_{n+1} - x_n$ then $\sum_n y_n$ is unconditionally summable to x (we are assuming here that A carries an invariant metric, which we may by [1, Ch. IX, Section 3, Proposition 2]). Since each y_n is in B, x is in B.

THEOREM 2. *Every regular adic torsion Σ-group is Σ-torsion.*

PROOF. Let (A, S, Σ) be an adic torsion Σ-group. Since (A, S, Σ) is complete, it is sufficient to show that if $(x_i: i \in I)$ is in S where I is countable then the x_i's are of bounded order. For each i in I, let n_i be the order of x_i, let $\Pi = \prod_i Z(n_i)$ have the product Σ-structure, and define $f: \Pi \to A$ by $f(t) = \sum_i t(i)x_i$. The fact that (A, S, Σ) is adic ensures that f is everywhere-defined. To see that f is a Σ-morphism, let $\sum_j t_j$ exist in Π. Then $\{j: t_j(i) \neq 0\}$ is finite for each i in I and hence, by the adicity of (A, S, Σ), the existence of $\sum_i x_i$ implies that of $\sum_{i,j} t_j(i)x_i$. Now on the one hand, as would hold in any Σ-group, $\sum_{i,j} t_j(i)x_i = \sum_i (\sum_j t_j(i)x_i)$, and on the other, by the regularity of (A, S, Σ), $\sum_{i,j} t_j(i)x_i = \sum_j (\sum_i t_j(i)x_i)$. Thus

$$f\Big(\sum_j t_j\Big) = \sum_i \Big(\sum_j t_j(i)x_i\Big) = \sum_j \Big(\sum_i t_j(i)x_i\Big) = \sum_j f(t_j)$$

as required. Since f is a Σ-morphism, its kernel, K say, is a Σ-closed subgroup of Π.

Now Π, as a countable product of finite groups with the discrete topology, is a compact metric group; also the Σ-structure of unconditional summability on Π coincides with the product Σ-structure on Π [1, Ch. III, Section 5, No. 4]. By the above lemma, K is a topologically closed subgroup of Π. Hence Π/K is a compact group, which is also torsion since it is isomorphic to a subgroup of A. It follows that Π/K is of bounded exponent (see Morris [4, Theorem 18] for example). This implies that the x_i's are of bounded order, as required.

From Theorems 1 and 2, we have immediately:

THEOREM 3. *Primary decomposition holds for all regular adic torsion Σ-groups.*

We have been unable to determine whether Theorems 2 and 3 continue to hold when adicity is replaced by completeness.

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BUREAU OF INDUSTRY ECONOMICS
CANBERRA, A.C.T. 2600, AUSTRALIA

PURE MATHEMATICS DEPARTMENT, UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA N2L 3G1