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UNIFORM APPROXIMATION BY POLYNOMIALS WITH VARIABLE EXPONENTS

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Introduction. We examine questions related to approximating functions by sums of the form

$$(1)\quad \sum_{i=1}^{n} a_i x^{\delta_i} \quad a_i, \delta_i \text{ real.}$$

We focus on approximations to functions given by the integral transformation

$$(2)\quad f(x) = \int_0^\infty x^t d\gamma(t)$$

where γ is a positive measure. Approximations to this class of functions (Laplace transforms in the variable $-\ln x$) are particularly well behaved (see Theorem 1). Questions concerning existence, uniqueness and characterization of such approximations have been thoroughly examined in the equivalent setting of exponential sum approximations (see [**3**], [**4**], [**6**] and [**9**]). Less well studied is the order of convergence of the approximation. This is the problem we address. Part of the motivation for using sums of the form (1), which we shall call Gaussian sums, stems from the observation that all analytic functions with Taylor series expansion having positive coefficients are of the form (2). Thus, we may ask the apparently natural question of how well e^x may be approximated by Gaussian sums. We have called these approximations "Gaussian" because of the close connection to Gaussian quadrature (see Section 4).

The first section of the paper collects together the necessary characterizations of best approximants. We then derive a number of theorems that compare Gaussian approximations to polynomial approximations and to other Gaussian approximations. For example: if f and g are transforms of type (2) associated with positive measures γ and β respectively and if $\beta - \gamma$ is also a positive measure then the best Gaussian approximation to f is better than the best Gaussian approximation to g of comparable order. We

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also show that Gaussian approximations to functions of the form $\int_0^1 x^t d\gamma(t)$ are very much better than polynominal approximations, that such approximations to $1 + x + \ldots + x^n$ are considerably better while little improvement is to be expected in approximating functions like e^x.

1. Notation. Let Π_n be the real algebraic polynomials of degree at most n. Let $\mathscr{G}_n$ be the set

$$\left\{ \sum_{i=1}^{n} a_i x^{\delta_i} | a_i, \delta_i \ real \right\}.$$

For a continuous function f let

$$(1) \quad E_n(f: [a, b]) = \inf_{p_n \in \Pi_n} \|f - p_n\|_{[a,b]}$$

and for $a \geqq 0$ let

$$(2) \quad G_n(f: [a, b]) = \inf_{g_n \in \mathscr{G}_n} \|f - g_n\|_{[a,b]}$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a, b]$. When we talk about best approximations they will always be in this norm. The polynomial attaining the infimum in (1) will be denoted by $p_n^* = p_n^*(f: [a, b])$. If the infimum is attained in (2) by an element of $\mathscr{G}_n$ then that element will be denoted by $g_n^* = g_n^*(f: [a, b])$. The class $\mathscr{G}_n$ is not closed, for example:

$$\lim_{\delta \to 0} \frac{x^\delta - 1}{\delta} = \ln x$$

and we cannot in general expect g_n^* to exist (see [**9**], pp. 42-46).

The class of functions most amenable to approximation from $\mathscr{G}_n$ is the class of functions f given by the transformation

$$f(x): = \int_0^\infty x^t d\gamma(t), \quad \int_0^\infty d\gamma(t) < \infty$$

where γ is a non-negative measure. We will represent the class by Γ.

We will say that $f - g$ has an alternant of length m on $[a, b]$ if $f - g$ achieves its maximum modulus on $[a, b]$ at $m + 1$ points $a \leqq \tau_1 < \tau_2 < \ldots < \tau_{m+1} \leqq b$ and if

$$\operatorname{sign}(f(\tau_i) - g(\tau_i)) = -\operatorname{sign}(f(\tau_{i+1}) - g(\tau_{i+1})).$$

2. Existence and characterization. The first theorem guarantees the existence and uniqueness of best approximations from $\mathscr{G}_n$ to functions in Γ.

