Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 678–685

ON A FAMILY OF GENERALIZED NUMERICAL RANGES

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1. Introduction and basic properties of operator radii. Throughout this note, an operator will always mean a bounded linear operator acting on a Hilbert space X into itself, unless otherwise stated. The class C_ρ $(0 < \rho < \infty)$ of operators, considered by Sz.-Nagy and Foiaş [**5**], is defined as follows: An operator T is in C_ρ if $T^n x = \rho P U^n x$ for all $x \in X$, $n = 1, 2, \ldots$, where U is a unitary operator on some Hilbert space Y containing X as a subspace, and P is the orthogonal projection of Y onto X. In [**2**] Holbrook defined the operator radii $w_\rho(\cdot)$ $(0 < \rho \leqq \infty)$ as the generalized Minkowski distance functionals on the Banach algebra of bounded linear operators on X, i.e.,

$$w_\rho(T) = \inf\{u : u > 0 \text{ and } u^{-1}T \in C_\rho\}, \qquad 0 < \rho < \infty,$$

and $w_\infty(T) = r(T)$, the spectral radius of T [**2**, Theorem 5.1].

The numerical range of an operator is a useful tool in operator theory, mainly because it is convex and its closure contains the spectrum. In the development, first comes the numerical range, then the numerical radius. With this in mind, it seems natural to ask whether there exists a family of generalized numerical ranges corresponding to the operator radii of Holbrook in such a way that naturally defined numerical radii coincide with his radii. We shall answer this question partially in this note.

A generalized numerical range of an operator T is defined as

$$W_\rho(T) = \bigcap_v \{u\colon |u - v| \leqslant w_\rho(T - vI), u \text{ and } v \in \mathbf{C}\}, \quad 1 \leqslant \rho \leqslant \infty,$$

where $\mathbf{C}$ denotes the complex plane, and I the identity operator. For a fixed operator T, the family $\{W_\rho(T) : 1 \leqq \rho \leqq \infty\}$ turns out to be a complete chain whose minimal element is $W_\infty(T) = \Sigma(T)$, the convex hull of the spectrum of T, and whose maximal element $W_\rho(T)$ $(1 \leqq \rho \leqq 2)$ is the closure of the usual numerical range of T. We discuss consequences of imposing growth conditions upon an operator radius of the resolvent of T, rather than upon its norm as usual. From this we obtain some new characterizations of a self-adjoint operator, and some new expressions of $r(T)$ are given. We also investigate the consequences of requiring $w_\rho(T) = r(T)$ or $W_\rho(T) = \Sigma(T)$, which yields a study of generalized normaloid, spectraloid and convexoid operators.

Received January 2, 1973 and in revised forms, April 11, 1973 and May 1, 1973.

Let

$$W(T) = \{(Tx, x)/\|x\|^2 : 0 \neq x \in X\}^-$$
$$(=\{(Tx, x) : x \in X \text{ and } \|x\| = 1\}^-)$$

denote the closure of the usual numerical range and

$$w(T) = \sup\{|u| : u \in W(T)\}$$

the usual numerical radius of T. For later reference we cite the following primary results from [**2**; **3**]:

(a) If $0 < \rho < \infty$, then $w_\rho(T) > 0$ unless $T = 0$, and $w_\rho(T) \leqq 1$ if and only if $T \in C_\rho$;

(b) $w_1(T) = \|T\|$, $w_2(T) = w(T)$ and $\lim_{\rho\to\infty} w_\rho(T) = w_\infty(T) = r(T)$;

(c) if $0 < \rho \leqq \infty$, then $w_\rho(cT) = |c|w_\rho(T)$, $w_\rho(T) < \infty$, $w_\rho(T) \geqq w_\rho(I)r(T)$, $w_\rho(T) \leqq w_\rho(I)\|T\|$ and $w_\rho(T^n) \leqq w_\rho(T)^n$, $n = 1, 2, \ldots,$ where $c \in \mathbf{C}$, $w_\rho(I) = 1$ if $\rho \geqq 1$, and $(2 - \rho)/\rho$ if $0 < \rho \leqq 1$;

(d) if $0 < \rho < \alpha \leqq \infty$, then $w_\alpha(T) \leqq w_\rho(T)$;

(e) if $0 < \beta < \rho \leqq \infty$ and $w_\rho(T) = w_\beta(T)$, then $w_\alpha(T) = w_\beta(T)$ whenever $\beta \leqq \alpha \leqq \infty$.

