J. Austral. Math. Soc. (Series A) 31 (1981), 405–414

ON THE BEHAVIOUR NEAR THE BOUNDARY OF SOLUTIONS OF A SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

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(Received 26 September 1980)

Communicated by G. R. Morris

Abstract

The purpose of this note is to investigate traces of the function $\ln(1 + |u|)$, where u is a solution of a semi-linear partial differential equation of elliptic type, belonging to an appropriate Sobolev space. This article complements the results of Chabrowski and Thompson (1980), and Mihailov (1972), (1976).

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 35 D 10; secondary 35 J 25.

1.

Consider the semi-linear operator of the form

$$D_i(a_{ij}(x)D_ju) - b_i(x)D_iu - b(x, u) = 0 \tag{1}$$

in a bounded domain $Q \subset R_n$ with the boundary ∂Q of the class C^2, $D_iu = \partial/\partial x_i$. The summation convention that repeated indices indicate summation from 1 to n is followed here.

We make the following assumptions:

(A) There is a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \gamma|\xi|^2$$

for all $\xi \in R_n$ and all $x \in Q$.

https://doi.org/10.1017/S1446788700024198 Published online by Cambridge University Press

(B) The coefficients a_{ij} are of the class $C^1(\bar{Q})$. The coefficients b_i are measurable on Q and

$$\int_Q b_i(x)^2 r(x)dx < \infty, \qquad i = 1, \ldots, n,$$

where $r(x)$ denotes the distance function defined by $r(x) = \text{dist}(x, \partial Q)$.

(C) The function $b(x, u)$ is defined for $(x, u) \in Q \times R$ and satisfies the Carathéodory conditions; that is,

(i) for almost all $x \in Q$, $b(x, \cdot)$ is a continuous function on R, and

(ii) for every fixed $u \in R$, $b(\cdot, u)$ is a measurable function on Q.

Moreover we assume that

$$|b(x, u)| \leqslant L\frac{|u| + 1}{r(x)^{1+\alpha}},$$

where L and α are nonnegative constants, $0 \leqslant \alpha < 1$.

Remark. Under assumptions (i) and (ii) $b(x, u(x))$ is a measurable function and

$$b(x, \cdot)\colon L^1_{\text{loc}}(Q) \to L^1_{\text{loc}}(Q)$$

is continuous.

In the sequel we use the notion of a generalized solution involving the Sobolev space $W^{1,2}_{\text{loc}}(Q)$.

A function $u(x)$ is said to be a weak solution of the equation (1) in Q if $u \in W^{1,2}_{\text{loc}}(Q)$ and u satisfies

$$(2) \qquad \int_Q a_{ij}(x)D_j u D_i v\, dx + \int_Q b_i(x) D_i u v\, dx + \int_Q b(x, u) v\, dx = 0$$

for every $v \in W^{1,2}_{\text{loc}}(Q)$ with compact support in Q.

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $Q_\delta = Q \cap \{x; \min|x - y| > \delta\}$, with the boundary ∂Q_δ, possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping $x_0 = x_0(x_\delta)$ is given by the formula $x_0 = x_\delta + \delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed x_δ introduce the sets

$$A_\varepsilon = \partial Q_\delta \cap \{x; |x - x_\delta| < \varepsilon\},$$
$$B_\varepsilon = \{x; x = \tilde{x}_\delta + \delta\nu_\delta(\tilde{x}_\delta), \tilde{x}_\delta \in \partial Q_\delta \cap (x; |x - x_\delta| < \varepsilon)\}$$

and put

$$\frac{dS_\delta}{dS_0} = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the Lebesgue measure of a set A. Mikhailov (1976) proved that there is a positive constant γ_0 such that

$$\gamma_0^{-2} < \frac{dS_\delta}{dS_0} < \gamma_0^2 \tag{3}$$

and

$$\lim_{\delta \to 0} \frac{dS_\delta}{dS_0}(x_\delta) = 1 \tag{4}$$

uniformly with respect to $x_0 \in \partial Q$.

