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SEMIGROUP COMPACTIFICATIONS OF DIRECT AND SEMIDIRECT PRODUCTS

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ABSTRACT. A classical result of I. Glicksberg and K. de Leeuw asserts that the almost periodic compactification of a direct product $S\times T$ of abelian semigroups with identity is (canonically isomorphic to) the direct product of the almost periodic compactifications of S and T. Some efforts have been made to generalize this result and recently H. D. Junghenn and B. T. Lerner have proved a theorem giving necessary and sufficient conditions for an F-compactification of a semidirect product $S\otimes_\sigma T$ to be a semidirect product of compactifications of S and T. A different such theorem is presented here along with a number of corollaries and examples which illustrate its scope and limitations. Some behaviour that can occur for semidirect products, but not for direct products, is exposed.

Preliminaries and theorem. Let S be a semitopological semigroup. If S has an identity or a specified left or right identity, this identity will usually be denoted by e, but sometimes it will be denoted by 0 or 1. Let $C(S)$ be the C^*-algebra of bounded, continuous, complex-valued functions on S. The translation operators R_t and L_s are defined by

$$R_tf(s)=L_sf(t)=f(ts),\qquad s,t\in S,\qquad f\in C(S).$$

For a C^*-subalgebra F of $C(S)$, S_F denotes the spectrum of F furnished with the weak * topology from F^*; let $\delta:S\to S_F$ be the evaluation mapping. If F is left translation invariant (i.e., $L_sf\in F$ if $f\in F$ and $s\in S$) and contains the constant functions, then F is called *left m-introverted* if the function $f_x:s\to x(L_sf)$ is in F for all $f\in F$ and $x\in S_F$. For left m-introverted F a binary operation $(x',x)\to x'x$ can be defined on S_F (by $x'xf=x'(f_x)$) relative to which the pair $(\psi,X)=(\delta,S_F)$ has the following properties:

(i) X is a compact Hausdorff space and a semigroup, and for each $x\in X$, the map $x'\to x'x$, $X\to X$ is continuous;

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(ii) $\psi : S \to X$ is a continuous homomorphism with image dense in X and for each $s \in S$, the map $x \to \psi(s)x$, $X \to X$ is continuous; and

(iii) $\psi^*C(X) = F$.

Any pair (ψ, X) satisfying (i) and (ii) is called a *right topological compactification* of S and an *F-compactification* if, in addition, (iii) holds. We shall refer to (δ, S_F) as the *canonical F-compactification*, and note that F-compactifications are unique in the following sense: if (ψ_1, X_1) also satisfies (i)–(iii), then there is an isomorphism ϕ of S_F onto X_1 such that $\phi \circ \delta = \psi_1$. See [1] for the general theory of F-compactifications.

Let S and T be semigroups with identity and let $S_1 = S \otimes_\sigma T$ be a *semidirect product* of them, i.e., S_1 is homeomorphic to $S \times T$ and there is a homomorphism σ of T into the semigroup of endomorphisms of S such that $\sigma(e)$ is the identity map and $\sigma(t)e = e$ for all $t \in T$; multiplication in S_1 is given by $(s, t)(s', t') = (s\sigma(t)s', tt')$. If S and T are semitopological semigroups, we shall assume that $S \otimes_\sigma T$ is as well, i.e., that the maps $s \to \sigma(t')s$, $S \to S$ and $(s, t) \to s\sigma(t)s'$, $S \times T \to S$ are continuous for all $s' \in S$, $t' \in T$. (See, however, Example 8 ahead, where a semidirect product of semitopological semigroups is not semitopological.)

Suppose now that $S_1 = S \otimes_\sigma T$ is a semidirect product with right topological compactification (ψ, X); then ψ yields homomorphisms ψ_1 and ψ_2 of S and T, respectively, into X

$$\psi_1(s) = \psi(s, e) \quad \text{for} \quad s \in S, \qquad \psi_2(t) = \psi(e, t) \quad \text{for} \quad t \in T.$$

Let S_ψ and T_ψ denote the closures in X of $\psi_1(S)$ and $\psi_2(T)$, respectively. Our next theorem investigates the possibility of identifying X with a semidirect product $S_\psi \otimes_\sigma T_\psi$. The proof uses ideas from [2, 8]. Other work in this vein appears in [3, 4, 6, 7, 10, 11].

