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PARA-BLASCHKE ISOPARAMETRIC HYPERSURFACES IN A UNIT SPHERE $S^{n+1}(1)$*

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Abstract. Let $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i \otimes \theta_j$ and $\mathbf{B} = \rho^2 \sum_{i,j} B_{ij}\theta_i \otimes \theta_j$ be the Blaschke tensor and the Möbius second fundamental form of the immersion $\mathbf{x}$. Let $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$ be the para-Blaschke tensor of $\mathbf{x}$, where λ is a constant. If $\mathbf{x} : M^n \mapsto S^{n+1}(1)$ is an n-dimensional para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ and $\mathbf{x}$ has three distinct Blaschke eigenvalues one of which is simple or has three distinct Möbius principal curvatures one of which is simple, we obtain the full classification theorems of the hypersurface.

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1. Introduction. In Möbius differential geometry, Wang **[18]** studied invariants of hypersurfaces in a unit sphere $S^{n+1}(1)$ under the Möbius transformation group. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n-dimensional immersed hypersurface without umbilical points in $S^{n+1}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature of the immersion $\mathbf{x}$. By putting $\rho^2 = \frac{n}{n-1}\{\sum_{i,j} h_{ij}^2 - nH^2\}$, Wang **[18]** defined *the Möbius metric* , *the Möbius form*, *the Blaschke tensor* and *the Möbius second fundamental form* of the immersion $\mathbf{x}$ by $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$, $\Phi = \rho \sum_i C_i\theta_i$, $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij}\theta_i \otimes \theta_j$ and $\mathbf{B} = \rho^2 \sum_{i,j} B_{ij}\theta_i \otimes \theta_j$, respectively, where

$$C_i = -\rho^{-2}\left\{H_{,i} + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\right\}, \tag{1.1}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}\{\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\} \\ &- \frac{1}{2}\rho^{-2}(|\nabla(\log\rho)|^2 - 1 + H^2)\delta_{ij}, \end{aligned} \tag{1.2}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{1.3}$$

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and Hess_{ij}, ∇ are the Hessian matrix and the gradient with respect to the induced metric $d\mathbf{x}\cdot d\mathbf{x}$. It was proved that g, Φ, $\mathbf{A}$ and $\mathbf{B}$ are the Möbius invariants (see [**18**]). We should notice that it is one of the important aims to characterize submanifolds in terms of Möbius invariants. Concerning this topic, there are many important results, one can see [**1**, **2** and **5–20**]. Recently, by making use of the two important Möbius invariants, *the Blaschke tensor* $\mathbf{A}$ and *the Möbius second fundamental form* $\mathbf{B}$ of the immersion $\mathbf{x}$, Cheng, Li and Qi [**6**] and Zhong and Sun [**19**] defined a symmetric $(0, 2)$ tensor $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$ which is so-called *the para-Blaschke tensor* of $\mathbf{x}$, where λ is a constant. An eigenvalue of the Blaschke tensor is called *a Blaschke eigenvalue* of $\mathbf{x}$, an eigenvalue of the Möbius second fundamental form is called *a Möbius principal curvature* of $\mathbf{x}$ and an eigenvalue of the para-Blaschke tensor is called *a para-Blaschke eigenvalue* of $\mathbf{x}$. It is reasonable to introduce the definition: A hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points is called *a Blaschke isoparametric hypersurface*, or *a Möbius isoparametric hypersurface*, or *a para-Blaschke isoparametric hypersurface*, if the Möbius form $\Phi \equiv 0$ and the Blaschke eigenvalues, or the Möbius principal curvatures, or the para-Blaschke eigenvalues of the immersion x are constants. In [**11**], Li and Wang investigated and completely classified hypersurfaces $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points and with vanishing Möbius form Φ in $S^{n+1}(1)$, which satisfy $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$. Li and Zhang [**12**] generalized this topic to general submanifolds. It should be noted that the condition $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$ implies that the para-Blaschke eigenvalues of $\mathbf{x}$ are all equal. If $\mathbf{x}$ has two distinct constant para-Blaschke eigenvalues, the classification theorem was obtained by Zhong and Sun [**19**].

Let $\mathbf{H}^{n+1}$ be an $(n + 1)$-dimensional hyperbolic space defined by

$$\mathbf{H}^{n+1} = \left\{(y_0, y_1) \in \mathbf{R}^+ \times \mathbf{R}^{n+1} \mid -y_0^2 + y_1 \cdot y_1 = -1\right\}.$$

Let $\sigma : \mathbf{R}^{n+1} \mapsto S^{n+1}(1)\backslash\{(-1, 0)\}$ and $\tau : \mathbf{H}^{n+1} \mapsto S^{n+1}_+(1)$ be defined by

$$\sigma(u) = \left(\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2}\right), \quad u \in \mathbf{R}^{n+1}, \tag{1.4}$$

$$\tau(y_0, y_1) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad (y_0, y_1) \in \mathbf{H}^{n+1}, \tag{1.5}$$

respectively, where $S^{n+1}_+(1)$ is the open hemisphere in $S^{n+1}(1)$ whose first coordinate is positive.

If $\lambda = 0$, we notice that para-Blaschke isoparametric hypersurfaces reduce to Blaschke isoparametric hypersurfaces. Li and Peng [**13**] obtained the following:

THEOREM 1.1. *Let* $\mathbf{x}$ *be an n-dimensional immersed Blaschke isoparametric hypersurface in a unit sphere* $S^{n+1}(1)$ *with three distinct Blaschke eigenvalues one of which is simple. Then,* $\mathbf{x}$ *is locally Möbius equivalent to*

(1) *$CSS(p, q, r)$ for some constants p, q, r, $p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$, or*
(2) *Cartan's non-minimal isoparametric hypersurfaces in S^4 with three principal curvatures, that is, the non-minimal tube of constant radius over a standard Veronese minimal immersion of $S^2(\sqrt{3})$ into S^4, or*
(3) *one of the hypersurfaces as indicated in Example* 3.4 *where* $k = 3$ *and* $\tilde{y}_1 : M_1 \mapsto S^4(r)$ *is one of Cartan's non-minimal isoparametric hypersurfaces with*

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three principal curvatures μ_1, μ_2 and μ_3 satisfying $\lambda\mu_i = \frac{1}{r^2}$ for some $i \in \{1, 2, 3\}$.

If $\lambda \neq 0$, we consider the immersed para-Blaschke isoparametric hypersurfaces in a unit sphere $S^{n+1}(1)$ with three distinct Blaschke eigenvalues. We may obtain the following:

THEOREM 1.2. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ and $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$, $(\lambda \neq 0)$, be the para-Blaschke tensor of $\mathbf{x}$. If $\mathbf{x}$ is of three distinct Blaschke eigenvalues one of which is simple, then $\mathbf{x}$ is locally Möbius equivalent to:*

(1) *a hypersurface with constant mean curvature and constant scalar curvature in $S^{n+1}(1)$, or*
(2) *the image of σ of a hypersurface with constant mean curvature and constant scalar curvature in $\mathbf{R}^{n+1}$, or*
(3) *the image of τ of a hypersurface with constant mean curvature and constant scalar curvature in $\mathbf{H}^{n+1}$, or*
(4) *$CSS(p, q, r)$ for some constants p, q, r, $p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$, or*
(5) *one of the hypersurfaces as indicated in Example 3.4 where $k = 3$ and $\tilde{y}_1 : M_1 \mapsto S^4(r)$ is one of Cartan's non-minimal isoparametric hypersurfaces with three principal curvatures satisfying $\lambda\mu_i = \frac{1}{r^2}$ for some $i \in \{1, 2, 3\}$.*

For Möbius isoparametric hypersurface with three distinct Möbius principal curvatures in a unit sphere $S^{n+1}(1)$, Hu and co-authors [**8**] and [**9**] obtained the following:

THEOREM 1.3. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed Möbius isoparametric hypersurface with three distinct Möbius principal curvatures one of which is simple. Then $\mathbf{x}$ is locally Möbius equivalent to*

(1) *$CSS(p, q, r)$ for some constants p, q, r, or*
(2) *an open part of the image of σ of the cone $\bar{x} : N^3 \times \mathbf{R}^+ \mapsto \mathbf{R}^5$ defined by $\bar{x}(\varphi, t) = t\varphi$, where $t \in \mathbf{R}^+$ and $\varphi : N^3 \mapsto S^4 \hookrightarrow \mathbf{R}^5$ is minimal isoparametric immersion in S^4 with three principal curvatures, or*
(3) *one of the hypersurfaces as indicated in Example 3.4 where $k = 3$, $r = \sqrt{\frac{6n}{n-1}}$, $\lambda = 0$ and $\tilde{y}_1 : M_1 \mapsto S^4(r)$ is Cartan's minimal isoparametric hypersurfaces with vanishing scalar curvature and three principal curvatures of values $\pm\sqrt{\frac{n-1}{2n}}, 0$.*

If $\mathbf{x}$ is an immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ with three distinct Möbius principal curvatures one of which is simple, we obtain the following:

THEOREM 1.4. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$. If $\mathbf{x}$ is of three distinct Möbius principal curvatures one of which is simple, then $\mathbf{x}$ is locally Möbius equivalent to:*

(1) *a hypersurface with constant mean curvature and constant scalar curvature in $S^{n+1}(1)$, or*

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(2) *the image of* σ *of a hypersurface with constant mean curvature and constant scalar curvature in* $\mathbf{R}^{n+1}$, *or*
(3) *the image of* τ *of a hypersurface with constant mean curvature and constant scalar curvature in* $\mathbf{H}^{n+1}$, *or*
(4) $CSS(p, q, r)$ *for some constants* p, q *and* r, *or*
(5) *an open part of the image of* σ *of the cone* $\bar{x} : N^3 \times \mathbf{R}^+ \mapsto \mathbf{R}^5$ *defined by* $\bar{x}(\varphi, t) = t\varphi$, *where* $t \in \mathbf{R}^+$ *and* $\varphi : N^3 \mapsto S^4 \hookrightarrow \mathbf{R}^5$ *is minimal isoparametric immersion in* S^4 *with three principal curvatures, or*
(6) *one of the hypersurfaces as indicated in Example* 3.4 *where* $k = 3$, $r = \sqrt{\frac{6n}{n-1}}$, $\lambda = 0$ *and* $\tilde{y}_1 : M_1 \mapsto S^4(r)$ *is Cartan's minimal isoparametric hypersurfaces with vanishing scalar curvature and three principal curvatures of values* $\pm\sqrt{\frac{n-1}{2n}}, 0$.

2. Möbius invariants and fundamental formulas. In this section, we review the Möbius invariants and fundamental formulas on Möbius geometry of hypersurfaces in $S^{n+1}(1)$, for more details, see Wang [**18**].

Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n-dimensional hypersurface of $S^{n+1}(1)$ without umbilical points. We use the following range of indices throughout this paper:

$$1 \leq i, j, k \leq n.$$

For an immersed hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1) \hookrightarrow \mathbf{R}^{n+2}$ of $S^{n+1}(1)$ without umbilical points, we define its Möbius position vector $Y : M \mapsto \mathbf{L}^{n+3}$ by $Y = \rho(1, \mathbf{x})$, where $\rho^2 = \frac{n}{n-1}\{\sum_{i,j} h_{ij}^2 - nH^2\}$. Let Δ be the Laplace–Beltrami operator of Möbius metric $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$. We define $N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y$, then the structure equations on M with respect to the Möbius metric g can be written as follows:

$$dY = \sum_i \omega_i Y_i, \tag{2.1}$$

$$dN = \sum_i \psi_i Y_i + \phi E_{n+1}, \tag{2.2}$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1} E_{n+1}, \tag{2.3}$$

$$dE_{n+1} = -\phi Y - \sum_i \omega_{in+1} Y_i, \tag{2.4}$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \phi\}$ are 1-forms on M with

$$\omega_{ij} + \omega_{ji} = 0. \tag{2.5}$$

By exterior differentiation of these equations, we get

$$\sum_i \omega_i \wedge \psi_i = 0, \quad \sum_i \omega_{in+1} \wedge \omega_i = 0, \tag{2.6}$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \tag{2.7}$$

$$d\psi_i = \sum_j \omega_{ij} \wedge \psi_j + \omega_{in+1} \wedge \phi, \tag{2.8}$$

$$d\phi = -\sum_i \omega_{in+1} \wedge \psi_i, \tag{2.9}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{in+1} \wedge \omega_{jn+1} - \omega_i \wedge \psi_j - \psi_i \wedge \omega_j, \tag{2.10}$$

$$d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1} - \omega_i \wedge \phi, \tag{2.11}$$

where

$$\psi_i = \sum_j A_{ij}\omega_j, \quad A_{ij} = A_{ji}, \quad \omega_{in+1} = \sum_j B_{ij}\omega_j, \quad B_{ij} = B_{ji}, \quad \phi = \sum_i C_i\omega_i, \tag{2.12}$$

and A_{ij}, B_{ij} and C_i are locally defined functions and satisfy (1.1), (1.2) and (1.3). We have

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.13}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \quad \mathrm{tr}\mathbf{A} = \frac{1}{2n}(1+n^2R). \tag{2.14}$$

Let $C_{i,j}$, $A_{ij,k}$ and $B_{ij,k}$ be the covariant derivative of C_i, A_{ij} and B_{ij}. We define them by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.15}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.16}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.17}$$

From the structure equations (2.1)–(2.4), we infer

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.18}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{2.19}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.20}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.21}$$

where R_{ijkl} denotes the curvature tensor with respect to the Möbius metric g on M and $n(n-1)R = \sum_{i,j} R_{ijij}$ is the Möbius scalar curvature of the immersion $\mathbf{x} : M \to S^{n+1}(1)$.

Since the Möbius form $\Phi = \sum_i C_i\omega_i E_{n+1} \equiv 0$, by (2.18)–(2.20), we have for all indices i, j and k that

$$A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}A_{kj} = \sum_k B_{kj}A_{ki}. \tag{2.22}$$

Denote by $\mathbf{D} = \sum_{i,j} D_{ij}\omega_i \otimes \omega_j$ the $(0, 2)$ para-Blaschke tensor, then

$$D_{ij} = A_{ij} + \lambda B_{ij}, \quad 1 \le i, j \le n, \tag{2.23}$$

where λ is a constant. The covariant derivative of D_{ij} is defined by

$$\sum_k D_{ij,k}\omega_k = dD_{ij} + \sum_k D_{ik}\omega_{kj} + \sum_k D_{kj}\omega_{ki}. \tag{2.24}$$

From (2.23), we have

$$D_{ij,k} = A_{ij,k} + \lambda B_{ij,k}, \quad D_{ij,k} - D_{ik,j} = A_{ij,k} - A_{ik,j} + \lambda(B_{ij,k} - B_{ik,j}). \tag{2.25}$$

From (2.22), we have for all indices i, j and k that

$$D_{ij,k} = D_{ik,j}. \tag{2.26}$$

3. Propositions and typical examples. Throughout this section, we shall make the following convention on the ranges of indices:

$$1 \le a, b \le m_1, \quad m_1 + 1 \le p, q \le m_1 + m_2,$$
$$m_1 + m_2 + 1 \le \alpha, \beta \le m_1 + m_2 + m_3 = n, \quad 1 \le i, j, k \le n.$$

We may prove the following:

PROPOSITION 3.1. *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n-dimensional hypersurface with vanishing Möbius form in a unit sphere $S^{n+1}(1)$.*

(1) *If the multiplicity of a Blaschke eigenvalue is constant and greater than* 1*, then this Blaschke eigenvalue is constant along its leaf.*
(2) *If the multiplicity of a Möbius principal curvature is constant and greater than* 1*, then this Möbius principal curvature is constant along its leaf.*