It is well known that the distance function $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small (for the proof, see Gilbarg and Trudinger (1977), p. 382). Denote by $\rho(x)$ the extension of the function $r(x)$ to $\bar{Q}$ satisfying the following properties: $\rho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$, $\rho \in C^2(\bar{Q})$, $\rho(x) \geq \frac{3}{4}\delta_0$ in Q_{δ_0}, $\gamma_1^{-1}r(x) \leq \rho(x) \leq \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x; \rho(x) = \delta\}$ for $\delta \in (0, \delta_0)$ and finally $\partial Q = \{x; \rho(x) = 0\}$, $\rho(x) > 0$ on Q.

Introduce the surface integrals

$$M_1(\delta) = \int_{\partial Q} \ln(1 + |u(x_\delta(x))|)dS_x$$

and

$$M(\delta) = \int_{\partial Q_\delta} \ln(1 + |u(x)|)dS_x,$$

where the values of $u(x_\delta(x))$ on ∂Q and $u(x)$ on ∂Q_δ are understood in the sense of traces (see Kufner, John and Fucik (1977), Chapter 6). Of course we suppose that $u \in W^{1,2}_{\text{loc}}(Q)$. By Lemma 4 in Chabrowski and Thompson (1980) $M_1(\delta)$ and $M(\delta)$ are absolutely continuous on $[\delta_1, \delta_0]$ for every $0 < \delta_1 < \delta_0$.

Our first objective is to characterise the continuity of M_1 and M on the interval $[0, \delta_0]$. This characterisation will be described below. However we shall first mention some preliminary results.

LEMMA 1. *Suppose that $M_1(\delta)$ is a bounded function on $(0, \delta_0]$. Then for every $0 \leq \alpha < 1$ there is a positive constant C such that*

$$\int_{Q_\delta} \frac{\ln(1 + |u(x)|)}{(\rho(x) - \delta)^\alpha} dx \leq C$$

for every $\delta \in (0, \delta_0/2]$.

https://doi.org/10.1017/S1446788700024198 Published online by Cambridge University Press

LEMMA 2. *Let* $u \in W^{1,2}_{\text{loc}}(Q)$ *and*

$$\int_Q |D\ln(1+|u(x)|)|^2 r(x)\,dx < \infty,$$

then for every $\alpha \in (0, 1)$, *there is a positive constant* C *such that*

$$\int_{Q_\delta} \frac{\ln(1+|u(x)|)}{(\rho(x)-\delta)^\alpha}\,dx \leqslant C$$

for every $\delta \in (0, \delta_0/2]$.

For proofs see Chabrowski and Thompson (1980), Lemmas 5 and 6.

THEOREM 1. *Let* u *be a solution of* (1) *belonging to* $W^{1,2}_{\text{loc}}(Q)$, *then the following conditions are equivalent*:

I. $M(\delta)$ *is a bounded function on* $(0, \delta_0]$;

II. $\int_Q |D\ln(1+|u(x)|)|^2 r(x)\,dx < \infty$;

III. $M_1(\delta)$ *is a continuous function on* $[0, \delta_0]$.

PROOF. The proof follows by a simple modification of the proof of Theorem 1 in Chabrowski and Thompson (1980).

Set

$$f_\beta(t) = \begin{cases} t & \text{for } t \geqslant 0, \\ -\beta t & \text{for } t < 0, \end{cases}$$

where β is a positive number.

Now we define

$$v(x) = \begin{cases} \dfrac{(\rho-\delta)}{1+f_\beta(u)} & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta. \end{cases}$$

By standard properties of weak derivatives it is evident that v is an admissible test function in (2), hence

$$-\int_{Q_\delta \cap (u>0)} a_{ij}\frac{D_iuD_ju}{(1+u)^2}(\rho-\delta)\,dx + \int_{Q_\delta \cap (u<0)} \beta a_{ij}\frac{D_iuD_ju}{(1-\beta u)^2}(\rho-\delta)\,dx$$
$$+\int_{Q_\delta \cap (u>0)} b_i\frac{D_iu}{(1+u)}(\rho-\delta)\,dx + \int_{Q_\delta \cap (u<0)} b_i\frac{D_iu}{(1-\beta u)}(\rho-\delta)\,dx$$
$$+\int_{Q_\delta} b(x,u)\frac{(\rho-\delta)}{1+f_\beta(u)}\,dx + \int_{Q_\delta} a_{ij}\frac{D_iuD_j\rho}{(1+f_\beta(u))}\,dx = 0.$$