THEOREM 1. *Let f be an element of $\Gamma - \mathcal{G}_n$ defined on $[\alpha, 1]$, $0 < \alpha < b$.*

(a) *There exists a unique best approximation $g_n^* \in \mathcal{G}_n$ to f on $[\alpha, 1]$.*

(b) *$f - g_n^*$ has an alternant of length $2n$. This also characterizes g_n^*.*

(c) *$g_n^* \in \Gamma$, that is, there exist positive a_i, δ_i, $0 < \delta_1 < \ldots < \delta_n$, so that*

$$g_n^*(x) = \sum_{i=1}^{n} a_i x^{\delta_i}.$$

(d) $f(\alpha) - g_n^*(\alpha) = f(1) - g_n^*(1) = \|f - g_n^*\|_{[\alpha,1]}$.

The above results follow from the corresponding results for approximations by sums of exponentials to completely monotonic functions via the change of variables $x = e^{-y}$. These results are all to be found in [**4**] (see also [**3**], [**6**] and [**7**]). Conclusion (c) of the above theorem characterizes functions in Γ:

THEOREM 2. *A continuous function f is an element of Γ if and only if for some $\alpha \in (0, 1)$ and for all n, $g_n^*(f\colon [\alpha, 1])$ is an element of Γ.*

Proof. We need only comment on the "only if" part of this theorem. If $g_n^* \in \Gamma$ then

$$g_n^*(x) = \int_0^\infty x^t d\gamma_n(t)$$

where γ_n is a discrete positive measure and where

$$g_n^*(1) = \int_0^\infty d\gamma_n(t) \leqq 2\|f\|_{[\alpha,1]}.$$

It follows from Helly's Theorem [**10**, p xii] that

$$f(x) = \int_0^\infty x^t d\gamma(t)$$

where γ is a subsequential limit of the γ_n.

Descartes' rule of signs tells us that a sum of the form

$$\sum_{i=1}^{n} a_i x^{\xi_i} \quad \text{with } \xi_i < \xi_{i+1}$$

can have no more positive zeros than the sequence $\{a_1, \ldots, a_m\}$ has sign changes. It follows that, for any continuous f, if there exists $g_n \in \mathcal{G}_n$ so that $f - g_n$ has an alternant of length $2n$ on $[\alpha, 1]$ then g_n is the unique best approximant to f from $\mathcal{G}_n$ on $[\alpha, 1]$. Also, from Descartes' rule of signs we

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can deduce that if there exist $2n + 1$ points $\alpha \leqq \tau_1 < \ldots < \tau_{2n+1} \leqq 1$ and $g_n \in \mathcal{G}_n$ so that

$$\operatorname{sign}(f(\tau_i) - g_n(\tau_i)) = -\operatorname{sign}(f(\tau_{i+1}) - g_n(\tau_{i+1}))$$

then we have a de la Vallée Poussin type conclusion, namely

$$G_n(f: [\alpha, 1]) \geqq \min_{i=1,2,\ldots,2n+1} |f(\tau_i) - g_n(\tau_i)|.$$

We are considering approximations on an interval $[\alpha, 1]$ where $a > 0$. That the right hand endpoint of the interval be 1 is a matter of convenience. It can be taken to be any positive number by rescaling. However, taking $\alpha > 0$ is necessary. If $\alpha = 0$ then particular care must be taken. This case corresponds to exponential approximation on $[0, \infty)$ (see [**4**]). In all that follows we will be assuming that α is between zero and one.

3. Comparison theorems. The rate of approximation to a function f in Γ depends only on bounds on the measure γ defining f in the following sense.