Proof. (1) Let A_i be the Blaschke eigenvalues of $\mathbf{x}$ with constant multiplicities. We choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ such that E_i is a unit principal vector with respect to A_i. From (2.16), we have

$$A_{ij,k} = E_k(A_i)\delta_{ij} + \Gamma^j_{ik}(A_i - A_j), \tag{3.1}$$

where Γ^j_{ik} is the Levi–Civita connection for the Möbius metric g given by

$$\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k, \quad \Gamma^j_{ik} = -\Gamma^i_{jk}. \tag{3.2}$$

From (2.22), we know that $A_{ii,j} = A_{ij,i}$. Thus, from (3.1), we get

$$E_j(A_i) = \Gamma^j_{ii}(A_i - A_j), \quad \text{for } i \neq j. \tag{3.3}$$

Without loss of generality, we may assume that A_1 is the Blaschke eigenvalue of $\mathbf{x}$ with constant multiplicity m_1 and $m_1 \geq 2$, that is, for $1 \leq a \leq m_1$, we have $A_a = A_1$. From (3.3), we have

$$E_a(A_1) = \Gamma_{11}^{a}(A_1 - A_a) = 0, \quad \text{for} \ \ a \neq 1,$$

and

$$E_1(A_1) = E_1(A_a) = \Gamma_{aa}^{1}(A_a - A_1) = 0, \quad \text{for} \ \ a \neq 1.$$

Thus,

$$E_a(A_1) = 0, \quad \text{for any} \ \ a.$$

This implies that A_1 is constant along its leaf.

(2) Since the Möbius second fundamental form is also Codazzi tensor, by the same method, we see that (2) is true. We complete the proof of Proposition 3.1.

Proposition 3.2. *Let* $\mathbf{x} : M \mapsto S^{n+1}(1)$ *be an* $n(n \geq 4)$*-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere* $S^{n+1}(1)$ *and* $\mathbf{D} = \mathbf{A} + \lambda\mathbf{B}$ *be the para-Blaschke tensor of* $\mathbf{x}$.

(1) *If* $\mathbf{x}$ *has three distinct Blaschke eigenvalues* A_1, A_2 *and* A_3 *one of which is simple and* $\lambda \neq 0$, *then either* A_1, A_2 *and* A_3 *are constants or* $A_{ap,n} = 0$ *for every* a, p.
(2) *If* $\mathbf{x}$ *has three distinct Möbius principal curvatures* B_1, B_2 *and* B_3 *one of which is simple, then either* B_1, B_2 *and* B_3 *are constants or* $B_{ap,n} = 0$ *for every* a, p.

Proof. (1) Let A, B and D denote the $n \times n$-symmetric matrices (A_{ij}), (B_{ij}) and (D_{ij}), respectively, where A_{ij}, B_{ij} and D_{ij} are defined by (1.2), (1.3) and (2.23). From (2.22) and (2.23), we know that $BA = AB$, $DA = AD$ and $BD = DB$. We may choose a local orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ such that $A_{ij} = A_i\delta_{ij}$, $B_{ij} = B_i\delta_{ij}$ and $D_{ij} = D_i\delta_{ij}$, where A_i, B_i and D_i are the Blaschke eigenvalues, the Möbius principal curvatures and the para-Blaschke eigenvalues of the immersion $\mathbf{x}$.

From (2.13) and (3.2), the curvature tensor of $\mathbf{x}$ may be given by (see **[14]**)

$$R_{ijkl} = E_l\big(\Gamma_{ik}^{j}\big) - E_k\big(\Gamma_{il}^{j}\big) + \sum_m \Gamma_{im}^{j}\Gamma_{lk}^{m} - \sum_m \Gamma_{im}^{j}\Gamma_{kl}^{m} + \sum_m \Gamma_{ik}^{m}\Gamma_{ml}^{j} - \sum_m \Gamma_{il}^{m}\Gamma_{mk}^{j}. \quad (3.4)$$

Since $\mathbf{x}$ has three distinct Blaschke eigenvalues A_1, A_2 and A_3 one of which is simple and $n \geq 4$, without loss of generality, we may assume that $m_3 = 1$, $m_1 m_2 \geq 2$ and $m_2 \geq 2$.

From (2.14) and (2.23), we have

$$m_1 A_1 + m_2 A_2 + m_3 A_3 = \mathrm{tr}\mathbf{D}, \quad (3.5)$$

$$m_1 A_1^2 + m_2 A_2^2 + m_3 A_3^2 - 2\left(\sum_a D_a\right) A_1 - 2\left(\sum_p D_p\right) A_2 - 2\left(\sum_\alpha D_\alpha\right) A_3 = \frac{n-1}{n}\lambda^2 - \sum_i D_i^2. \quad (3.6)$$

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Since we assume that $D_1, D_2, \ldots, D_n$ are constants, we get

$$m_1 dA_1 + m_2 dA_2 + m_3 dA_3 = 0, \tag{3.7}$$

$$\xi_1 dA_1 + \xi_2 dA_2 + \xi_3 dA_3 = 0, \tag{3.8}$$

where $\xi_1 = m_1 A_1 - \sum_a D_a$, $\xi_2 = m_2 A_2 - \sum_p D_p$ and $\xi_3 = m_3 A_3 - \sum_\alpha D_\alpha$. Since $\lambda \neq 0$, we know that all of ξ_1, ξ_2 and ξ_3 are not zero. We consider two cases:

If at least one of $m_2\xi_3 - m_3\xi_2, m_3\xi_1 - m_1\xi_3$ and $m_1\xi_2 - m_2\xi_1$ is zero, from (3.7) and (3.8), we easily know that A_1, A_2 and A_3 are constants.

If all of $m_2\xi_3 - m_3\xi_2, m_3\xi_1 - m_1\xi_3$ and $m_1\xi_2 - m_2\xi_1$ are not zero, from (3.7) and (3.8), we easily see that

$$\frac{dA_1}{m_2\xi_3 - m_3\xi_2} = \frac{dA_2}{m_3\xi_1 - m_1\xi_3} = \frac{dA_3}{m_1\xi_2 - m_2\xi_1}. \tag{3.9}$$

From Proposition 3.1 and (3.9), we have

$$E_p(A_2) = E_p(A_1) = E_p(A_3) = 0, \tag{3.10}$$

and from (3.1), we have

$$\Gamma^p_{ab} = \Gamma^\alpha_{ab} = 0, a \neq b, \quad \Gamma^\alpha_{pq} = 0, p \neq q, \quad \Gamma^p_{aa} = \Gamma^p_{bb}, \quad \Gamma^\alpha_{aa} = \Gamma^\alpha_{bb}, \tag{3.11}$$

$$\Gamma^p_{a\alpha} = \frac{A_{ap,\alpha}}{A_1 - A_2}, \quad \Gamma^a_{\alpha p} = \frac{A_{\alpha a,p}}{A_3 - A_1}, \quad \Gamma^\alpha_{pa} = \frac{A_{p\alpha,a}}{A_2 - A_3}. \tag{3.12}$$

(i) If $m_1 \geq 2$, from Proposition 3.1 and (3.9), we have

$$E_a(A_1) = E_a(A_2) = E_a(A_3) = 0. \tag{3.13}$$

From (3.1), (3.3), (3.10) and (3.13), we have

$$\Gamma^p_{ab} = \Gamma^a_{pq} = 0, \quad \Gamma^a_{nn} = \Gamma^p_{nn} = 0, \tag{3.14}$$

$$\Gamma^n_{aa} = \frac{E_n(A_1)}{A_1 - A_3}, \quad \Gamma^n_{pp} = \frac{E_n(A_2)}{A_2 - A_3}. \tag{3.15}$$