THEOREM. *Let $S_1 = S \otimes_\sigma T$, (ψ, X), S_ψ and T_ψ be as in the preceding paragraph, and let $\mu : S_\psi \times T_\psi \to X$, $(x, y) \to xy$ be the restriction to $S_\psi \times T_\psi$ of the semigroup multiplication from $X \times X$ into X. Then there is a semidirect product $S_\psi \otimes_\sigma T_\psi$ that is canonically isomorphic to X if and only if μ is an injection and is continuous (i.e., jointly continuous). In case S_1 is a direct product, this conclusion still holds if the identity of S, resp. T, is assumed to be merely a right, resp. left, identity.*

Proof. The canonical aspect of the isomorphism of the theorem means, of course, that $(\psi_1(s), \psi_2(t)) \in S_\psi \otimes_\sigma T_\psi$ shall correspond to $\psi(s, t) \in X$ and that $\sigma(\psi_2(t))\psi_1(s) = \psi_1(\sigma(t)s)$ for all $(s, t) \in S_1$. (See Example 1 ahead in this regard.) Thus, it is clear that the conditions on μ must be satisfied if $X \simeq S_\psi \otimes_\sigma T_\psi$. So, suppose the conditions are satisfied. We conclude first that the map $\mu : (x, y) \to xy$ of $S_\psi \times T_\psi$ into X is a homeomorphism. This follows from the fact that the image of $S_\psi \times T_\psi$ must be compact, since μ is continuous, and must equal X, since the image contains

$$\psi(S_1)=\{\psi(s,t)=\psi_1(s)\psi_2(t) \mid (s,t)\in S_1\},$$

which is dense in X. One can now complete the proof as in [8; proof of Theorem 3.1]: give $S_\psi\times T_\psi$ the multiplication

$$(x,y)(x',y')=\mu^{-1}(\mu(x,y)\mu(x',y'))$$

and, for $x\in S_\psi$, $y\in T_\psi$, define $\sigma(y)x$ to be the first coordinate of $(\psi_1(e),y)(x,\psi_2(e))$.

An unusual aspect of Example 3 ahead prompts us to formulate a proposition; for it, it is convenient to refer to the distal compactification and the compactifications of [1; Chapter III] as the *standard compactifications* and to refer to their associated left m-introverted C^*-subalgebras as the *standard algebras.*

REMARKS. Let (n, NS_1) be a standard compactification of a semitopological semigroup S_1, and let $F(S_1)$ be its associated standard algebra. If $S_1=S\otimes_\sigma T$ is a semidirect product it follows from general considerations that the closure T_n of $n(\{e\}\times T)$ in NS_1 is canonically isomorphic to NT; see [2; pp. 168–9], for example. If $S_1=S\times T$ is a direct product, then $n(S\times\{e\})^-=S_n\simeq NS$ as well, and it follows as in [2; p. 169] that the map

$$\mu:(x,y)\to xy, \qquad NS\times NT\to NS_1$$

is an injection; hence to conclude $NS_1\simeq NS\times NT$, one need only prove that μ is continuous. When $S_1=S\otimes_\sigma T$ is a semidirect product, one can conclude that μ is an injection if T_n is a group, and that S_n is a homomorphic, not necessarily isomorphic, image of NS; see Example 6, and also [8].

PROPOSITION. *Let S and T be semitopological semigroups with right and left identities, respectively, and let $S_1=S\times T$ be their direct product. Let (n, NS_1) and $(n', N'S_1)$ be two of the standard compactifications with associated standard algebras $F(S_1)$ and $F'(S_1)$, respectively, and suppose $F(S_1)\supset F'(S_1)$. If $NS_1\simeq NS\times NT$, then $N'S_1\simeq N'S\times N'T$.*

Proof. Since NS_1 is a direct product, $NS_1\simeq NS\times NT$ and the map $(x,y)\to xy$, $NS\times NT\to NS_1$ is continuous. Since $F(S_1)\supset F'(S_1)$, there is a canonical continuous homomorphism of NS_1 onto $N'S_1$ (given by the adjoint of the inclusion map), which maps NS onto $N'S$ and NT onto $N'T$ and yields the continuity of

$$\mu\ :(x,y)\to xy, \qquad N'S\times N'T\to N'S_1.$$

That μ is an injection is established in the remarks above, and the proof is complete.

