CHARACTERS OF CARTESIAN PRODUCTS OF ALGEBRAS

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Introduction. Let R be a commutative ring with identity 1. A *character* of an R-algebra E is a homomorphism from E onto R, regarded as an algebra over itself. If $(E_\alpha)_{\alpha \in A}$ is a family of R-algebras indexed by a set A and if

$$E = \prod_{\alpha \in A} E_\alpha,$$

then for every $\beta \in A$ and every character v_β of E_β, $v_\beta \circ pr_\beta$ is a character of E where pr_β is the projection homomorphism from E onto E_β. Further if A is finite and if the only idempotents of R are 0 and 1 (equivalently, if R is not the direct sum of two proper ideals), it is easy to see that every character of E is of this form. In general, it is natural to ask:

(1) *Is every character of*

$$E = \prod_{\alpha \in A} E_\alpha$$

of the form $v_\beta \circ pr_\beta$ for some $\beta \in A$, where v_β is a character of E_β?

If each E_α is R, E is simply the R-algebra of all R-valued functions with domain A; we shall denote this algebra by R^A, the set of its characters by $M(R^A)$, and its identity element by e. Since the only character of the R-algebra R is the identity map, (1) becomes for R^A:

(2) *Is every character of R^A a projection?*

Question (1) appears more general than (2), but we shall see in § 1, as a consequence of an extension theorem of Buck, that an affirmative answer to (2) implies an affirmative answer to (1).

Recently, by a measure-theoretic argument, Bialynicki-Birula and Żelazko **(1)** answered (1) in the affirmative if R is an infinite field, if each E_α has an identity, and if A satisfies a certain set-theoretic condition. The author obtained his results independently (without the hypothesis that each E_α possess an identity) as corollaries of a density theorem concerning a suitable weak uniform structure imposed on the set of characters of R^A. These results are given in §§ 2 and 3. In §§ 4 and 5 we shall prove that if R is finite and A infinite, question (2) has a negative answer, but that if R is a principal domain having at least two non-associated extremal elements (for example, if R is the integers) and if A satisfies a certain set-theoretic condition, the

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questions have an affirmative answer. These results are applied in the remaining two sections: in § 6 we show that the only compact principal domains are the known ones, namely, finite fields and valuation rings of locally compact fields whose topology is given by a discrete valuation of rank 1; and in § 7 we give conditions on the algebra $\mathfrak{C}(T)$ of all real-valued continuous functions on topological space T which are both necessary and sufficient for every connected component of T to be open.

1. The extension theorem. Buck's extension theorem (**5**, Theorem 1) may be stated in its most general form as follows (as observed in (**8**, p. 74), one of the hypotheses of Buck's original version is superfluous).

THEOREM A. *Let R be a commutative ring with identity, E an R-algebra, H an ideal of E, F an R-algebra with identity. If f is a homomorphism from H onto F, there exists a unique homomorphism g from E onto F extending f.*

Consequently, we see that an affirmative answer to question (2) implies an affirmative answer to question (1):

THEOREM 1. *Let $(E_\alpha)_{\alpha \in A}$ be a family of R-algebras indexed by A. If every character of R^A is a projection, then for every character u of*

$$E = \prod_{\alpha \in A} E_\alpha$$

there exist $\beta \in A$ and a character v_β of E_β such that $u = v_\beta \circ pr_\beta$.

Proof. First, let us assume each E_α has an identity e_α. The restriction of u to the subalgebra

$$F = \prod_{\alpha \in A} Re_\alpha$$

of E is a character of F since F contains the identity of E. As F is canonically isomorphic with R^A, it follows from the hypothesis that there exists $\beta \in A$ such that $u(i_\beta(e_\beta)) = 1$, where i_β is the canonical injection map from E_β into E. Hence if $v_\beta = u \circ i_\beta$, v_β is a character of E_β, and u and $v_\beta \circ pr_\beta$ coincide on the ideal $i_\beta(E_\beta)$ of E. Therefore, as $v_\beta \circ pr_\beta$ is a character of E, the uniqueness part of Buck's theorem ensures $u = v_\beta \circ pr_\beta$. In the general case, let $E_\alpha{}^+$ be the R-algebra obtained by adjoining an identity to E_α. As E_α is an ideal in $E_\alpha{}^+$, E is an ideal in

$$G = \prod_{\alpha \in A} E_\alpha^+.$$

By Buck's theorem, there exists a character of G extending u, and an application of the preceding result completes the proof.