From (3.12), we have

$$\Gamma^p_{an} = \frac{A_{ap,n}}{A_1 - A_2}, \quad \Gamma^p_{nb} = \frac{A_{bp,n}}{A_3 - A_2}, \quad \Gamma^n_{bq} = \frac{A_{bq,n}}{A_1 - A_3}, \quad \Gamma^n_{qb} = \frac{A_{bq,n}}{A_2 - A_3}. \tag{3.16}$$

Thus, from (3.4), (3.11) and (3.14)–(3.16), we have

$$\begin{aligned} R_{apbq} =& E_q(\Gamma^p_{ab}) - E_b(\Gamma^p_{aq}) + \sum_m \Gamma^p_{am}\Gamma^m_{qb} - \sum_m \Gamma^p_{am}\Gamma^m_{bq} + \sum_m \Gamma^m_{ab}\Gamma^p_{mq} - \sum_m \Gamma^m_{aq}\Gamma^p_{mb} \\ =& \Gamma^p_{an}\Gamma^n_{qb} - \Gamma^p_{an}\Gamma^n_{bq} + \Gamma^n_{ab}\Gamma^p_{nq} - \Gamma^n_{aq}\Gamma^p_{nb} \\ =& \frac{A_{ap,n}A_{bq,n} + A_{aq,n}A_{bp,n} - E_n(A_1)E_n(A_2)\delta_{ab}\delta_{pq}}{(A_1 - A_3)(A_2 - A_3)}. \end{aligned} \tag{3.17}$$

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On the other hand, from (2.21), we have

$$R_{apbq} = (B_aB_p + A_a + A_p)\delta_{ab}\delta_{pq} = (B_aB_p + A_1 + A_2)\delta_{ab}\delta_{pq}. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\begin{aligned} &A_{ap,n}A_{bq,n} + A_{aq,n}A_{bp,n} \\ &\quad = \{(A_1 - A_3)(A_2 - A_3)(B_aB_p + A_1 + A_2) + E_n(A_1)E_n(A_2)\}\delta_{ab}\delta_{pq}. \end{aligned}$$

Putting

$$\varrho_{a,p} = (A_1 - A_3)(A_2 - A_3)(B_aB_p + A_1 + A_2) + E_n(A_1)E_n(A_2), \tag{3.19}$$

we get

$$A_{ap,n}A_{bq,n} + A_{aq,n}A_{bp,n} = \varrho_{a,p}\delta_{ab}\delta_{pq}.$$

If $a = b$, from $B_p = \frac{1}{\lambda}(D_p - A_2)$, we have

$$2A_{ap,n}A_{aq,n} = \varrho_{a,p}\delta_{pq}, \tag{3.20}$$

and

$$\varrho_{a,p} = (A_1 - A_3)(A_2 - A_3)\left(\frac{B_a}{\lambda}D_p + A_1 + \left(1 - \frac{B_a}{\lambda}\right)A_2\right) + E_n(A_1)E_n(A_2). \tag{3.21}$$

If $p = q$, from $B_a = \frac{1}{\lambda}(D_a - A_1)$, we have

$$2A_{ap,n}A_{bp,n} = \varrho_{a,p}\delta_{ab}, \tag{3.22}$$

and

$$\varrho_{a,p} = (A_1 - A_3)(A_2 - A_3)\left(\frac{B_p}{\lambda}D_a + A_2 + \left(1 - \frac{B_p}{\lambda}\right)A_1\right) + E_n(A_1)E_n(A_2). \tag{3.23}$$

Since $m_1 \geq 2$ and $m_2 \geq 2$, we may consider two cases:

If at least one of B_a and B_p is zero, for example $B_a = 0$, from (3.21), we know that $\varrho_{a,p}$ is irrelevant to p. Assume that exists one p_0 such that $A_{ap_0,n} \neq 0$ for any a, $1 \leq a \leq m_1$. By (3.20), we have $A_{ap,n} = 0$ for $p(p \neq p_0)$. By (3.20) again, if $p = q$, then $A^2_{ap,n} = \frac{\varrho_{a,p}}{2}$ for any p. Since $\varrho_{a,p}$ is irrelevant to p, we have $A^2_{ap_0,n} = \frac{\varrho_{a,p_0}}{2} = \frac{\varrho_{a,p}}{2} = A^2_{ap,n} = 0$ for $p_0, p(p \neq p_0)$. Thus, $A_{ap_0,n} = 0$, this is a contradiction. Therefore, we have $A_{ap,n} = 0$ for any p and a. If, for example $B_p = 0$, from (3.22), (3.23) and by the same assertion, we have $A_{ap,n} = 0$ for any a and p.

If $B_a \neq 0$ and $B_p \neq 0$, from (3.21) and (3.23), we know that $\varrho_{a,p}$ depends on a, p. If $D_1 = D_2 = \cdots = D_n$, from $B_a = \frac{1}{\lambda}(D_a - A_1)$ and $B_p = \frac{1}{\lambda}(D_p - A_2)$, we know that for any a, all B_a are equal and for any p, all B_p are equal. From (3.21) and (3.23), we see that for any a and p, all $\varrho_{a,p}$ are equal. By the same proof as above, we know that $A_{ap,n} = 0$ for any a and p.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, since $m_2 \geq 2$ and $m_1 \geq 2$, we may prove that there exists at most one p such that $\varrho_{a,p} \neq 0$ for any a, $1 \leq a \leq m_1$ and there exists at most one a such that $\varrho_{a,p} \neq 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$.

In fact, assume that exists more than one p, for example p_1, p_2 $(p_1 \neq p_2)$, such that $\varrho_{a,p_1} \neq 0$, $\varrho_{a,p_1} \neq 0$. By (3.20), we have $A^2_{ap,n} = \frac{\varrho_{a,p}}{2}$ for any p. Thus, $A^2_{ap_1,n} = \frac{\varrho_{a,p_1}}{2} \neq 0$, $A^2_{ap_2,n} = \frac{\varrho_{a,p_2}}{2} \neq 0$. By (3.20) again, we see that $A_{ap_1,n}A_{ap_2,n} = 0$, this is a contradiction. Thus, we know that there exists at most one p such that $\varrho_{a,p} \neq 0$ for any a, $1 \leq a \leq m_1$. By the same proof as above, we also know that there exists at most one a such that $\varrho_{a,p} \neq 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$.

If for all p, $\varrho_{a,p} = 0$, $1 \leq a \leq m_1$, by (3.20), we have $A_{ap,n} = 0$ for any p and a.

If there exists one p_0 such that $\varrho_{a,p_0} \neq 0$, $\varrho_{a,p} = 0$, $(p \neq p_0)$, in this case, we must have that there exists one a_0 such that $\varrho_{a_0,p} \neq 0$, $\varrho_{a,p} = 0$, $(a \neq a_0)$. In fact, if for all a, $\varrho_{a,p} = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$, this is in contradiction with $\varrho_{a,p_0} \neq 0$. Thus, for $1 \leq a \leq m_1$, from (3.23), we have

$$\frac{B_p}{\lambda}D_a + A_2 + \left(1 - \frac{B_p}{\lambda}\right)A_1 = -\frac{E_n(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)}, \quad p \neq p_0. \tag{3.24}$$

Since $B_p \neq 0$, by (3.24), we know that $D_a = D_b$ for any a, b, $1 \leq a, b \leq m_1$. Since $B_a = \frac{1}{\lambda}(D_a - A_1)$, we easily see that $B_a = B_b$.

By the same assertion as above, from (3.21), we have $D_p = D_q$ and $B_p = B_q$ for any p, q, $m_1 + 1 \leq p, q \leq m_1 + m_2$. Thus, from (3.21) and (3.23), we see that $\varrho_{a,p_0} = \varrho_{a,p}$. This is in contradiction with the assumption that $\varrho_{a,p_0} \neq 0$, $\varrho_{a,p} = 0$, $(p \neq p_0)$. Thus, the case that there exists one p_0 such that $\varrho_{a,p_0} \neq 0$, $\varrho_{a,p} = 0$, $(p \neq p_0)$ does not occur.