Acknowledgement. The author wishes to express his gratitude to Professor Gustavus E. Huige for his advice and valuable suggestions, and to the referee for his comments and suggesting the references to the papers by C. Davis, *The shell of a Hilbert-space operator*, Acta Sci. Math. *29* (1968), 69–86; *31* (1970), 301–318, and also that by T. Furuta and R. Nakamoto, *On the numerical ranges of an operator*, Proc. Japan Acad. *47* (1971), 279–284. None of these references was known to the author when this paper was written.

2. A family of generalized numerical ranges. A generalized numerical range of an operator T, in symbols $W_\rho(T)$, is defined by

$$W_\rho(T) = \bigcap_v \{u : |u - v| \leqslant w_\rho(T - vI), u \text{ and } v \in \mathbf{C}\}, \quad 1 \leqslant \rho \leqslant \infty.$$

THEOREM 1. *$W_\rho(\cdot)$ has the following properties:*

(1) *$W_\rho(T)$ is a compact convex subset of $\mathbf{C}$;*

(2) *$W_\rho(T)$ is nonempty; in fact, $W_\rho(T) \supseteqq \Sigma(T)$;*

(3) *$W_\rho(cT + bI) = cW_\rho(T) + b$, b and $c \in \mathbf{C}$.*

Proof. (1) The closedness of $W_\rho(T)$ is easily verified. It is also bounded since $|u| \leqq w_\rho(T) < \infty$ for every $u \in W_\rho(T)$. To show the convexity, let u_1 and $u_2 \in W_\rho(T)$. For any number t with $0 \leqq t \leqq 1$ and any $v \in \mathbf{C}$, we have

$$|tu_1 + (1 - t)u_2 - v| \leqq t|u_1 - v| + (1 - t)|u_2 - v| \leqq tw_\rho(T - vI) + (1 - t)w_\rho(T - vI) = w_\rho(T - vI).$$

Hence $W_\rho(T)$ is convex.

(2) Let $u \in \Sigma(T)$. Then $u - v \in \Sigma(T - vI)$ and hence $|u - v| \leqq r(T - vI) \leqq w_\rho(T - vI)$ for every $v \in \mathbf{C}$, i.e., $u \in W_\rho(T)$.

(3) The case when $c = 0$ is trivial, i.e., $W_\rho(bI) = \bigcap_v \{u : |u - v| \leqq |b - v|,$ u and $v \in \mathbf{C}\} = \{b\}$. We shall suppose therefore that $c \neq 0$. So,

$$u \in W_\rho(cT + bI) \Leftrightarrow |u - v| \leqq w_\rho(cT + (b - v)I)$$

for all $v \in \mathbf{C} \Leftrightarrow |(u - b)/c + (b - v)/c| \leqq w_\rho(T + (b - v)I/c)$ for all

$$v \in \mathbf{C} \Leftrightarrow (u - b)/c \in W_\rho(T) \Leftrightarrow u \in cW_\rho(T) + b.$$

For an operator T we shall define $w_\rho^0(T)$ as

$$w_\rho^0(T) = \sup\{|u| : u \in W_\rho(T)\}, \qquad 1 \leqq \rho \leqq \infty.$$

In view of Theorem 1, the following results are easily obtained:

$$r(T) \leqq w_\rho^0(T) \leqq w_\rho(T), w_\infty^0(T) = r(T) \quad \text{and} \quad w_\rho^0(cT) = |c| w_\rho^0(T), \; c \in \mathbf{C}.$$

Indeed, it is readily verified that

$$W_\rho(T) = \bigcap_v \{u : |u - v| \leqslant w_\rho^0(T - vI), u \text{ and } v \in \mathbf{C}\}, \quad 1 \leqslant \rho \leqslant \infty.$$

We will show later that $w_2(T) = w_\rho^0(T)$, $1 \leqq \rho \leqq 2$.