2. Algebras over fields. Let K be a field equipped with the discrete topology. K is then a topological field whose associated uniform structure is the discrete uniform structure. Let $\mathfrak{U}_K(A)$ be the weakest uniform structure on A such that each $f \in K^A$ is uniformly continuous, let $\mathfrak{W}_K(A)$ be the weakest uniform structure on the set $\mathfrak{F}(K^A, K)$ of all K-valued functions on K^A such that $u \to u(f)$ is uniformly continuous on $\mathfrak{F}(K^A, K)$ for all $f \in K^A$, and let $\mathfrak{B}_K(A)$ be the uniform structure induced on $M(K^A)$ by $\mathfrak{W}_K(A)$. As the uniform structure of K is complete and separated, $\mathfrak{W}_K(A)$ is a complete, separated uniform structure (**4**, § 1, Proposition 2, and Theorem 1). A familiar argument shows $M(K^A)$ is closed: if $\mathfrak{F}$ is a filter on $M(K^A)$ converging to $u \in \mathfrak{F}(K^A, K)$ and if $f, g \in K^A$, then $\mathfrak{F}(fg) \to u(fg)$, $\mathfrak{F}(f) \to u(f)$, and $\mathfrak{F}(g) \to u(g)$; for any $F \in \mathfrak{F}$, $(fg)(F) \subseteq f(F)g(F)$, so $\mathfrak{F}(fg)$ is the filter base for a filter finer than that generated by $\mathfrak{F}(f)\cdot\mathfrak{F}(g)$; hence as $\mathfrak{F}(f)\cdot\mathfrak{F}(g) \to u(f)u(g)$, so also $\mathfrak{F}(fg) \to u(f)u(g)$, and therefore $u(fg) = u(f)u(g)$. Similarly, u is linear. $\mathfrak{F}(e) \to u(e)$, but as $v(e) = 1$ for all $v \in M(K^A)$, $u(e) = 1$. Hence $u \in M(K^A)$. $M(K^A)$ is therefore a complete separated uniform space. For any finite subset Γ of K^A, let $U(\Gamma) = [(\alpha, \beta) \in A \times A: f(\alpha) = f(\beta)$ for all $f \in \Gamma]$, $V(\Gamma) = [(u, v) \in M(K^A) \times M(K^A): u(f) = v(f)$ for all $f \in \Gamma]$. The collection of sets $U(\Gamma)$ [respectively, $V(\Gamma)$] forms a fundamental system of entourages for $\mathfrak{U}_K(A)$ [respectively, $\mathfrak{B}_K(A)$] as Γ ranges through all finite subsets of K^A. For each $\alpha \in A$ let $\alpha^\wedge$ be the projection $f \to f(\alpha)$ on K^A. Then $\alpha \to \alpha^\wedge$ is clearly a uniform structure isomorphism from A into $M(K^A)$, and we shall denote by $A^\wedge$ the image of A under this map.

THEOREM 2. *For any field K, $A^\wedge$ is dense in $M(K^A)$.*

Proof. Let $u \in M(K^A)$; we shall prove there exists a filter on $A^\wedge$ converging to u. Let H be the kernel of u, a proper ideal of K^A. For each finite subset Γ of H let $F(\Gamma) = [\alpha^\wedge \in A^\wedge: f(\alpha) = 0$ for all $f \in \Gamma]$. Clearly $F(\Gamma_1) \cap F(\Gamma_2) = F(\Gamma_1 \cup \Gamma_2)$, so to prove the sets $F(\Gamma)$ form a filter base for a filter $\mathfrak{F}$ on $A^\wedge$, it suffices to prove $F(\Gamma) \neq \phi$ for all finite subsets Γ of H. Suppose $F(\Gamma) = \phi$ for some $\Gamma = \{f_1, \ldots, f_n\} \subseteq$ H; we define $g_1, \ldots, g_n$ inductively by letting $g_1 = e$ and, for $j > 1$, letting g_j be the characteristic function of

$$\left[\alpha \in A: \left(\sum_{k=1}^{j-1} f_k g_k\right)(\alpha) = 0\right].$$