THEOREM 3. *Let f and h be elements of Γ where*

$$f(x) = \int_0^\infty x^t d\gamma(t) \quad \text{and } h(x) = \int_0^\infty x^t d\beta(t).$$

If $\beta - \gamma$ is a non-negative measure then

$$G_n(f: [\alpha, 1]) \leqq G_n(h: [\alpha, 1]).$$

Proof. We may suppose that $f, h \notin \mathcal{G}_n$. Let g_n^* be the best approximant to h and let $\tau_1, \ldots, \tau_{2n}$ be the points in $[\alpha, 1]$ at which $h - g_n^* = 0$. Let $q_n^* \in \mathcal{G}_n$ interpolate f at $\tau_1, \ldots, \tau_{2n}$. Such a q_n^* exists and is an element of Γ (see [**4**]). Furthermore,

$$\operatorname{sign}(f(x) - q_n^*(x)) = \operatorname{sign}(h(x) - g_n^*(x))$$

at every point in $[\alpha, 1]$ (see [**6**, p. 167]). Thus, if

$$G_n(f: [\alpha, 1]) > G_n(h: [\alpha, 1])$$

then there exists $c > 1$ so that

$$f - q_n^* = ch - cg_n^*$$

at $2n + 1$ points in $[\alpha,1]$. Since $c\beta - \gamma$ is a positive measure and $q_n^* \in \Gamma$ there exists a positive measure μ so that

$$ch(x) - f(x) + q_n^*(x) = \int_0^\infty x^t d\mu(t).$$

It follows that

$$cg_n^*(x) - \int_0^\infty x^t d\mu(t)$$

has $2n + 1$ zeros in $[\alpha, 1]$. It is, however, not possible for an element of $\mathcal{G}_n$ to interpolate a function in $\Gamma - \mathcal{G}_n$ at more than $2n$ points. This can be seen by replacing $\int_0^\infty x^t d\mu(t)$ by a sufficiently close interpolating approximation of the form $\Sigma a_i x^{\delta_i}$, $a_i, \delta_i > 0$, and appealing to Descartes' rule of signs. (Alternatively one may use [**6**, p. 164].) This contradiction finishes the proof.

COROLLARY 1. *Let*

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \quad \textit{and} \quad h(x) = \sum_{i=0}^{\infty} b_i x^i.$$

If $b_i \geqq a_i \geqq 0$ *for all* i *then*

$$G_n(f(x); [\alpha, 1]) \leqq G_n(h(x): [\alpha, 1]).$$

The above theorem is really an interpolation result. We state it as such in the next corollary.

COROLLARY 2. *Let* γ, β *and* $\beta - \gamma$ *be non-negative measures. Let* $\tau_1, \ldots, \tau_{2n}$ *be* $2n$ *(not necessarily distinct) points in* $[\alpha,1]$ *and let* $q_n, g_n \in \mathcal{G}_n$ *interpolate* $\int_0^\infty x^t d\gamma(t)$ *and* $\int_0^\infty x^t d\beta(t)$ *respectively at* $\tau_1, \ldots, \tau_2 n$. *Then, for* $x \in [\alpha, 1]$

$$|q_n(x) - \int_0^\infty x^t d\gamma(t)\,| \leqq |g_n(x) - \int_0^\infty x^t d\beta(t)\,|.$$

The proof of the corollary is virtually identical to the proof of Theorem 3. Multiple roots are handled by taking limits. The analogous results for rational approximations to Stieltjes transforms (functions of the form $\int_0^\infty 1/(x+\tau)\; d\gamma(t)$) are established in [**2**]. This result, of course, also applies to exponential sum approximation to completely monotone functions by changing variables.

The next results provide some comparisons between Gaussian and polynomial approximations.