EXAMPLES. We give some examples which show how the conditions on m, as in the theorem, can fail to be satisfied. For them, and also for the corollaries which follow, terms and notation not defined here can be bound in [1].

The requirement that μ be an injection implies that $S_\psi \cap T_\psi = \{\psi(e, e)\}$.

1. Let $S = T = \{e, a\}$ be the group with 2 elements, and let $\psi: S \times T \to (S \times T)/H = X$ be the quotient map, where H is the normal subgroup $\{(e, e), (a, a)\} \subset S \times T$. Then $S_\psi = T_\psi = X \simeq \{e, a\}$. Note that X is the direct product of a homomorphic image of S (the trivial one) and T_ψ, but cannot be a direct or semidirect product of S_ψ and T_ψ.

2. Let S and T be non-compact, locally compact groups and let X be the one-point compactification of $S \times T$. Then at least $\psi_1(S) \cap \psi_2(T) = \{\psi(e, e)\}$, but S_ψ and T_ψ each contain the point at ∞ as well.

3. Let $0 < b < 1$ and let $S = [b, 1]$ with semigroup operation defined by $s \circ s' = \max\{b, ss'\}$, where ss' is the ordinary product of real numbers. Let T be the natural numbers, which act on S by the formula $\sigma(j)s = \max\{s^j, b\}$; so $S_1 = S \otimes_\sigma T$ is a semidirect product with multiplication

$$(s, j)(s', j') = (s \circ \sigma(j)s', jj').$$

This example appears in [8] and it is shown there that the almost periodic compactification AS_1 is not a semidirect product. The example is of interest here because so many of the hypotheses for μ (as in the theorem) are satisfied; namely, considering $(\psi, X) = (a, A(S_1))$, we have μ continuous by [1; Theorem III.9.4], and $S_\psi \cap T_\psi = \{\psi(1, 1)\}$ by the remarks preceding the proposition above and by the fact that, if $f \in C(S) = AP(S)$, then $f_1 \in AP(S_1)$, where

$$f_1(s, j) = \begin{cases} f(s), & \text{if } \quad j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

However, μ is not an injection. For, if $f \in AP(S_1)$ and $\psi_2(j_\alpha) \to y \in AT \backslash \psi_2(T)$, then a subnet of $\{L_{(1,j_\alpha)}f\}$ must converge pointwise (in fact, uniformly) to an $h \in C(S_1)$. One sees that

$$h(r, 1) = \lim_\alpha L_{(1,j_\alpha)}f(r, 1) = \begin{cases} \lim_\alpha f(1, j_\alpha), & \text{if} \quad r = 1, \\ \lim_\alpha f(b, j_\alpha), & \text{if} \quad r < 1. \end{cases}$$

Hence, the continuity of the restriction to $S \times \{1\}$ of h forces

$$\lim_\alpha f(1, j_\alpha) = \lim_\alpha f(b, j_\alpha);$$

thus $(1, 1)(1, y) = (b, 1)(1, y)$.

One is led to another interesting aspect of this example. If (u, US_1) is the compactification arising from $LUC(S_1)$, the left uniformly continuous functions in $C(S_1)$, Theorem 3.5 in [8] says that $US_1 \simeq S \otimes_\sigma UT$ (and it is easy to verify this directly). The point to be made here is that, even though $LUC(S_1) \supset AP(S_1)$, hence there is a canonical continuous homomorphism of $US_1 \simeq S_u \otimes_\sigma UT$ onto AS_1, it does not follow that $AS_1 \simeq S_a \otimes_\sigma AT$. One wonders if behaviour like this could happen for a semidirect product of groups. The proposition above says such behaviour cannot occur if S_1 is a direct product.

Returning to the present example, we make the further observations that the idea of the previous paragraph (the part showing μ is not an injection) also shows that the *left* topological compactification arising from $RUC(S_1)$, the right uniformly continuous functions in $C(S_1)$, is not a semidirect product; further, it shows that the weakly almost periodic compactification WS_1 is not a semidirect product.