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Using the assumptions (B) and (C) and the fact that $u \in W^{1,2}_{\text{loc}}(Q)$, letting $\beta \to \infty$ we obtain

$$-\int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j \ln(1+u_+)(\rho-\delta)\,dx$$
$$+\int_{Q_\delta} b_i D_i \ln(1+u_+)(\rho-\delta)\,dx + \int_{Q_\delta} b(x,u)\frac{(\rho-\delta)}{1+u_+}\,dx$$
$$+\int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j\rho\,dx = 0,$$

where

$$u_+(x) = \max(0, u(x)).$$

Now by the assumption (A)

$$\begin{aligned}\int_{Q_\delta} |D\ln(1+u_+)|^2(\rho-\delta)\,dx \\ \leqslant \gamma\Bigg[\int_{Q_\delta} b_i D_i \ln(1+u_+)(\rho-\delta)\,dx \\ + \int_{Q_\delta\cap(u>0)} b(x,u)\frac{(\rho-\delta)}{1+u_+}\,dx \\ + \int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j\rho\,dx\Bigg].\end{aligned} \tag{5}$$

Note that

$$\int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j\rho\,dx = \int_{\delta Q_\delta} a_{ij} \ln(1+u_+) D_j\rho D_i\rho dS_x$$
$$-\int_{Q_\delta} D_i(a_{ij}D_j\rho)\ln(1+u_+)\,dx.$$

From (5) and the assumptions (A), (B) and (C), applying Young's inequality

$$\begin{aligned}\int_{Q_\delta} |D\ln(1+u_+)|^2(\rho-\delta)\,dx \leqslant \gamma^2\int_{\partial Q_\delta} \ln(1+u_+)dS_x \\ +2\gamma^2\int_{Q_\delta} b_i^2(\rho-\delta)\,dx + \tfrac{1}{2}\int_{Q_\delta}|D\ln(1+u_+)|(\rho-\delta)\,dx \\ + C_1\int_{Q_\delta}\ln(1+u_+)\,dx + \gamma L\int_{Q_\delta} r(x)^{-1-\alpha}(\rho(x)-\delta)\,dx \\ +\gamma L\int_{Q_\delta\cap(u>0)} \frac{u_+(\rho-\delta)}{(1+u_+)r^{\alpha+1}}\,dx,\end{aligned} \tag{6}$$

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where

$$C_1 = \sup_Q |D_i(a_{ij}D_j\rho)|.$$

The analogous inequality can be derived for $u_-(u_-(x)) = \max(0, -u(x))$ if in the definition of v we replace the function f_β by

$$\tilde{f}_\beta(t) = \begin{cases} -t & \text{for } t \leqslant 0, \\ \beta t & \text{for } t > 0. \end{cases}$$

Thus we arrive at the inequality

$$\begin{aligned} \int_{Q_\delta} |D\ln(1+u_-)|^2(\rho-\delta)\,dx &\leqslant \gamma^2 \int_{\partial Q_\delta} \ln(1+u_-)dS_x \\ &+ 2\gamma^2 \int_{Q_\delta} b_i^2(\rho-\delta)\,dx + \tfrac{1}{2}\int_{Q_\delta} |D\ln(1+u_-)|(\rho-\delta)\,dx \\ &+ C_1 \int_{Q_\delta} \ln(1+u_-)\,dx + \gamma L \int_{Q_\delta} r^{-1-\alpha}(\rho-\delta)\,dx \\ &+ \gamma L \int_{Q_\delta \cap (u<0)} \frac{u_-(\rho-\delta)}{(1+u_-)r^{\alpha+1}}\,dx. \end{aligned} \tag{7}$$

Adding (6) and (7) and applying Lemma 1 and the Monotone Convergence Theorem we obtain I $\Rightarrow$ II.

