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Functions of bounded k^{th} variation and Stieltjes type integrals

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This thesis is mainly concerned with two concepts, bounded k^{th} variation, and generalised Riemann-Stieltjes integration. The definition of bounded k^{th} variation is a natural generalisation of bounded variation as defined in the classical sense. More specifically, let us denote by $\pi(x_0, x_1, \ldots, x_n)$ a subdivision of the closed interval $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. Then the total k^{th} variation of a bounded function f on $[a, b]$ is defined by

$$V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} |Q_{k-1}(f; x_{i+1}, \ldots, x_{i+k})-Q_{k-1}(f; x_i, \ldots, x_{i+k-1})| ,$$

where the $(k-1)^{\text{th}}$ divided difference $Q_{k-1}(f; x_s, \ldots, x_{s+k-1})$ is defined as

$$\sum_{i=s}^{s+k-1} \left| f(x_i) \Big/ \prod_{\substack{j=s \\ j\neq i}}^{s+k-1} (x_i-x_j) \right| .$$

If $V_k(f; a, b) < \infty$, we will say that f is of bounded k^{th} variation on $[a, b]$, and write $f \in BV_k[a, b]$.

Just as the concept of bounded variation plays a fundamental role in the classical theory of Riemann-Stieltjes integration, so the concept of bounded k^{th} variation will play a fundamental role in the theory of

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generalised Riemann-Stieltjes integration.

Several properties of bounded k^{th} variation are studied, perhaps the most interesting and useful being that a function of bounded k^{th} variation can be expressed as a difference of two functions, each of which is a $0, 1, 2, \ldots, k$-convex function - see [2, Definition 1 (b)]. It is also shown that the sequence $\{BV_k[a, b]\}$ is contracting, and that

$$\bigcap_{k=1}^{\infty} BV_k[a, b] = C^{(\infty)}[a, b] ,$$ the class of infinitely differentiable functions on $[a, b]$.

The k^{th} order Riemann-Stieltjes integral $\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}$ is defined, the usual linearity properties are obtained, and existence theorems for the integral are presented. For example, we prove that the integral above exists when f is continuous and g is of bounded k^{th} variation, and that a modified form of the integral exists when f is merely quasi-continuous. Various modifications of the integral are considered, one of these being of particular interest as it exhibits properties of the well known Dirac delta function. A comparison of our integral with one defined by Burkill [1] is also included.

The thesis concludes with a discussion of the representation of bounded linear functionals as k^{th} order Riemann-Stieltjes integrals.

References

[1] J.C. Burkill, "An integral for distributions", *Proc. Cambridge Philos. Soc.* 53 (1957), 821-824.

[2] A.M. Russell, "Functions of bounded *k*th variation", *Proc. London Math. Soc.* (3) 26 (1973), 547-563.

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