LEMMA 1.

$$\lim_{t\to\infty} w_\rho^0(T + tI) - t = \sup \operatorname{Re} W_\rho(T), \quad 1 \leqslant \rho \leqslant \infty.$$

In particular, $\lim_{t\to\infty} w(T + tI) - t = \sup \operatorname{Re} W(T)$ *and*

$$\lim_{t\to\infty} r(T + tI) - t = \sup \operatorname{Re} \Sigma(T) = \sup \operatorname{Re} W_\infty(T).$$

Proof. Since $W_\rho(T + tI) = W_\rho(T) + t$, $w_\rho^0(T + tI) \geqq \operatorname{Re}(u + t) = \operatorname{Re} u + t$ for $u \in W_\rho(T)$. Hence $w_\rho^0(T + tI) - t \geqq \sup \operatorname{Re} W_\rho(T)$. On the other hand, if $a + bi = u \in W_\rho(T)$, where a and b are real, then $|u + t| - t - \operatorname{Re} u = |(a + t) + bi| - (a + t)$. So, given $\epsilon > 0$, we have $|u + t| - t - \operatorname{Re} u \leqq \epsilon$ for large $t > 0$. Therefore, $\lim_{t\to\infty} w_\rho^0(T + tI) - t \leqq \sup \operatorname{Re} W_\rho(T) + \epsilon$. Since ϵ was arbitrary, we have $\lim_{t\to\infty} w_\rho^0(T + tI) - t \leqq \sup \operatorname{Re} W_\rho(T)$ and hence the equality holds. Now, with the aid of previous remarks, the particular cases are clear.

THEOREM 2. $W_2(T) = W(T)$ *and* $W_\infty(T) = \Sigma(T)$.

Proof. If $u \in W(T)$, then $|u - v| \leqq w(T - vI) = w_2(T - vI)$ for every $v \in \mathbf{C}$ and hence $u \in W_2(T)$. If $u \notin W(T)$, because of the convexity of $W(T)$, we may assume without loss of generality that $W(T)$ lies in the left half-plane $\operatorname{Re} z \leqq 0$, and that $u > 0$. For large $t > 0$, $w(T + tI) - t < u$ by Lemma 1, i.e., $w_2(T + tI) - t < u$ and hence $u \notin W_2(T)$. The second result follows similarly.

It was proved in [**7**, Theorem 4] that $W(T) = W_1(T)$ by using a remarkable result, $\lim_{t\to\infty} \|T + tI\| - t = \sup \operatorname{Re} W(T)$ which is due to Lumer [**7**, Lemma 2]. Therefore, we conclude that for $1 \leqq \rho \leqq 2$, $W(T) = W_\rho(T)$ and $w(T) = w_\rho^0(T)$. Also, for a fixed operator T, the family $\{W_\rho(T) : 1 \leqq \rho \leqq \infty\}$ is a complete chain with the minimal and the maximal elements $W_\infty(T) = \Sigma(T)$ and $W_\rho(T) = W(T)$, $1 \leqq \rho \leqq 2$, respectively.

Because of above remarks, from now on we will consider $W_\rho(T)$ (and $w_\rho^0(T)$) only for $2 \leqq \rho \leqq \infty$. We shall give another expression of $W_\rho(T)$ as follows. Let $X_\rho(T) = \bigcup \{E_u(T) : u \in W_\rho(T)\}$, where $E_u(T) = \{x : (Tx, x) = u\|x\|^2, x \in X\}$. Clearly, $E_u(T) \cap E_v(T) = \{0\}$ if $u \neq v$, and $X_\rho(T) \subseteq X_2(T) = X$. Also, if $x \in X_\rho(T)$, then $cx \in X_\rho(T)$ for $c \in \mathbf{C}$. It follows easily that

$$\begin{aligned} W_\rho(T) &= \{(Tx, x)/\|x\|^2 : 0 \neq x \in X_\rho(T)\}^- \\ &= \{(Tx, x) : x \in X_\rho(T) \text{ and } \|x\| = 1\}^-. \end{aligned}$$

In view of above remarks and the fact that $T = 0$ if and only if $(Tx, x) = 0$ for all $x \in X$, we note that if $w_\rho^0(T) = 0$, then $T = 0$ if and only if $(Tx, x) = 0$ for all $x \in X\backslash X_\rho(T)$.