4. Let S be a commutative semigroup with identity and let (n, NS) be one of the standard compactifications. Suppose NS is not a topological semigroup, i.e., the associated standard algebra $F(S)$ is not contained in $AP(S)$. The map

$$\nu:(s,t)\to s+t, \qquad S\times S\to S$$

is a homomorphism whose adjoint injects $F(S)$ into $F(S\times S)$, $\nu^*f(s,t)=f(s+t)$ for $f\in F(S)$ and $(s,t)\in S\times S$. Then, writing $(\psi, X)=(n, N(S\times S))$, one sees that supposing μ (of the theorem) to be continuous forces the contradiction that NS is topological. (The idea for this line of argument came from [7; Remark 5.2(b)].)

Corollary 1(b) ahead shows how the theorem above deals with compactifications of a direct product $S\times T$ when S is a compact topological group. The next three examples concern the cases where S is compact, but not required to be a group, or where S (or T) is a compact topological group, but the product is allowed to be semidirect.

5. Let S_1 be the left group $[0,1]\times R$: multiplication is given by $(s,t)(s',t')=(s,t+t')$. It follows from [2] that, if $(\psi, X)=(w, WS_1)$, then μ is not continuous; $WS_1\neq[0,1]_\psi\times R_\psi\simeq[0,1]\times WR$.

6. Let $S_1=C\otimes_\sigma T$ be the euclidean group of the complex plane C, $(z,w)\ (z',w')=(z+wz', ww')$, i.e. $\sigma(w)z=wz$ for $z\in C$, $w\in T$. (Here $T=\{w\in C \mid |w|=1\}$.) Then, if $(\psi, X)=(a, AS_1)$ is the almost periodic compactification, $C_\psi=\{\psi(0,1)\}$ and $T_\psi\simeq T$, so $X\simeq C_\psi\otimes_\sigma T_\psi\simeq T$. If $(\psi, X)=(w, WS_1)$ is the weakly almost periodic compactification, then $T_\psi\simeq T$, C_ψ is isomorphic to the one-point compactification of C and $X\simeq C_\psi\otimes_\sigma T_\psi\simeq C_\psi\otimes_\sigma T$. In view of Example 4, the following fact about S_1 seems quite amazing: if $(\psi, X)=(w, W(S_1\times S_1))$, then $(S_1)_\psi\simeq WS_1$ and $X\simeq WS_1\times WS_1$. See [10] for all this, and see [3, 11] for further examples of the last phenomenon.

7. Let S be the torus and let τ be an automorphism of S such as that given by the formula $\tau(z,w)=(z^2w, zw)$. Let Z be the integers. We then get a

semidirect product $S_1 = S \otimes_\sigma Z$, with

$$((z, w), n)((z', w'), n') = ((z, w)\tau^n(z', w'), n+n').$$

We have not been able to analyse WS_1. However, (a) and (b) of Corollary 2 apply to S_1.

8. Let S be the integers, let T be the natural numbers and let $S_1 = S \otimes_\sigma T$ be the semidirect product with multiplication $(s, t)(s', t') = (s + ts', tt')$. Let S_0 and T_0 be the one-point compactifications of S and T, respectively; with the added points acting as zeros, S_0 and T_0 are semitopological semigroups. It is easy to extend the product in $S \otimes_\sigma T$ to $S_0 \times T_0$ so that $S_0 \otimes_\sigma T_0$ becomes a compactification of S_1. However $S_0 \otimes_\sigma T_0$ is not a semitopological semigroup.

Corollaries. We now present some corollaries to the theorem.

COROLLARY 1. *Let S and T be semitopological semigroups with right and left identities, respectively, and let $S_1 = S \times T$ be their direct product.*

(a) *If $(\psi, X) = (a, AS_1)$ is the almost periodic compactification, then $X \simeq S_\psi \times T_\psi \simeq AS \times AT$.*

(b) *If S is a compact topological group and $(\psi, X) = (p, PS_1)$ is the compactification arising from $LMC(S_1)$, the largest left m-introverted subalgebra of $C(S_1)$, then $X \simeq S_\psi \times T_\psi \simeq S \times PT$.*

(c) *If S is a compact semitopological semigroup and $(\psi, X) = (u, US_1)$ is the compactification arising from $LUC(S_1)$, the left uniformly continuous functions in $C(S_1)$, then $X \simeq S_\psi \times T_\psi \simeq S \times UT$.*

COROLLARY 2. *Let S and T be semitopological semigroups with identity, and let $S_1 = S \otimes_\sigma T$ be a semidirect product of them.*