To prove II $\Rightarrow$ III we note as in the first part of the proof

$$\begin{aligned} \int_{\partial Q_\delta} \ln(1+u_+)a_{ij}D_i\rho D_j\rho dS_x &= \int_{Q_\delta} D_i(a_{ij}D_j\rho)\ln(1+u_+)\,dx \\ &+ \int_{Q_\delta} a_{ij}D_i\ln(1+u_+)D_j\ln(1+u_+)(\rho-\delta)\,dx \\ &- \int_{Q_\delta} b_iD_i\ln(1+u_+)(\rho-\delta)\,dx \\ &- \int_{Q_\delta \cap (u>0)} b(x,u)\frac{(\rho-\delta)}{1+u_+}\,dx \end{aligned}$$

and the analogous formula remains true for u_-. First we prove the continuity of M. It is clear that it suffices to prove the continuity of $M(\delta)$ at 0. By the Dominated Convergence Theorem

$$\lim_{\delta\to 0} \int_{Q_\delta} a_{ij}D_i\ln(1+|u|)D_j\ln(1+|u|)(\rho-\delta)\,dx$$

exists. Since $1/\gamma \leqslant a_{ij}D_i\rho D_j\rho \leqslant \gamma$ is continuous on $\bar{Q}$ it follows by Lemma 2 and Assumption C that $M(\delta)$ is continuous at $\delta = 0$. Since $dS_\delta/dS_0 \to 1$

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uniformly as $\delta \to 0$, $M(\delta)$ is continuous and

$$M(\delta) - M_1(\delta) = \int_{Q_\delta} \ln(1 + |u(x_\delta(x))|)\left(\frac{dS_\delta}{dS_0} - 1\right) dS_0.$$

$M_1(\delta)$ is continuous proving II $\Rightarrow$ III.

Finally III $\Rightarrow$ I follows from the proof II $\Rightarrow$ III.

THEOREM 2. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *Assume one of the conditions* I, II *or* III *holds.*

There is a sequence $\delta_k \to 0$ as $k \to \infty$ and a Borel measure μ on ∂Q such that

$$\lim_{k\to 0} \int_{Q_\delta} \ln\left(1 + |u(x_{\delta_k}(x))|\right) g(x)\, dS_x = \int_{Q_\delta} g(x)\, d\mu(x)$$

for each $g \in C(\partial Q)$.

THEOREM 3. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *If one of the conditions* I, II *or* III *holds then the function*

$$G(\delta) = \int_{Q_\delta} \ln(1 + |u(x_\delta(x)|))\Psi(x)\, dS_x$$

is continuous on $[0, \delta_0]$ *for any Ψ in $C(\partial Q)$.*

PROOF. The proof is similar to that of Theorem 4 in Chabrowski and Thompson (1980), so we sketch the proof. By Lemma 4 in Chabrowksi and Thompson (1980) $G(\delta)$ is absolutely continuous on $[\delta_1, \delta_0]$ for any $\delta_1 > 0$ so it suffices to prove continuity at $\delta = 0$. Since

$$\Phi(x) = a_{ij}(x) D_i\rho D_j\rho$$

is uniformly continuous, $1/\gamma \leqslant \Gamma(x) \leqslant \gamma$, $M_1(\delta)$ is bounded, and the elements of $C^1(\bar{Q})$ restricted to ∂Q are dense in $C(\partial Q)$, it suffices to show that

$$\bar{G}(\delta) = \int_{\partial Q_\delta} \ln(1 + |u(x)|)\Psi(x)\Phi(x)\, dS_x$$

is continuous for each $\Psi \in C^1(\bar{Q})$.