In the next theorem, we observe that with $\|\cdot\|$ and $W(\cdot)$ in place of $w_\rho(\cdot)$ and $W_\rho(\cdot)$, respectively, (1), (2) and (3) are originally due to Lumer, Rota and Hildebrandt [**1**, p. 22], respectively. We shall omit proofs since they can be easily done by slight modifications of original results, previous remarks and the relation $r(T^n) = r(T)^n$, $n = 1, 2, \ldots,$ which is a well-known consequence of the spectral mapping theorem. Let $R(X)$ denote the set of invertible operators on X.

THEOREM 3. (1) $\lim_{t\to\infty} w_\rho(T + tI) - t = \sup \operatorname{Re} W(T)$, $1 \leqq \rho \leqq 2$;
(2) $r(T) = \inf_S\{w_\rho(S^{-1}TS) : S \in R(X)\}$, $1 \leqq \rho \leqq \infty$;
(3) $\Sigma(T) = \bigcap_S\{W_\rho(S^{-1}TS) : S \in R(X)\}$, $2 \leqq \rho \leqq \infty$;
(4) $r(T) = \inf_S\{w_\rho^0(S^{-1}TS) : S \in R(X)\}$, $2 \leqq \rho \leqq \infty$;
(5) $r(T) = \lim_{n\to\infty} w_\rho^0(T^n)^{1/n}$, $2 \leqq \rho \leqq \infty$;
(6) $r(T) = \lim_{n\to\infty} w_\rho(T^n)^{1/n}$, $1 \leqq \rho \leqq \infty$.

3. Growth conditions. Let us now investigate growth conditions on the resolvent of an operator T in terms of $w_\rho^0(\cdot)$ and $w_\rho(\cdot)$, rather than the usual case of norm. The theorem below is an extension, in the sense of operators in Hilbert space, of Theorem 2 [**7**], where the numerical range in an arbitrary Banach algebra with unit is studied. Incidentally, an operator T is invertible if there exists a scalar $v \neq 0$ such that $w_\rho^0(T - vI) < |v|$.

In what follows, let $d(v, N)$ denote the distance between a point v and a subset N in the plane $\mathbf{C}$. Also, let $|T|$ denote the minimum modulus of T, i.e., $|T| = \inf\{\|Tx\| : x \in X \text{ and } \|x\| = 1\}$. Clearly we have $d(0, W(T)) \leqq |T|$. It is well-known and easy to prove that $|T|\,\|T^{-1}\| = 1$ if T is invertible.

THEOREM 4. *Let D be a closed convex subset of* $\mathbf{C}$, *and ρ and α be fixed numbers in the described ranges.*

https://doi.org/10.4153/CJM-1974-064-9 Published online by Cambridge University Press

(1) *For* $2 \leqq \rho \leqq \infty$, *if* $w_\rho{}^0((T - vI)^{-1}) \leqq d(v, D)^{-1}$ *for all* $v \notin D$, *then* $D \supseteq W_\rho(T)$.

(2) *The following statements are equivalent:*

(i) $D \supseteq W(T)$;

(ii) $w_\alpha((T - vI)^{-1}) \leqq d(v, D)^{-1}$ *for all* $v \notin D$, $1 \leqq \alpha \leqq 2$;

(iii) $T - vI$ *is invertible and* $|T - vI| \geqq d(v, D)$ *for all* $v \notin D$.

Proof. (1) We need only show that every half-plane H which contains D also contains $W_\rho(T)$. By a preliminary translation and rotation we may choose H as the right half-plane $\operatorname{Re} z \geqq 0$. Since $D \subseteq H$, and $-v^{-1} \notin D$ if $v > 0$, we have $w_\rho{}^0((T + v^{-1}I)^{-1}) \leqq d(-v^{-1}, D)^{-1}$ by assumption. It follows by the homogeneity of $w_\rho{}^0(\cdot)$ that

$$w_\rho{}^0((vT + I)^{-1}) = v^{-1}w_\rho{}^0((T + v^{-1}I)^{-1}) \leqq v^{-1}d(-v^{-1}, D)^{-1} \leqq 1.$$

This shows that $\operatorname{Re} u \leqq 1$ for $u \in W_\rho((vT + I)^{-1})$. But we have

$$1 - W_\rho((vT + I)^{-1}) = W_\rho(I - (vT + I)^{-1}) = vW_\rho((vT + I)^{-1}T).$$

So, $0 \leqq \operatorname{Re} u$ for $u \in W_\rho((vT + I)^{-1}T)$. By letting $v \to 0$, we see that $0 \leqq \operatorname{Re} u$ for $u \in W_\rho(T)$. Therefore, $H \supseteq W_\rho(T)$.