(ii) If $m_1 = 1$, from (3.3) and (3.11), we have

$$\Gamma^1_{pq} = \Gamma^n_{pq} = 0, \quad \Gamma^p_{nn} = \Gamma^p_{11} = 0, \tag{3.25}$$

$$\begin{aligned} \Gamma^1_{nn} &= \frac{E_1(A_3)}{A_3 - A_1}, \quad \Gamma^1_{pp} = \frac{E_1(A_2)}{A_2 - A_1}, \\ \Gamma^n_{11} &= \frac{E_n(A_1)}{A_1 - A_3}, \quad \Gamma^n_{pp} = \frac{E_n(A_2)}{A_2 - A_3}. \end{aligned} \tag{3.26}$$

From (3.4), (3.11), (3.12), (3.25) and (3.26), by the similar calculation as in (i), we have

$$2A_{1p,n}A_{1q,n} = \upsilon_p\delta_{pq}, \tag{3.27}$$

for any p and q, where

$$\begin{aligned} \upsilon_p = (A_1 - A_2)(A_1 - A_3)\Bigg\{&\frac{B_n}{\lambda}D_p + A_3 + \left(1 - \frac{B_n}{\lambda}\right)A_2 \\ &+ \frac{E_1(A_2)E_1(A_3)}{(A_1 - A_2)(A_1 - A_3)} + \frac{[E_n(A_2) - E_n(A_3)]E_n(A_2)}{(A_2 - A_3)^2} - \frac{E_n(E_n(A_2))}{A_2 - A_3} + \frac{E_n(A_2)}{(A_2 - A_3)^2}\Bigg\}. \end{aligned} \tag{3.28}$$

Since $m_1 = 1$ and $m_2 \geq 2$, we may consider two cases:

If $B_n = 0$, from (3.28), we know that υ_p is irrelevant to p. By the same proof as in (i), we see that $A_{1p,n} = 0$ for any p.

If $B_n \neq 0$, from (3.28), we know that υ_p depends on p. If $D_1 = D_2 = \cdots = D_n$, from (3.28), we see that for any p, all υ_p are equal. By the same proof as in (i), we see that $A_{1p,n} = 0$ for any p.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, since $m_2 \geq 2$, by the same proof as in (i), we easily know that there exists at most one p such that $\upsilon_p \neq 0$.

If for any p, $\upsilon_p = 0$, by (3.27), we have $A_{1p,n} = 0$.

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If there is p_0, such that $\upsilon_{p_0} \neq 0$ and $\upsilon_p = 0$, for other $p(p \neq p_0)$, we have

$$\upsilon_{p_0} = \upsilon_{p_0} - \upsilon_p = (A_1 - A_2)(A_1 - A_3)\frac{B_n}{\lambda}(D_{p_0} - D_p). \tag{3.29}$$

On the other hand, since $m_1 = 1$, $m_3 = 1$ and $A_{ij,k}$ is symmetric for all indices i, j and k, interchanging 1 and n in the above equations, we also have

$$2A_{np,1}A_{nq,1} = \omega_p \delta_{pq}, \tag{3.30}$$

where

$$\omega_p = (A_3 - A_2)(A_3 - A_1)\left\{\frac{B_1}{\lambda}D_p + A_1 + \left(1 - \frac{B_1}{\lambda}\right)A_2 + \frac{E_n(A_2)E_n(A_1)}{(A_3 - A_2)(A_3 - A_1)} + \frac{[E_1(A_2) - E_1(A_1)]E_1(A_2)}{(A_2 - A_1)^2} - \frac{E_1(E_1(A_2))}{A_2 - A_1} + \frac{E_1(A_2)}{(A_2 - A_1)^2}\right\}. \tag{3.31}$$

If $B_1 = 0$, from (3.31), we know that ω_p is irrelevant to p. By the same assertion as above, we know that $A_{1p,n} = 0$ for any p.

If $B_1 \neq 0$, from (3.31), we know that ω_p depends on p. If $D_1 = D_2 = \cdots = D_n$, from (3.31), we see that for any p, all ω_p are equal. By the same assertion as above, we see that $A_{1p,n} = 0$ for any p.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, since $m_2 \geq 2$, by the same assertion as above, we know that there exists at most one p such that $\omega_p \neq 0$.

If for any p, $\omega_p = 0$, by (3.30), we have $A_{1p,n} = 0$. Otherwise, we may prove that $\omega_{p_0} \neq 0$ for the above p_0 in (3.29). In fact, by (3.27), we have $A^2_{1p_0,n} = \frac{\upsilon_{p_0}}{2} \neq 0$. On the other hand, by (3.30), we have $A^2_{np_0,1} = \frac{\omega_{p_0}}{2}$. Since $A_{1p_0,n} = A_{np_0,1}$, we have $\omega_{p_0} = \upsilon_{p_0} \neq 0$. By (3.31), we also have

$$\upsilon_{p_0} = \omega_{p_0} = \omega_{p_0} - \omega_p = (A_3 - A_2)(A_3 - A_1)\frac{B_1}{\lambda}(D_{p_0} - D_p). \tag{3.32}$$

Thus, from (3.29) and (3.32), we have

$$(A_1 - A_3)\left\{(A_1 - A_2)\frac{B_n}{\lambda} + (A_3 - A_2)\frac{B_1}{\lambda}\right\}(D_{p_0} - D_p) = 0,$$

that is

$$\{(A_1 - A_2)(D_n - A_3) + (A_3 - A_2)(D_1 - A_1)\}(D_{p_0} - D_p) = 0.$$

If $D_{p_0} = D_p$, by (3.29), we have $\upsilon_{p_0} = \upsilon_p$, this contradicts with $\upsilon_{p_0} \neq 0, \upsilon_p = 0, (p \neq p_0)$. Therefore,

$$(A_1 - A_2)(D_n - A_3) + (A_3 - A_2)(D_1 - A_1) = 0. \tag{3.33}$$

Thus,

$$(2A_3 - A_2 - D_n)dA_1 - (A_3 + A_1 - D_n - D_1)dA_2 + (2A_1 - A_2 - D_1)dA_3 = 0. \tag{3.34}$$

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If there is a point such that at this point,

$$2A_3 - A_2 - D_n = 0, \tag{3.35-1}$$

$$A_3 + A_1 - D_n - D_1 = 0, \tag{3.35-2}$$

$$2A_1 - A_2 - D_1 = 0. \tag{3.35-3}$$

From (3.35-1)–(3.35-3), we have $A_3 - A_2 = A_2 - A_1$ at this point. By (3.33), we have

$$A_1 - A_3 - D_1 + D_n = 0, \tag{3.36}$$

at this point. From (3.35-1)–(3.35-3) and (3.36), we see that $A_1 = A_2 = A_3$ at this point. This contradicts with the assumption of (1) in Proposition 3.2. Thus, the coefficients of (3.34) are not simultaneously zero at any point. From (3.9) and (3.34), we easily know that $dA_1 = dA_2 = dA_3 = 0$, that is A_1, A_2 and A_3 are constants.

(2) Since **x** has three distinct Möbius principal curvatures B_1, B_2 and B_3 one of which is simple and $n \geq 4$, without loss of generality, we may assume that $m_3 = 1$, $m_1 m_2 \geq 2$ and $m_2 \geq 2$.

From (2.14), we have

$$\frac{m_1 dB_1}{B_3 - B_2} = \frac{m_2 dB_2}{B_1 - B_3} = \frac{m_3 dB_3}{B_2 - B_1}. \tag{3.37}$$

(i) If $m_1 \geq 2$, from Proposition 3.1 and (3.37), by the same method in the proof of (1), we will obtain that

$$2B_{ap,n}B_{aq,n} = \varrho'_{a,p}\delta_{pq}, \tag{3.38}$$

where

$$\varrho'_{a,p} = (B_1 - B_3)(B_2 - B_3)\big(D_a + D_p + B_1B_2 - \lambda(B_1 + B_2)\big) + E_n(B_1)E_n(B_2). \tag{3.39}$$

Since $m_1 \geq 2$ and $m_2 \geq 2$, if $D_1 = D_2 = \cdots = D_n$, from (3.39), we see that for any a and p, all $\varrho'_{a,p}$ are equal. By the same proof as in (1), we know that $B_{ap,n} = 0$ for any a and p.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, by the same method in the proof of (1), we may obtain that there exists at most one p such that $\varrho'_{a,p} \neq 0$ for any a, $1 \leq a \leq m_1$.