THEOREM 4. *Let* $f \in \Gamma$ *and let* $\alpha > 0$. *Then*

$$G_{n+1}(f(x): [\alpha, 1]) \leqq \inf_{p_{2n} \in \Pi_{2n}} \|f(x) - p_{2n}(\ln(x))\|_{[\alpha, 1]}.$$

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THEOREM 5. *Let $f \in \Gamma$ and let $\nu > 1$. Then*

$$G_{n+1}\left(f\left(\frac{1}{x}\right): [1, \nu]\right) \leqq E_{2n}\left(f\left(\frac{1}{x}\right): [1, \nu]\right).$$

Proof. We first prove Theorem 4. Let $g^*_{n+1} \in \mathcal{G}_{n+1}$ be the best approximation to $f(x)$ on $[\alpha, 1]$ and let p_{2n} be any polynomial of degree $2n$. Suppose that

$$\| f(x) - p_{2n}(\ln(x))\|_{[\alpha,1]} < \| f(x) - g^*_{n+1}(x)\|_{[\alpha,1]}.$$

Then, by Theorem 1, $g^*_{n+1}(x) - p_{2n}(\ln(x))$ has at least $2n + 2$ positive zeros and hence, $g^*_{n+1}(e^y) - p_{2n}(y)$ has at least $2n + 2$ real zeros. Since

$$g^*_{n+1}(x) = \sum_{i=1}^{n+1} a_i x^{\delta_i} \quad \text{where } a_i \geqq \delta_i \geqq 0$$

we arrive at the contradiction that the function

$$(g^*_{n+1}(e^y) - p_{2n}(y))^{(2n+1)} = \sum_{i=1}^{n+1} (\delta_i)^{2n+1} a_i e^{\delta_i y} > 0$$

has a real zero.

Theorem 5 is proved analogously. We need only note that the best approximation to $f(1/x)$ from $\mathcal{G}_{n+1}$ is of the form

$$\sum_{i=1}^{n+1} a_i x^{-\delta_i} \quad \text{where } a_i, \delta_i \geqq 0$$

and hence, has non-vanishing derivatives of all orders.

4. Gauss-Padé approximants. Let $f \in \Gamma - \mathcal{G}_n$ be given by

$$f(x) = \int_0^b x^t d\gamma(t).$$

Associated with γ is the Gaussian quadrature rule with positive nodes $z_1, \ldots, z_n$ and positive weights $\omega_1, \ldots, \omega_n$ which has the property that

$$\int_0^b p_{2n-1}(t)d\gamma(t) = \sum_{i=1}^{n} \omega_i p_{2n-1}(z_i), \quad p_{2n-1} \in \Pi_{2n-1}.$$

In the next theorem we show that

$$f(x) - \sum_{i=1}^{n} \omega_i x^{z_i} = O((x - 1)^{2n})$$

and derive error bounds for this Padé-type approximation.

THEOREM 6. *Let*

$$f(x) = \int_0^1 x^t d\gamma(t) \in \Gamma$$

and let

$$g_n(x) = \sum_{i=1}^{n} \omega_i x^{z_i}$$

where the ω_i *and* z_i *are as above. Then, for* $0 < x \leqq 1$

$$|f(x) - g_n(x)| \leqq \frac{(\ln x)^{2n}}{4^{2(n-1)}(2n)!} \int_0^1 d\gamma(t).$$

Proof. For fixed $x \leqq 1$ and any $p_{2n-1} \in \Pi_{2n-1}$

$$|f(x) - g_n(x)| \leqq 2 \| x^t - p_{2n-1}(t) \|_{[0,1]} \int_0^1 d\gamma(t).$$

Since, with respect to the variable t,

$$(x^t)^{(2n)} = (\ln x)^{2n} e^{t \ln x}$$

we know that there exists $p^*_{2n-1} \in \Pi_{2n-1}$ so that

$$\| x^t - p^*_{2n-1}(t) \|_{[0,1]} \leqq \frac{8 (\ln x)^{2n}}{4^{2n}(2n)!}.$$

(See [**8**, p. 38].)

This result is reminiscent of a similar theorem for Padé approximants due to Baker [**1**, p. 191].