Then if $h = \Sigma_1^n f_j g_j$, $h \in H$ and, since $F(\Gamma) = \phi$, $h(\alpha) \neq 0$ for all $\alpha \in A$. But then $e = h\cdot(e/h) \in H$, so $H = K^A$ which is impossible. Finally, the filter $\mathfrak{F}$ thus defined converges to u: if $\Gamma = \{f_1, \ldots, f_n\} \subseteq K^A$, for each j let $h_j = f_j - u(f_j)e$; then $\Gamma_0 = \{h_1, \ldots, h_n\} \subseteq H$. $F(\Gamma_0) \subseteq V(\Gamma)(u)$, for if $\alpha^\wedge \in F(\Gamma_0)$ and if $1 \leqslant j \leqslant n$, $\alpha^\wedge(f_j) - u(f_j) = \alpha^\wedge(h_j + u(f_j)e) - u(f_j) = \alpha^\wedge(h_j) = h_j(\alpha) = 0$, by definition of $F(\Gamma_0)$. Hence $\mathfrak{F} \to u$, and the proof is complete.

COROLLARY 1. *If K is a field, every character of K^A is a projection if and only if $\mathfrak{U}_K(A)$ is complete.*

COROLLARY 2. *If K is a field, $(E_\alpha)_{\alpha\in A}$ a family of K-algebras indexed by A, and if $\mathfrak{U}_K(A)$ is complete, then for every character v of $E = \Pi_\alpha E_\alpha$ there exist $\beta \in A$ and a character v_β of E_β such that $v = v_\beta \circ pr_\beta$.*

3. The theorems of Bialynicki-Birula and Żelazko. An *Ulam measure* on set A is a non-zero, countably additive set-function λ, defined on the class of all subsets of A, taking on only the values 0 and 1, such that $\lambda(X) = 0$ for all finite subsets X of A; an *Ulam ultrafilter* on A is an ultrafilter $\mathfrak{U}$ such that the intersection of any countable subfamily of $\mathfrak{U}$ is again a member of $\mathfrak{U}$; a *point ultrafilter* on A is simply the (Ulam) ultrafilter of all subsets of A containing a given point of A. If λ is an Ulam measure, the sets X such that $\lambda(X) = 1$ form an Ulam ultrafilter which is not a point ultrafilter; conversely any Ulam ultrafilter which is not a point ultrafilter defines an Ulam measure. Thus A admits no Ulam measure if and only if every Ulam ultrafilter on A is a point ultrafilter.

Bialynicki-Birula and Żelazko **(1)** proved the following results (under the additional hypothesis in Theorem B that each E_α possessed an identity):

THEOREM B. *Let K be an infinite field, $(E_\alpha)_{\alpha \in A}$ a family of K-algebras indexed by a set A which either admits no Ulam measure or has cardinality not greater than that of K. Then for every character u of $E = \Pi_\alpha E_\alpha$, there exist $\beta \in A$ and a character v_β of E_β such that $u = v_\beta \circ pr_\beta$.*

THEOREM C. *If K is an infinite field admitting no Ulam measure, then A admits no Ulam measure if and only if every character of K^A is a projection.*