From (2) taking

$$v(x) = \begin{cases} \dfrac{\Psi(x)(\rho(x) - \delta)}{1 + f_\beta(u)} & \text{for } x \in Q, \\ 0 & \text{for } x \in Q - Q_\delta, \end{cases}$$

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where f_β is the function defined in the proof of Theorem 1, then letting $\beta \to \infty$, we obtain

$$\int_{\partial Q_\delta} \ln(1+u_+)\Phi\Psi \, dS_x = \int_{Q_\delta} D_i(a_{ij}D_j\rho)\ln(1+u_+)\, dx$$
$$+ \int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j \ln(1+u_+)(\rho-\delta)\, dx$$
$$- \int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j \Psi(\rho-\delta)\, dx$$
$$- \int_{Q_\delta} b_i D_i \ln(1+u_+)(\rho-\delta)\, dx$$
$$+ \int_{Q_\delta \cap (u>0)} b(x,u)\frac{(\rho-\delta)}{1+u_+}\, dx.$$

The analogous formula for u_- is true. The integrands on the right side of the last equality are dominated by functions belonging to $L^1(Q)$. Hence the result follows from the Dominated Convergence Theorem.

2.

In this section using the previous technique we derive a global estimate for the gradient of $\ln(1+|u|)$. The most interesting feature of this estimate lies in the fact that we do not require any knowledge of the boundary values.

Throughout this section we adopt Assumption (A) and the assumptions:

(B′) The coefficients a_{ij} are of the class $C^1(\bar{Q})$. The coefficients b_i are measurable on Q and

$$\int_Q b_i^2(x) r(x)^p \, dx < \infty, \qquad i = 1, \ldots, n,$$

where $1 < p < \infty$;

(C′) The function $b(x,u)$ defined on $Q \cap R$ satisfies the Carathéodory conditions (see Assumption (C) in Section 1) and

$$|b(x,u)| \leqslant C(x)|u|,$$

where $C(x)$ is a non negative function on Q such that

$$\int_Q C(x) r(x)^p \, dx < \infty,$$

$1 < p < \infty$.

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THEOREM 4. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *Then*

$$\int_Q |D\ln(1+|u|)|^2 r(x)^p \, dx < C,$$

where C is a positive constant depending on p and the coefficients of (1).

PROOF. As a test function in (2) we take

$$v(x) = \begin{cases} \dfrac{(\rho(x)-\delta)^p}{1+f_\beta(u)} & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta. \end{cases}$$

Thus, letting $\beta \to \infty$ we obtain

$$\begin{aligned} &-\int_{Q_\delta} a_{ij} D_i \ln(1+u_+) D_j \ln(1+u_+)(\rho-\delta)^p \, dx \\ &+\int_{Q_\delta} b_i D_i \ln(1+u_+)(\rho-\delta)^p \, dx + \int_{Q_\delta \cap (u>0)} b(x,u) \frac{(\rho-\delta)^p}{1+u_+} \, dx \\ &+p\int_{Q_\delta} a_{ij} D_i \rho \ln(1+u_+) D_j \rho (\rho-\delta)^{p-1} \, dx = 0. \end{aligned}$$

Now

$$\begin{aligned} \int_{Q_\delta} |D\ln(1+u_+)|^2(\rho-\delta)^p \, dx \leqslant \gamma\Bigg[&\int_{Q_\delta} b_i D_i \ln(1+u_+)(\rho-\delta)^p \, dx \\ &+p\int_{Q_\delta} a_{ij} D_i \rho \ln(1+u_+) D_j \rho (\rho-\delta)^{p-1} \, dx \\ &+\int_{Q_\delta} C(x) \frac{u_+}{1+u_+} (\rho-\delta)^p \, dx \Bigg]. \end{aligned}$$

Applying Young's inequality we obtain

$$\begin{aligned} \frac{1}{3}\int_{Q_\delta} |D\ln(1+u_+)|^2(\rho-\delta)^p \, dx &\leqslant 3\gamma \sum_{i=1}^n \int_{Q_\delta} b_i^2 (\rho-\delta)^p \, dx \\ &+\int_{Q_\delta} C(x)(\rho-\delta)^p \, dx + 3C_2^2\gamma \int_{Q_\delta} (\rho-\delta)^{p-2} \, dx, \end{aligned}$$

where

$$C_2 = p \sup_{i,Q} |a_{ij} D_j \rho|.$$

A similar inequality can be derived for u_- and this completes the proof.

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