(2) (i) $\Rightarrow$ (ii). The condition $D \supseteq W(T)$ implies that $(T - vI)^{-1}$ exists and $||(T - vI)^{-1}|| = |T - vI|^{-1}$ for all $v \notin D$ by above remark. Hence $w_\alpha((T - vI)^{-1}) \leqq d(v, D)^{-1}$ for all $v \notin D$.

(ii) $\Rightarrow$ (i). We need only show that $D \supseteq W(T)$ if $w((T - vI)^{-1}) \leqq d(v, D)^{-1}$ for all $v \notin D$. But this is a special case of (1).

(i) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (ii). $w_\alpha((T - vI)^{-1}) \leqq ||(T - vI)^{-1}|| = |T - vI|^{-1} \leqq d(v, D)^{-1}$ for all $v \notin D$.

THEOREM 5. *Let* α *and* ρ *be fixed numbers in the described ranges. Then the following statements are equivalent:*

(1) T *is self-adjoint;*

(2) $w_\alpha((T + viI)^{-1}) \leqq |v^{-1}|$ *for all real* $v \neq 0$, $1 \leqq \alpha \leqq 2$;

(3) $T + viI$ *is invertible and* $|T + viI| \geqq |v|$ *for all real* $v \neq 0$. *Moreover, if this is the case, then*

$$r(T) = \lim_{t\to\infty} w_\rho(T + tI) - t, \quad 1 \leqslant \rho \leqslant 2.$$

Proof. We need first the following two consequences of Theorem 4. Let D be the upper half-plane. Then it is readily verified that

$$\operatorname{Im} W(T) \geqq 0 \Leftrightarrow w_\alpha((T + viI)^{-1}) \leqq v^{-1}$$

for all $v > 0 \Leftrightarrow T + viI$ is invertible and $|T + viI| \geqq v$ for all $v > 0$. Similarly, if D is the lower half-plane, then $\operatorname{Im} W(T) \leqq 0 \Leftrightarrow w_\alpha((T - viI)^{-1}) \leqq v^{-1}$ for all $v > 0 \Leftrightarrow T - viI$ is invertible and $|T - viI| \geqq v$ for all $v > 0$. Now, T is self-adjoint $\Leftrightarrow \operatorname{Im}(Tx, x) = 0$ for all $x \in X \Leftrightarrow \operatorname{Im} W(T) = 0 \Leftrightarrow$,

$w_\alpha((T + viI)^{-1}) \leqq |v^{-1}|$ for all real $v \neq 0 \Leftrightarrow T + viI$ is invertible and $|T + viI| \geqq |v|$ for all real $v \neq 0$. The last assertion follows from Theorem 3 and the fact that $r(T) = w(T)$ if T is self-adjoint.

Note that Theorem 5 is an improvement of Nieminen's result [**6**, Theorem 1] which says that if the spectrum $\sigma(T)$ of T is real and $||(T - viI)^{-1}|| \leqq |v^{-1}|$ for all real $v \neq 0$, then T is self-adjoint. Incidentally, we can also show, in much the same way as above, that the following are valid:

(1) $\operatorname{Re} W(T) \geqq 0 \Leftrightarrow w_\alpha((T + uI)^{-1}) \leqq u^{-1}$ for all $u > 0$, $1 \leqq \alpha \leqq 2 \Leftrightarrow T + uI$ is invertible and $|T + uI| \geqq u$ for all $u > 0$;

(2) $\operatorname{Re} W(T) \leqq 0 \Leftrightarrow w_\alpha((T - uI)^{-1}) \leqq u^{-1}$ for all $u > 0$, $1 \leqq \alpha \leqq 2 \Leftrightarrow T - uI$ is invertible and $|T - uI| \geqq u$ for all $u > 0$ (this case should be compared with [**6**, Lemma 2]); and

(3) $w(T) \leqq 1 \Leftrightarrow w_\alpha((T - cI)^{-1}) \leqq (|c| - 1)^{-1}$ for all $c \in \mathbf{C}$ with $|c| > 1$, $1 \leqq \alpha \leqq 2 \Leftrightarrow T - cI$ is invertible and $|T - cI| \geqq |c| - 1$ for all $c \in \mathbf{C}$ with $|c| > 1$.