By Corollary 2 of Theorem 1, to prove Theorem B it suffices to show that either of its hypotheses concerning A ensures $\mathfrak{U}_K(A)$ is complete. If the cardinality of A is not greater than that of K, there exists a one-to-one function $g \in K^A$. $U(\{g\})$ is then the diagonal in $A \times A$, so $\mathfrak{U}_K(A)$ is the discrete uniform structure and hence is complete. Suppose A admits no Ulam measure. Let $\mathfrak{F}$ be a Cauchy filter on A and let $\mathfrak{U}$ be an ultrafilter containing $\mathfrak{F}$. $\mathfrak{U}$ is an Ulam ultrafilter: let $(\lambda_n)_{n\geqslant 0}$ be a sequence of distinct non-zero elements of K. If $(F_n)_{n\geqslant 0}$ is any decreasing sequence of members of $\mathfrak{U}$ such that $F_0 = A$, let $g(\alpha) = \lambda_n$ for all $\alpha \in F_n - F_{n+1}$, $g(\alpha) = 0$ for all $\alpha \in F = \bigcap_{n\geqslant 0} F_n$, and let $C \in \mathfrak{U}$ be $U(\{g\})$-small. If $C \cap (F_n - F_{n+1}) \neq \phi$, $C \cap F_{n+1} = \phi$ by definition of g, which is impossible. Hence $C \subseteq F$, so $F \in \mathfrak{U}$. Thus by hypothesis, as every Ulam ultrafilter is a point ultrafilter, there exists $\beta \in A$ which is contained in each member of $\mathfrak{U}$. β is then an adherent point of the Cauchy filter $\mathfrak{F}$, so $\mathfrak{F}$ converges to β and the proof is complete.

To prove Theorem C, it suffices by Theorem B and Corollary 1 of Theorem 2 to show that if $\mathfrak{U}_K(A)$ is complete and if every Ulam ultrafilter on K is a point ultrafilter, then every Ulam ultrafilter $\mathfrak{U}$ on A is a point ultrafilter. For each $f \in K^A$, $[L \subseteq K : f^{-1}(L) \in \mathfrak{U}]$ is clearly an Ulam ultrafilter on K, so there exists $\lambda \in K$ such that $f^{-1}(\lambda) \in \mathfrak{U}$. But $f^{-1}(\lambda)$ is $U(\{f\})$-small; it follows easily that $\mathfrak{U}$ is a Cauchy filter on A and therefore converges. As the topology defined by $\mathfrak{U}_K(A)$ is the discrete topology, $\mathfrak{U}$ is therefore a point ultrafilter.

4. Algebras over finite rings. We next ask for what other commutative rings R with identity does question (2) (and therefore question (1)) have an essentially affirmative answer. We first consider finite rings.

Let R be a finite commutative ring with identity 1. If γ and δ are idempotents in R, we write $\gamma \geqslant \delta$ if $\gamma\delta = \delta$, and obtain thus the usual partial ordering of idempotents; idempotent ϵ is *minimal* if $\epsilon \neq 0$ and if $\delta \leqslant \epsilon$ implies $\delta = 0$ or $\delta = \epsilon$. Let $(\epsilon_j)_{1 \leqslant j \leqslant n}$ be the set of all minimal idempotents. Then if $j \neq k$, $\epsilon_j \epsilon_k = 0$, and clearly $1 = \sum_1^n \epsilon_j$ (for otherwise, as R is finite, idempotent $1 - \sum_1^n \epsilon_j \geqslant$ some minimal idempotent not in $(\epsilon_j)_{1 \leqslant j \leqslant n}$.) Then R is the direct sum of ideals $(R\epsilon_j)_{1 \leqslant j \leqslant n}$, and every idempotent is the sum of a subfamily of $(\epsilon_j)_{1 \leqslant j \leqslant n}$ (so there exist exactly 2^n idempotents in R). If X is a subset of A, let $\phi_X \in R^A$ be its characteristic function. If u is a character of R^A, $u(\phi_X)$ is then an idempotent in R since ϕ_X is an idempotent in R^A.

THEOREM 3. *Let R be a finite commutative ring with identity, $\epsilon_1, \ldots, \epsilon_n$ $(n \geqslant 1)$ its minimal idempotents, and let Φ be the class of all ultrafilters on A. For each character u of R^A and for $1 \leqslant j \leqslant n$, let $\mathfrak{F}_{u,j} = [X \subseteq A : u(\phi_X) \geqslant \epsilon_j]$. Then $u \rightarrow (\mathfrak{F}_{u,1}, \ldots, \mathfrak{F}_{u,n})$ is a one-to-one map from $M(R^A)$ onto Φ^n. Hence if A is finite and has m members, $M(R^A)$ has m^n members; if A is infinite with cardinality $\aleph$, $M(R^A)$ has cardinality* exp $(\exp(\aleph))$.