THEOREM 7. *Let*

$$f(x) = \int_0^2 \frac{x}{1+xt} d\gamma(t)$$

then, for $x > 0$,

$$\left| f(x) - \sum_{i=1}^{n} \frac{\omega_i x}{1+z_i x} \right| \leqq \frac{2x^{2n+1}}{[1+x+\sqrt{1+2x}]^{2n-1}[1+2x]} \int_0^2 \gamma(t).$$

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This a slight variation of Baker's result. We include it because its proof is identical to the proof of Theorem 6 using the fact that

$$E_n\left(\frac{1}{x+c} : [0,2]\right) = \frac{(|c+1| - \sqrt{c^2+2c})^n}{c^2 + 2c}$$

and as such is considerably different from Baker's proof. It is worth noting that this method also establishes that

$$\sum_{i=1}^{n} \frac{\omega_i x}{1+z_i x}$$

is the (n, n) Padé approximant to f.

Theorem 6 can be applied to the function $(x - 1)/\ln x$ which has an essential singularity at zero and, hence, cannot be approximated by polynomials on $[\alpha,1]$ with a rate of convergence greater than ρ_α^{-n}.

5. Approximations to e^x and $1 + \dots + x^n$. Let

$$T_n^\alpha(x) = \prod_{i=1}^{n} (x - z_i)$$

be the n[th] Cebycev polynomial shifted to the interval $[\alpha,1]$ and normalized to have lead coefficient 1. Let

$$T_n^\alpha(x) = a_0^\alpha + \dots + a_{n-1}^\alpha x^{n-1} + x^n,$$

let

$$M_{\alpha,n} = \max\{|a_0^\alpha|, \dots, |a_{n-1}^\alpha|\}$$

and let

$$m_{\alpha,n} = \min\{|a_0^\alpha|, \dots, |a_{n-1}^\alpha|\}.$$

THEOREM 8.

$$\frac{2(1-\alpha)^{2n}}{M_{\alpha,2n}4^{2n}} \leqq G_n(1 + x + \dots + x^n: [\alpha, 1]) \leqq \frac{2(1 - \alpha)^{2n}}{m_{\alpha,2n}4^{2n}}.$$

If $c = (3 + \alpha)/(1 - \alpha)$ *then*

$$G_n(1 + x + \dots + x^n : [\alpha, 1]) \geqq 1/(c + \sqrt{c^2 - 1})^{2n}$$

and if $d = (1 + \alpha)/(1-\alpha)$ *then*

$$G_n(1 + x + \dots + x^n: [\alpha, 1]) \leqq 2/(d + \sqrt{d^2 - 1})^{2n}.$$

Proof. Consider

$$T^{\alpha}_{2n}(\sqrt{x}) = \sum_{i=0}^{n} a^{\alpha}_{2i}(x)^{i} + \sum_{i=0}^{n-1} a^{\alpha}_{2i+1}(\sqrt{x})^{2i+1}$$

$$= p^{\alpha}_{n}(x) + g^{\alpha}_{n}(x)$$

where $p^{\alpha}_{n} \in \Pi_n$ and $g^{\alpha}_{n} \in \mathcal{G}_n$. By Theorem 1 part (b) $-g^{\alpha}_{n}$ is the best approximation from $\mathcal{G}_n$ to p^{α}_{n} on $[\alpha,1]$. Also,

$$\| p^{\alpha}_{n}(x) + g^{\alpha}_{n}(x) \|_{[\alpha,1]} = 2(1 - \alpha)^{2n}/4^{2n}.$$

By Corollary 1

$$G_n(p^{\alpha}_{n}/M_{\alpha,2n}: [\alpha, 1]) \leqq G_n(1 + x + \ldots + x^n: [\alpha, 1]) \leqq G_n(p^{\alpha}_{n}/m_{\alpha,2n}: [\alpha, 1])$$

and hence,

$$\frac{2(1-\alpha)^{2n}}{M_{\alpha,2n}4^{2n}} \leqq G_n(1 + x + \ldots + x^n: [\alpha, 1]) \leqq \frac{2(1-\alpha)^{2n}}{m_{\alpha,2n}4^{2n}}.$$