The following is an analogous result to Theorem 4 which is useful likewise.

THEOREM 6. *Let D be a compact subset of $\mathbf{C}$ such that $W_\rho(T) \subseteq D$. Then there exists an operator S such that $w_\rho{}^0(T - vI) \leqq r(S - vI)$ for all $v \in \mathbf{C}$. If operators T and S are such that $w_\rho{}^0(T - vI) \leqq r(S - vI)$ for all $v \in \mathbf{C}$, then $W_\rho(T) \subseteq \Sigma(S)$.*

Proof. We observe that if D is a nonempty compact subset of $\mathbf{C}$, then there exists some (diagonal) operator S on a separable Hilbert space such that $\sigma(S) = D$ [**4**, Problem 48]. D is nonempty by assumption, thus $W_\rho(T - vI) = W_\rho(T) - v \subseteq \sigma(S) - v = \sigma(S - vI)$ for all $v \in \mathbf{C}$. Hence $w_\rho{}^0(T - vI) \leqq r(S - vI)$ for all $v \in \mathbf{C}$. On the other hand, if the condition is satisfied, then

$$W_\rho(T) \subseteq \bigcap_v \{u: |u - v| \leqslant r(S - vI), u \text{ and } v \in \mathbf{C}\} = \Sigma(S)$$

and hence the proof is complete.

COROLLARY 1. *Let α be a fixed number in the described range, and $2 \leqq \rho \leqq \infty$. Then the following statements are equivalents:*

(1) *$W(T) = W_\rho(T)$;*
(2) *$w(T - vI) = w_\rho{}^0(T - vI)$ for all $v \in \mathbf{C}$;*
(3) *$w_\alpha((T - vI)^{-1}) \leqq d(v, W_\rho(T))^{-1}$ for all $v \notin W_\rho(T)$, $1 \leqq \alpha \leqq 2$;*
(4) *$|T - vI| \geqq d(v, W_\rho(T))$ for all $v \notin W_\rho(T)$.*

The proof can be done by using the same techniques as in the proofs of Theorem 4 and 6. We note that a normal operator T satisfies any one of above four equivalent conditions.

4. Generalized normaloid, spectraloid and convexoid operators.

Recall that an operator T is normaloid if $||T|| = r(T)$, spectraloid if $w(T) = r(T)$, and convexoid if $W(T) = \Sigma(T)$ [**4**, p. 114]. In view of $w_\rho(T)$ and $W_\rho(T)$, we may naturally generalize these operators as follows: An operator T is ρ-normaloid if $w_\rho(T) = r(T)$, $1 \leqq \rho < \infty$, and ρ-convexoid if $W_\rho(T) = \Sigma(T)$, $2 \leqq \rho < \infty$.

THEOREM 7. *Let α, β and γ be fixed numbers in the described ranges, and $1 \leqq \rho < \infty$. Then the following statements are equivalent:*

(1) *T is ρ-normaloid;*

(2) $w_\alpha(T) = w_\rho(T)$, $\rho < \alpha \leqq \infty$;

(3) $w_\beta(T^n) = w_\rho(T)^n$, $\rho \leqq \beta < \infty$ *and every integer* $n \geqq 1$;

(4) $w_\gamma(T^n) = w_\rho(T)^n$, $\rho < \gamma \leqq \infty$ *and every integer* $n \geqq 1$.