Proof. $\mathfrak{F}_{u,j}$ is an ultrafilter: as $u(\phi_A) = 1 \geqslant \epsilon_j$ and $u(\phi_\phi) = 0 < \epsilon_j$, $A \in \mathfrak{F}_{u,j}$ and $\phi \notin \mathfrak{F}_{u,j}$; if $X, Y \in \mathfrak{F}_{u,j}$, $u(\phi_{X \cap Y}) = u(\phi_X)u(\phi_Y) \geqslant \epsilon_j^2 = \epsilon_j$, so $X \cap Y \in \mathfrak{F}_{u,j}$; if $X \in \mathfrak{F}_{u,j}$ and $Y \supseteq X$, $\phi_X = \phi_X \phi_Y$, so $u(\phi_X) = u(\phi_X)u(\phi_Y)$, that is, $u(\phi_Y) \geqslant u(\phi_X) \geqslant \epsilon_j$, and therefore $Y \in \mathfrak{F}_{u,j}$; finally, if $X \notin \mathfrak{F}_{u,j}$, $u(\phi_X)\epsilon_j = 0$ by minimality of ϵ_j, so

$$u(\phi_{A-X})\epsilon_j = [u(\phi_{A-X}) + u(\phi_X)]\epsilon_j = u(\phi_A)\epsilon_j = \epsilon_j,$$

that is, $u(\phi_{A-X}) \geqslant \epsilon_j$, and therefore $A - X \in \mathfrak{F}_{u,j}$. Thus $\mathfrak{F}_{u,j}$ is an ultrafilter. Next, suppose $\mathfrak{F}_{u,j} = \mathfrak{F}_{v,j}$ for $1 \leqslant j \leqslant n$. Given subset X of A,

$$\begin{aligned} u(\phi_X) = \sum[\epsilon_j : \epsilon_j \leqslant u(\phi_X)] = \sum[\epsilon_j ; X \in \mathfrak{F}_{u,j}] &= \sum[\epsilon_j : X \in \mathfrak{F}_{v,j}] \\ &= \sum[\epsilon_j : \epsilon_j \leqslant v(\phi_X)] = v(\phi_X). \end{aligned}$$

As R is finite, the functions ϕ_X generate R^A; hence $u = v$. Thus the map is one-to-one. Next, let $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ be any n (not necessarily distinct) ultrafilters on A, and let $1 \leqslant j \leqslant n$. As R is finite, for each $f \in R^A$ there exists one and only one $\lambda_{f,j} \in R$ such that $f^{-1}(\lambda_{f,j}) \in \mathfrak{F}_j$. If $f, g \in R^A$ and if $\mu \in R$, there exists

$$\alpha \in f^{-1}(\lambda_{f,j}) \cap g^{-1}(\lambda_{g,j}) \cap (f+g)^{-1}(\lambda_{f+g,j}) \cap (fg)^{-1}(\lambda_{fg,j}) \cap (\mu f)^{-1}(\lambda_{\mu f,j})$$

since $\mathfrak{F}_j$ is a filter. Hence

$$\begin{aligned} \lambda_{f+g,j} &= (f+g)(\alpha) = f(\alpha) + g(\alpha) = \lambda_{f,j} + \lambda_{g,j}, \\ \lambda_{fg,j} &= (fg)(\alpha) = f(\alpha)g(\alpha) = \lambda_{f,j}\lambda_{g,j}, \end{aligned}$$

and

$$\lambda_{\mu f,j} = (\mu f)(\alpha) = \mu f(\alpha) = \mu\lambda_{f,j}.$$

Let

$$u(f) = \sum_{j=1}^{n} \lambda_{f,j}\epsilon_j.$$

Then clearly from the above u is linear, and for any $f, g \in R^A$,

$$u(f)u(g) = \left(\sum_{j=1}^{n} \lambda_{f,j}\epsilon_j\right)\left(\sum_{k=1}^{n} \lambda_{g,k}\epsilon_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_{f,j}\lambda_{g,k}\epsilon_j\epsilon_k$$
$$= \sum_{j=1}^{n} \lambda_{f,j}\lambda_{g,j}\epsilon_j = \sum_{j=1}^{n} \lambda_{fg,j}\epsilon_j = u(fg).$$