If for all p, $\varrho'_{a,p} = 0$, $1 \leq a \leq m_1$, by (3.39), we easily see that $B_{ap,n} = 0$ for any a and p. Otherwise, by the same method in the proof of (1), we also conclude.

(ii) If $m_1 = 1$, by the same method in the proof of (1), we have

$$2B_{1p,n}B_{1q,n} = \upsilon'_p\delta_{pq}, \tag{3.40}$$

for any p and q, where

$$\upsilon'_p = (B_1 - B_2)(B_1 - B_3)\Bigg\{D_p + D_n + B_2B_3 - \lambda(B_2 + B_3) + \frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} + \frac{[E_n(B_2) - E_n(B_3)]E_n(B_2)}{(B_2 - B_3)^2} - \frac{E_n(E_n(B_2))}{B_2 - B_3} + \frac{E_n(B_2)}{(B_2 - B_3)^2}\Bigg\}. \tag{3.41}$$

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Since $m_1 = 1$ and $m_2 \geq 2$, if $D_1 = D_2 = \cdots = D_n$, from (3.41), we see that for any p, all υ'_p are equal. By the same proof as in (1), we know that $B_{1p,n} = 0$ for any p.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, we also see that there exists at most one p such that $\upsilon'_p \neq 0$.

If for any p, $\upsilon'_p = 0$, by (3.41), we have $B_{1p,n} = 0$. Otherwise, by the same method in the proof of (1), we see that B_1, B_2 and B_3 are constants. This completes the proof of Proposition 3.2.

EXAMPLE 3.3. [**7**, **8**]. For any natural number p, q, $p + q < n$ and real number $r \in (0, 1)$, consider the immersed hypersurface $u : S^p(r) \times S^q(\sqrt{1 - r^2}) \times \mathbf{R}^+ \times \mathbf{R}^{n-p-q-1} \mapsto \mathbf{R}^{n+1}$

$$u = (tu', tu'', u'''),$$
$$u' \in S^p(r) \subset \mathbf{R}^{p+1}, \quad u'' \in S^q(\sqrt{1 - r^2}) \subset \mathbf{R}^{q+1}, \quad u''' \in \mathbf{R}^{n-p-q-1},$$

then $\mathbf{x} = \sigma \circ u : S^p(r) \times S^q(\sqrt{1 - r^2}) \times \mathbf{R}^+ \times \mathbf{R}^{n-p-q-1} \mapsto S^{n+1}(1)$ is a hypersurface in $S^{n+1}(1)$ without umbilical points and with vanishing Möbius form, it is denoted by $CSS(p, q, r)$. From [**7**] and [**8**], by a direct calculation, we know that $CSS(p, q, r)$ has three distinct Möbius principal curvatures. In particular, if $p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$ then $CSS(p, q, r)$ has exactly three distinct Blaschke eigenvalues.

EXAMPLE 3.4. [**7**, **19**]. Let $\lambda \in \mathbf{R}$. For any integers n and k satisfying $n \geq 3$ and $2 \leq k \leq n - 1$, let $\tilde{y}_1 : M_1 \mapsto S^{k+1}(r) \subset \mathbf{R}^{k+2}$ be an immersed hypersurface without umbilical points such that the scalar curvature S_1 and the mean curvature H_1 of it satisfy

$$S_1 = \{nk(k-1) - (n-1)r^2\}/nr^2 + n(n-1)\lambda^2, \quad H_1 = -\frac{n}{k}\lambda.$$

Let $\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : \mathbf{H}^{n-k}(-1/r^2) \mapsto \mathbf{R}_1^{n-k+1}$ be the canonical embedding and $\tilde{M}^n = M_1 \times \mathbf{H}^{n-k}(-1/r^2)$, $\tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)$. We have that $\tilde{Y} : \tilde{M}^n \mapsto \mathbf{R}_1^{n+3}$ is an immersion, satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_1 = 0$ and inducing a Riemannian metric $g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2$. Obviously, $(\tilde{M}^n, g) = (M_1, d\tilde{y}_1^2) \times (\mathbf{H}^{n-k}(-1/r^2), \langle d\tilde{y}, d\tilde{y} \rangle_1)$ is a Riemannian manifold. Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0$, $\tilde{x}_2 = \tilde{y}_2/\tilde{y}_0$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$, then $|\tilde{\mathbf{x}}|^2 = 1$. Thus, $\tilde{\mathbf{x}} : \tilde{M}^n \mapsto S^{n+1}$ defines an immersed hypersurface without umbilical points. From [**7**] and [**19**], we know that the components of the Blaschke tensor, the Möbius second fundamental form and the para-Blaschke tensor of $\tilde{\mathbf{x}}$ are

$$A_{ij} = \left(\frac{1}{2r^2} - \frac{\lambda^2}{2}\right)\delta_{ij} - \lambda h_{ij}, \quad B_{ij} = h_{ij} + \lambda\delta_{ij},$$
$$D_{ij} = \left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_{ij}, \quad 1 \leq i, j \leq k,$$
$$A_{ij} = \left(-\frac{1}{2r^2} - \frac{\lambda^2}{2}\right)\delta_{ij}, \quad B_{ij} = \lambda\delta_{ij},$$
$$D_{ij} = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_{ij}, \quad k+1 \leq i, j \leq n,$$
$$A_{ij} = 0, \quad B_{ij} = 0, \quad D_{ij} = 0,$$
$$1 \leq i \leq k, k+1 \leq j \leq n, \quad \text{or} \quad 1 \leq j \leq k, k+1 \leq i \leq n.$$

Thus, if $\lambda = 0$, $\tilde{\mathbf{x}}$ has exactly two distinct constant Blaschke eigenvalues.

In addition, from [**7**], we know that $\tilde{\mathbf{x}}$ is Blaschke isoparametric with three Blaschke eigenvalues and four Möbius principal curvatures if and only if the corresponding hypersurface $\tilde{y}_1$ is a non-minimal Euclidean isoparametric with three distinct principal curvatures μ_1, μ_2 and μ_3 satisfying $\lambda\mu_i = \frac{1}{r^2}$ for some $i \in \{1, 2, 3\}$. From [**3**], we know that such hypersurfaces $\tilde{y}_1$ do exist. If $\tilde{\mathbf{x}}$ has a simple Blaschke eigenvalue, then $k = 3$.