(a) *If $(\psi, X) = (m, MS_1)$ is the strongly almost periodic compactification, then $X \simeq S_\psi \otimes_\sigma T_\psi \simeq S_\psi \otimes_\sigma MT$.*

(b) *If S is compact and $(\psi, X) = (d, DS_1)$ is the distal compactification, then $X \simeq S_\psi \otimes_\sigma T_\psi \simeq S_\psi \otimes_\sigma DT$.*

(c) *If T is a dense subsemigroup of a compact topological group G and $(\psi, X) = (w, WS_1)$ is the weakly almost periodic compactification, then $X \simeq S_\psi \otimes_\sigma T_\psi \simeq S_\psi \otimes_\sigma G$.*

Proofs. In Corollary 1, S_1 is a direct product, while, in Corollary 2, T_ψ is a group. Thus, it follows, using the remarks preceding the proposition above, that μ is an injection. The continuity of μ follows either from the (joint) continuity properties of multiplication in almost periodic compactifications and *LUC* compactifications [1; §§III.9 and III.5] or from Ellis' Theorem [5].

To establish the continuity of μ for Corollary 2(c), for example, one must note that, if $f \in WAP(S_1)$, then the restriction of f to $\{e\} \times T \simeq T$ is in $WAP(T)$,

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hence extends to a continuous function on G [1; Corollary III.15.7]. This implies $T_\psi \simeq G$. Since multiplication in weakly almost periodic compactifications is separately continuous [1; §III.8], $\mu : S_\psi \times T_\psi \to X$, $(x, y) \to xy$ is separately continuous and, by Ellis' theorem, jointly continuous. For Corollary 2(b) one must note that S_ψ is a compact semitopological semigroup that is a group, since it is a homomorphic image of DS; hence S_ψ is a compact topological group [5].

REMARKS. 1. Corollaries 1(a) and (b) appear in [2] and Corollary 2(a) appears in [8], while Corollary 1(c) improves Corollary 3 in [6] and is a variant of Theorem 3.5 in [8], Corollary 2(b) improves Theorem 5.1 in [7], and Corollary 2(c) improves part of Corollary 5 in [7].

2. The examples show some limitations to improvement of the corollaries. We mention only that Example 4 illustrates a major obstruction to a broad generalization of Corollary 1(a), and that Corollary 2(c) is the strongest general conclusion we have been able to devise for the weakly almost periodic compactification (but note Example 6 and the comments made there).

Note added in proof: 1. Since the present paper was submitted, we have encountered a preprint, "Semidirect product compactifications", by F. Dangello and R. Lindahl, in which are a number of interesting results. For example, if $S_1 = S \otimes_\sigma T$ is a semidirect product, the maximal semidirect product topological compactification is exhibited, i.e., a compactification (ψ, X) of S_1 is found such that X is a topological semigroup and (ψ, X) satisfies the conditions of the theorem above, and, if (ψ', X') is any other such compactification of S_1, then X maps canonically onto X'.

2. In Example 8, the point added to T acts on S_0 as the pointwise limit of the endomorphisms of S_0 corresponding to the members of T. The left translation $x \to yx$, $S_0 \otimes_\sigma T_0 \to S_0 \otimes_\sigma T_0$ is continuous for each $y \in S_0 \otimes_\sigma T_0$. (With the definition of semidirect product used here, the continuity of left translations in $S_0 \otimes_\sigma T_0$ follows directly, provided the left translations in S_0 and T_0 are continuous and the endomorphisms of S_0 involved are continuous.) However, the right translation $x \to xy$, $S_0 \otimes_\sigma T_0 \to S_0 \otimes_\sigma T_0$ is continuous if and only if the first coordinate of y is $0 \in S$ or the the point added to S. Thus, although $S \otimes_\sigma T$ is dense in $S_0 \otimes_\sigma T_0$, $S_0 \otimes_\sigma T_0$ is neither a right topological compactification (as defined above) of $S \otimes_\sigma T$ nor a left topological compactification (analogously defined) of $S \otimes_\sigma T$.

3. It has been pointed out to the author by Dr. M. Kunze that the algebraic part of the proof of the Theorem above has a longer history than was known to the author at the time this paper was written: it is used to show when a semigroup is a Zappa product of two of its subsemigroups. See M. Kunze, Lineare Parallelrechner, Habilitationsschrift, Darmstadt, 1982.

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