Proof. (1) $\Rightarrow$ (2) Because $w_\alpha(T) \leqq w_\rho(T) = r(T) \leqq w_\alpha(T)$. That (2) $\Rightarrow$ (1) follows immediately from (e). (1) $\Rightarrow$ (3) For $\rho \leqq \beta < \infty$ and every integer $n \geqq 1$, we have $w_\rho(T)^n = r(T)^n \leqq w_\beta(T^n) \leqq w_\beta(T)^n \leqq w_\rho(T)^n$. (3) $\Rightarrow$ (1) Since $w_\rho(T)^n = w_\beta(T^n) \leqq ||T^n||$ for every integer $n \geqq 1$,

$$w_\rho(T) \leqq \lim_{n\to\infty} ||T^n||^{1/n} = r(T).$$

The opposite inequality always holds. That (1) $\Leftrightarrow$ (4) is now clear.

In view of the property (3) in Theorem 7, it readily follows that if T is ρ-normaloid, so is T^k for any integer $k \geqq 1$. We note that if $\rho = \infty$, Corollary 1 gives characterizations of a convexoid operator. This yields an improvement of [**6**, Theorem 2 and 3]. It is clear that a convexoid operator is ρ-convexoid, however, using Theorem 4 and 6 we can say more.

THEOREM 8. (1) *T is ρ-convexoid if and only if $w_\rho{}^0(T - vI) = r(T - vI)$ for all $v \in \mathbf{C}$.*

(2) *If $w_\rho{}^0((T - vI)^{-1}) \leqq d(v, \Sigma(T))^{-1}$ for all $v \notin \Sigma(T)$, then T is ρ-convexoid.*

(3) *If $T - vI$ is ρ-normaloid for all $v \in \mathbf{C}$, $2 \leqq \rho < \infty$, then T is ρ-convexoid.*

It is well-known and easily verified that $d(v, \sigma(T))^{-1} = r((T - vI)^{-1})$ for all $v \notin \sigma(T)$. The next result indicates the relation among a convexoid, normaloid and spectraloid operators.

COROLLARY 2. *If T has a convex spectrum, i.e., $\sigma(T) = \Sigma(T)$, then the following statements are equivalent:*

(1) *T is convexoid;*

(2) *$(T - vI)^{-1}$ is ρ-normaloid for all $v \notin \sigma(T)$, $1 \leqq \rho \leqq 2$;*

(3) *$(T - vI)^{-1}$ is normaloid for all $v \notin \sigma(T)$;*

(4) *$(T - vI)^{-1}$ is spectraloid for all $v \notin \sigma(T)$.*

5. Remarks. In previous sections we have merely considered cases when $\rho \geqq 1$. Here, we shall explain the treatment of $\rho < 1$. It can be shown that $w(T) \leqq w_\rho(I)^{-1}w_\rho(T)$, $0 < \rho < 1$, where $w_\rho(I) = (2 - \rho)/\rho$ by [**2**, Theorem

4.3]. Hence $w(T) \leqq w_\rho(I)^{-1}w_\rho(T) \leqq \|T\|$ by (c). It follows easily that

$$W(T) = \bigcap_v \{u: |u - v| \leqslant w_\rho(I)^{-1}w_\rho(T - vI), u \text{ and } v \in \mathbf{C}\}, \quad 0 < \rho < 1.$$

Also, with this factor $w_\rho(I)^{-1}$ in mind, and some obvious modifications of proofs in previous sections, the cases when $\rho < 1$ can be treated rather easily. We give some results as follows. Let ρ be a fixed number in the range $0 < \rho < 1$.

(1) $\sup \operatorname{Re} W(T) = \lim_{t\to\infty} [w_\rho(I)^{-1} w_\rho(T + tI) - t]$;

(2) if D is a closed convex subset of $\mathbf{C}$, then $D \supseteq W(T)$ if and only if $w_\rho(I)^{-1}w_\rho((T - vI)^{-1}) \leqq d(v, D)^{-1}$ for all $v \notin D$; and

(3) T is self-adjoint if and only if $w_\rho(I)^{-1}w_\rho((T + viI)^{-1}) \leqq |v^{-1}|$ for all real $v \neq 0$.

Since $1 > w_\rho(I)^{-1}$, we see from (c), unless $T = 0$, that

$$w_\rho(T) > w_\rho(I)^{-1}w_\rho(T) \geqq r(T).$$

Also recall that $W_\rho(T) = W(T)$, $1 \leqq \rho \leqq 2$. These will enable us to explain why we choose the indicated ranges for ρ in the definitions of a ρ-normaloid operator and a ρ-convexoid operator.

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