If

$$\| p_n\|_{[\alpha,1]} \leqq c$$

then

$$| p_n(-1) | \leqq c\left(\frac{3+\alpha}{1-\alpha} + \sqrt{\left(\frac{3+\alpha}{1-\alpha}\right)^2 - 1}\right)^n.$$

This result may be found in [**8**, p. 43]. Thus, since

\ $$\| T_{2n}\|_{[\alpha,1]} = \frac{2(1-\alpha)^{2n}}{4^{2n}}$$

we have

$$M_{\alpha,2n} \leqq |T_{2n}(-1)| \leqq \frac{2(1-\alpha)^{2n}}{4^{2n}}\left(\frac{3+\alpha}{1-\alpha} + \sqrt{\left(\frac{3+\alpha}{1-\alpha}\right)^2 - 1}\right)^{2n}.$$

We note that

$$\mathrm{T}_{2n}(x) = \left(\frac{1-\alpha}{4}\right)^{2n}[(y + \sqrt{y^2-1})^{2n} + (y - \sqrt{y^2-1})^{2n}]$$

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where $y = (2x - (1 + \alpha))/(1 - \alpha)$. Thus,

$$m_{2n}^{\alpha} = T_{2n}^{\alpha}(0) = \left(\frac{1-\alpha}{4}\right)^{2n} \left[\left(\frac{1+\alpha}{1-\alpha} + \sqrt{\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1}\right)^{2n} + \left(\frac{1+\alpha}{1-\alpha}\right) - \sqrt{\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1}\right)^{2n}\right].$$

This finishes the proof.

If we take $\alpha = \frac{1}{2}$ we get

$$\frac{1}{(193.9\ldots)^n} = \frac{1}{(7 + \sqrt{48})^{2n}} \leqq G_n(1 + x + \ldots + x^n: [\tfrac{1}{2}, 1])$$

$$\leqq \frac{2}{(3 + \sqrt{8})^{2n}} = \frac{2}{(33.9\ldots)^n}.$$

By comparison

$$E_{n-1}(1 + x + \ldots + x^n: [\tfrac{1}{2}, 1]) = 2/8^n$$

and we observe that Gaussian approximation is moderately more effective than polynomial approximation in this instance.

COROLLARY 3. *If* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $a_k \geqq a_{k+1} \geqq 0$ *then*

$$G_n(f: [\alpha, 1]) \geqq \frac{a_n}{\left(\frac{3+\alpha}{1-\alpha} + \sqrt{\left(\frac{3+\alpha}{1-\alpha}\right)^2 - 1}\right)^{2n}}.$$

This corollary is a consequence of Corollary 1 and Theorem 8, we need only compare the approximation of f to the approximation of $a_n(1 + \ldots + x^n)$.

Applying the above to e^x on $[\frac{1}{2}, 1])$ shows that

$$G_n(e^x: [\tfrac{1}{2}, 1]) \geqq 1/(193.9\ldots)^n n!$$

whereas

$$E_{n-1}(e^x: [\tfrac{1}{2}, 1]) \leqq 6/8^n n!.$$

Finally, we observe that, for small n, $G_{n+1}(e^x: [\frac{1}{2}, 1])$ may be somewhat smaller than $E_n(e^x: [\frac{1}{2}, 1])$. This is illustrated in the following table. Theorem 4 guarantees that

$$G_{n+1}(e^x: [\tfrac{1}{2}, 1]) \leqq \inf_{p_{2n} \in \Pi_{2n}} \|e^x - p_{2n}(\ln x)\|_{[\frac{1}{2}, 1]}.$$

Computations were done using the IMSL routine IRATCU.

n	$E_n(e^x: [\frac{1}{2},1])$	$\inf \|e^x - p_{2n}(\ln x)\|$
1	3.3×10^{-2}	9.7×10^{-3}
2	1.4×10^{-3}	1.1×10^{-4}
3	4.3×10^{-5}	1.0×10^{-6}
4	1.1×10^{-6}	8.1×10^{-9}

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