EXAMPLE 3.5. [**7**, **19**]. Let $\lambda \in \mathbf{R}$. For any integers n and k satisfying $n \geq 3$ and $2 \leq k \leq n-1$, let $\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : M_1 \mapsto \mathbf{H}^{k+1}(-1/r^2) \subset \mathbf{R}_1^{k+2}$ be an immersed hypersurface without umbilical points such that the scalar curvature S_1 and the mean curvature H_1 of it satisfy

$$S_1 = -\{nk(k-1) + (n-1)r^2\}/nr^2 + n(n-1)\lambda^2, \quad H_1 = -\frac{n}{k}\lambda.$$

Let $\tilde{y}_2 : S^{n-k}(r) \mapsto \mathbf{R}^{n-k+1}$ be the canonical embedding and $\tilde{M}^n = M_1 \times S^{n-k}(r)$, $\tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)$. $\tilde{Y} : \tilde{M}^n \mapsto \mathbf{R}_1^{n+3}$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_1 = 0$ and inducing a Riemannian metric $g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2$. Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0, \tilde{x}_2 = \tilde{y}_2/\tilde{y}_0, \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$, $|\tilde{\mathbf{x}}|^2 = 1$, $\tilde{\mathbf{x}} : \tilde{M}^n \mapsto S^{n+1}$ is an immersed hypersurface without umbilical points. From [**7**] and [**19**], we know that

$$\begin{aligned}
&A_{ij} = -\left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_{ij} - \lambda h_{ij}, \quad B_{ij} = h_{ij} + \lambda\delta_{ij},\\
&D_{ij} = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_{ij}, \quad 1 \leq i, j \leq k,\\
&A_{ij} = \left(\frac{1}{2r^2} - \frac{\lambda^2}{2}\right)\delta_{ij}, \quad B_{ij} = \lambda\delta_{ij},\\
&D_{ij} = \left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_{ij}, \quad k+1 \leq i, j \leq n,\\
&A_{ij} = 0, \quad B_{ij} = 0, \quad D_{ij} = 0,\\
&1 \leq i \leq k, k+1 \leq j \leq n, \quad \text{or} \quad 1 \leq j \leq k, k+1 \leq i \leq n.
\end{aligned}$$

Thus, if $\lambda = 0$, $\tilde{\mathbf{x}}$ has exactly two distinct constant Blaschke eigenvalues.

In addition, from [**7**], we know that $\tilde{\mathbf{x}}$ is Blaschke isoparametric with three Blaschke eigenvalues and four Möbius principal curvatures if and only if the corresponding hypersurface $\tilde{y}_1$ is a non-minimal Euclidean isoparametric with three distinct principal curvatures μ_1, μ_2 and μ_3 satisfying $\lambda\mu_i = -\frac{1}{r^2}$ for some $i \in \{1, 2, 3\}$. But, from [**4**], we know that such a hypersurface $\tilde{y}_1$ does not exist, since if $\tilde{y}_1$ is isoparametric, then it has at most two principal curvatures.

4. Proof of theorems. We firstly state an important result due to Li and Wang [**11**]:

THEOREM 4.1. *For an immersed hypersurface* $\mathbf{x} : M \mapsto S^{n+1}(1)$ *without umbilical points and with vanishing Möbius form, if the para-Blaschke tensor* D *satisfies* $D = fg$ *for some function* f *on* M*, then* f *is constant and* $\mathbf{x}$ *is locally Möbius equivalent to one of the following:*

(1) *an immersed hypersurface* $\mathbf{x}: M \mapsto S^{n+1}(1)$ *with constant scalar curvature and constant mean curvature, or*
(2) *the image under* σ *of an immersed hypersurface in* $\mathbf{R}^{n+1}$ *with constant scalar curvature and constant mean curvature, or*
(3) *the image under* τ *of an immersed hypersurface in* $\mathbf{H}^{n+1}$ *with constant scalar curvature and constant mean curvature.*

Proof of Theorem 1.2. Let A_1, A_2 and A_3 be the three distinct Blaschke eigenvalues with multiplicities m_1, m_2 and m_3 and one of which is simple. We consider two cases:

(1) If all of the para-Blaschkes eigenvalues of $\mathbf{x}$ are equal, that is $D_1 = D_2 = \ldots = D_n$, by Theorem 4.1, we know that $\mathbf{x}$ is locally Möbius equivalent to one of an immersed hypersurface $\mathbf{x}: M \mapsto S^{n+1}(1)$ with constant scalar curvature and constant mean curvature, or the image under σ of an immersed hypersurface in $\mathbf{R}^{n+1}$ with constant scalar curvature and constant mean curvature, or the image under τ of an immersed hypersurface in $\mathbf{H}^{n+1}$ with constant scalar curvature and constant mean curvature.
(2) If not all of the para-Blaschkes eigenvalues of $\mathbf{x}$ are equal, by Proposition 3.2, we know that A_1, A_2 and A_3 are constants, or $A_{ap,n} = 0$ for any a and p.

We may prove that if $A_{ap,n} = 0$ for any a and p, then A_1, A_2 and A_3 are also constants. If fact, without loss of generality, we assume that $m_2 \geq 2$.

(i) If $m_1 = 1$, since $A_{1p,n} = 0$ for any p, putting $p = q$ in (3.27), we have $\upsilon_p = 0$. By (3.28),

$$\frac{B_n}{\lambda}D_p + A_3 + \left(1 - \frac{B_n}{\lambda}\right)A_2 \tag{4.1}$$
$$= -\frac{E_1(A_2)E_1(A_3)}{(A_1 - A_2)(A_1 - A_3)} - \frac{[E_n(A_2) - E_n(A_3)]E_n(A_2)}{(A_2 - A_3)^2} + \frac{E_n(E_n(A_2))}{A_2 - A_3} - \frac{E_n(A_2)}{(A_2 - A_3)^2}.$$

If $B_n = 0$, we see that $A_3 = D_n$ is constant. By (3.9), we have that A_1 and A_2 are constants.

If $B_n \neq 0$, by (4.1), we know that for any p, all D_p are equal. Thus, $\mathbf{x}$ has at most three distinct para-Blaschke eigenvalues D_1, D_p and D_n with multiplicities $1, m_2$ and 1.

(ii) If $m_1 \geq 2$, putting $p = q$ in (3.20) and $a = b$ in (3.22), we have $\varrho_{a,q} = 0$. Thus, for any a and p, by (3.21) and (3.23), we have

$$\frac{B_a}{\lambda}D_p + A_1 + \left(1 - \frac{B_a}{\lambda}\right)A_2 = -\frac{E_n(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)}, \tag{4.2}$$
$$\frac{B_p}{\lambda}D_a + A_2 + \left(1 - \frac{B_p}{\lambda}\right)A_1 = -\frac{E_n(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)}. \tag{4.3}$$

If at least one of B_p and B_a is zero, we easily see that A_1, A_2 and A_3 are constants.

If all of B_p and B_a are not zero, by (4.2) and (4.3), we easily see that for any a and b, $D_a = D_b$ and for any p and q, $D_p = D_q$. Thus, $\mathbf{x}$ has at most three distinct para-Blaschke eigenvalues D_a, D_p and D_n with multiplicities m_1, m_2 and 1.

Let D_a, D_p and D_n be the three constant para-Blaschke eigenvalues with multiplicities m_1, m_2 and 1. We choose a local orthonormal basis $\{E_1, \ldots, E_n\}$ such that E_i is the unit para-Blaschke tensor of D_i. By (2.24),

$$D_{ij,k} = E_k(D_i)\delta_{ij} + \Gamma^j_{ik}(D_i - D_j),$$

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where Γ^j_{ik} is the Levi–Civita connection of g given by $\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k$, $\Gamma^j_{ik} = -\Gamma^i_{jk}$. By (2.26), we have $D_{ii,j} = D_{ij,i}$. Thus,

$$E_j(D_i) = \Gamma^j_{ii}(D_i - D_j), \quad i \neq j. \tag{4.4}$$

If the number of the distinct para-Blaschke eigenvalues of D_a, D_p and D_n is two, when $m_1 = 1$, without loss of generality, we assume that $D_a = D_n \neq D_p$. By (4.4),

$$0 = E_1(D_p) = \Gamma^1_{pp}(D_p - D_a), \;\; 0 = E_n(D_p) = \Gamma^n_{pp}(D_p - D_n). \tag{4.5}$$

Thus, $\Gamma^1_{pp} = \Gamma^n_{pp} = 0$. On the other hand, by (2.16) and $A_{ii,j} = A_{ij,i}$, we have

$$E_j(A_i) = \Gamma^j_{ii}(A_i - A_j), \quad i \neq j, \tag{4.6}$$

where $\{E_1, \ldots, E_n\}$ a local orthonormal basis such that E_i is the unit Blaschke tensor of A_i. Thus,

$$E_1(A_2) = \Gamma^1_{pp}(A_2 - A_1) = 0, \;\; E_n(A_2) = \Gamma^n_{pp}(A_2 - A_3) = 0.$$

From Proposition 3.1, we have $E_p(A_2) = 0$, $2 \leq p \leq 1 + m_2$. Thus, A_2 is constant. By (3.9), we have that A_1 and A_3 are constants.

When $m_1 \geq 2$, without loss of generality, we assume that $D_a = D_p \neq D_n$. From (4.4),

$$0 = E_n(D_a) = \Gamma^n_{aa}(D_a - D_n).$$

Thus, $\Gamma^n_{aa} = 0$. On the other hand, by (4.6),

$$E_n(A_1) = \Gamma^n_{aa}(A_1 - A_3) = 0.$$

From Proposition 3.1, we have $E_a(A_1) = 0$, $1 \leq a \leq m_1$ and $E_p(A_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$. By (3.5) and (3.6), we have

$$\frac{E_i(A_1)}{m_2\xi_3 - m_3\xi_2} = \frac{E_i(A_2)}{m_3\xi_1 - m_1\xi_3} = \frac{E_i(A_3)}{m_1\xi_2 - m_2\xi_1}. \tag{4.7}$$

Thus, by (4.7), we have $E_a(A_3) = E_p(A_3) = E_n(A_3) = 0$ and A_3 is constant. By (3.9) again, we know that A_1 and A_2 are constants.

If the number of the distinct para-Blaschke eigenvalues of D_1, D_p and D_n is three, when $m_1 = 1$, by (4.4),

$$0 = E_1(D_p) = \Gamma^1_{pp}(D_p - D_1), \;\; 0 = E_n(D_p) = \Gamma^n_{pp}(D_p - D_n). \tag{4.8}$$

Thus, $\Gamma^1_{pp} = \Gamma^n_{pp} = 0$. On the other hand, by (4.6),

$$E_1(A_2) = \Gamma^1_{pp}(A_2 - A_1) = 0, \;\; E_n(A_2) = \Gamma^n_{pp}(A_2 - A_3) = 0.$$

From Proposition 3.1, we have $E_p(A_2) = 0$, $2 \leq p \leq 1 + m_2$. Thus, A_2 is constant. By (3.9) again, we know that A_1 and A_3 are constants.

When $m_1 \geq 2$, by (4.4),

$$0 = E_n(D_a) = \Gamma^n_{aa}(D_a - D_n).$$

Therefore, $\Gamma^n_{aa} = 0$. On the other hand, by (4.6),

$$E_n(A_1) = \Gamma^n_{aa}(A_1 - A_3) = 0.$$

From Proposition 3.1, we have $E_a(A_1) = 0$, $1 \le a \le m_1$ and $E_p(A_2) = 0$, $m_1 + 1 \le p \le m_1 + m_2$. From (4.7), we have $E_a(A_3) = E_p(A_3) = E_n(A_3) = 0$ and A_3 is constant. By (3.9) again, we know that A_1 and A_2 are constants.

Since A_1, A_2 and A_3 are constants and $n \ge 4$, from Theorem 1.1, we see that **x** is locally Möbius equivalent to $CSS(p, q, r)$ for some constants p, q, r, $p \ne q$ and $r \ne \frac{1}{\sqrt{2}}$, or one of the hypersurfaces as indicated in Example 3.4 where $k = 3$ and $\tilde{y}_1 : M_1 \mapsto S^4(r)$ is one of Cartan's non-minimal isoparametric hypersurfaces with three principal curvatures satisfying $\lambda\mu_i = \frac{1}{r^2}$ for some $i \in \{1, 2, 3\}$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.4. Let B_1, B_2 and B_3 be the three distinct Blaschke eigenvalues with multiplicities m_1, m_2 and m_3 and one of which is simple. We consider two cases:

(1) If all of the para-Blaschkes eigenvalues of **x** are equal, by Theorem 4.1, we see that Theorem 1.4 is true.

(2) If not all of the para-Blaschkes eigenvalues of **x** are equal, by Proposition 3.2, we know that B_1, B_2 and B_3 are constants, or $B_{ap,n} = 0$ for any a and p. In the latter case, without loss of generality, we assume that $m_2 \ge 2$.

(i) If $m_1 = 1$, since $B_{1p,n} = 0$ for any p, putting $p = q$ in (3.40), we have $\upsilon'_p = 0$. By (3.41),

$$\begin{aligned} &D_p + D_n + B_2B_3 - \lambda(B_2 + B_3) \qquad (4.9)\\ &= -\frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} - \frac{[E_n(B_2) - E_n(B_3)]E_n(B_2)}{(B_2 - B_3)^2} + \frac{E_n(E_n(B_2))}{B_2 - B_3} - \frac{E_n(B_2)}{(B_2 - B_3)^2}. \end{aligned}$$

Thus, we know that for any p, all D_p are equal and **x** has at most three distinct para-Blaschke eigenvalues D_1, D_p and D_n with multiplicities 1, m_2 and 1.

(ii) If $m_1 \ge 2$, putting $p = q$ in (3.38), we have $\varrho'_{a,q} = 0$. By (3.39), we have

$$D_a + D_p + B_1B_2 - \lambda(B_1 + B_2) = -\frac{E_n(B_1)E_n(B_2)}{(B_1 - B_3)(B_2 - B_3)}. \qquad (4.10)$$

By (4.10), we easily see that **x** has at most three distinct para-Blaschke eigenvalues D_a, D_p and D_n with multiplicities m_1, m_2 and 1.

If the number of the distinct para-Blaschke eigenvalues of D_a, D_p and D_n is two, when $m_1 = 1$, without loss of generality, we assume that $D_a = D_n \ne D_p$. By (4.5), we have $\Gamma^1_{pp} = \Gamma^n_{pp} = 0$. On the other hand, by (2.17) and $B_{ii,j} = B_{ij,i}$, we have

$$E_j(B_i) = \Gamma^j_{ii}(B_i - B_j), \quad i \ne j. \qquad (4.11)$$

Thus,

$$E_1(B_2) = \Gamma^1_{pp}(B_2 - B_1) = 0, \quad E_n(B_2) = \Gamma^n_{pp}(B_2 - B_3) = 0.$$

From Proposition 3.1 and (3.37), by the similar proof of Theorem 1.2, we know that B_1, B_2 and B_3 are constants.

When $m_1 \geq 2$, without loss of generality, we assume that $D_a = D_p \neq D_n$. From (4.4), we have $\Gamma^n_{aa} = 0$. On the other hand, by (4.11),

$$E_n(B_1) = \Gamma^n_{aa}(B_1 - B_3) = 0.$$

From Proposition 3.1, we have $E_a(B_1) = 0$, $1 \leq a \leq m_1$ and $E_p(B_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$. Combining with (3.37) and by the similar proof of Theorem 1.2, we know that B_1, B_2 and B_3 are constants.

If the number of the distinct para-Blaschke eigenvalues of D_1, D_p and D_n is three, when $m_1 = 1$, by (4.8), we have $\Gamma^1_{pp} = \Gamma^n_{pp} = 0$. On the other hand, by (4.11),

$$E_1(B_2) = \Gamma^1_{pp}(B_2 - B_1) = 0, \quad E_n(B_2) = \Gamma^n_{pp}(B_2 - B_3) = 0.$$

From Proposition 3.1 and (3.37), by the similar proof of Theorem 1.2, we know that B_1, B_2 and B_3 are constants.

When $m_1 \geq 2$, by (4.4), $\Gamma^n_{aa} = 0$. On the other hand, by (4.11),

$$E_n(B_1) = \Gamma^n_{aa}(B_1 - B_3) = 0.$$

From Proposition 3.1, we have $E_a(B_1) = 0$, $1 \leq a \leq m_1$ and $E_p(B_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$. Combining with (3.37), we know that B_1, B_2 and B_3 are constants.

Since B_1, B_2 and B_3 are constants, from Theorem 1.3, we see that Theorem 1.4 is true. This completes the proof of Theorem 1.4.

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