Also, clearly $\lambda_{e,j} = 1$. Hence $u \in M(R^A)$. For any subset X of A,

$$1 = \lambda_{\phi_X,j}$$

if and only if $X \in \mathfrak{F}_j$, and

$$0 = \lambda_{\phi_X,j}$$

if and only if $X \notin \mathfrak{F}_j$; therefore $u(\phi_X) = \sum[\epsilon_j : X \in \mathfrak{F}_j]$, and so $u(\phi_X) \geqslant \epsilon_j$ if and only if $X \in \mathfrak{F}_j$. Hence $\mathfrak{F}_{u,j} = \mathfrak{F}_j$ for $1 \leqslant j \leqslant n$, and therefore the map is onto Φ^n. If A is finite with m members, every ultrafilter on A is a point ultrafilter, and therefore Φ^n has m^n members. Suppose A is infinite with cardinality $\aleph$. Then Φ has cardinality exp (exp($\aleph$)) (**2**, Exercise 14(c), p. 73), so $M(R^A)$ has cardinality $[\exp(\exp(\aleph))]^n = \exp(\exp(\aleph))$.

We see therefore that the answer to question (2) is in general negative if R is finite:

COROLLARY. *Let R be a finite commutative ring with identity 1, A a set containing more than one element. Then every character of R^A is a projection if and only if A is finite and the only idempotents of R are 0 and 1.*

5. Algebras over integral domains.

THEOREM 4. *Let D be an integral domain, K its field of quotients. The following two conditions are both necessary and sufficient for every character of the D-algebra D^A to be a projection:*

(1) *Every character of the K-algebra K^A is a projection;*

(2) *For every $u \in M(D^A)$ and every $f \in D^A$ such that $f(\alpha) \neq 0$ for all $\alpha \in A$, $u(f) \neq 0$.*

Proof. Necessity: if u is a projection of D^A and if $f(\alpha) \neq 0$ for all $\alpha \in A$, then $u(f) \neq 0$; hence (2) is necessary. If (1) does not hold, there exist characters of K^A which are not projections. Then by Corollary 1 of Theorem 2, $\mathfrak{U}_K(A)$ is an incomplete uniform structure. Let $\mathfrak{F}$ be a non-convergent Cauchy filter on A for $\mathfrak{U}_K(A)$. Then for any $f \in D^A$, $f(\mathfrak{F})$ is a Cauchy filter base in D, hence $\lim f(\mathfrak{F})$ exists and lies in D since $D \subseteq K$ is closed and K complete. Clearly $u: f \rightarrow \lim f(\mathfrak{F})$ is a character of D^A, and for the characteristic function ϕ_α of any $\{\alpha\}$, $\alpha \in A$, $u(\phi_\alpha) = 0$ since $\mathfrak{F}$ is not convergent. Thus u is not a projection.

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Sufficiency: let $u \in M(D^A)$. If $f \in K^A$, f is the quotient g/h of functions $g, h \in D^A$ where $h(\alpha) \neq 0$ for all $\alpha \in A$, and so by (2) $u(h) \neq 0$. It is easy to see that if we define $v(g/h)$ to be $u(g)/u(h)$ for all $g, h \in D^A$ such that $h(\alpha) \neq 0$ for all $\alpha \in A$, then v is a well-defined character of K^A extending u. As v is a projection by (1), u is also a projection.

In discussing principal domains we shall use the terminology and results of § 1 of Bourbaki's *Algèbre*, chapter 7. Also, we assume as part of the definition of compactness that compact spaces are separated.

THEOREM 5. *Let D be a principal domain possessing at least two non-associated extremal elements π and σ. If A either admits no Ulam measure or has cardinality not greater than that of D, every character of D^A is a projection.*

Proof. By Theorem 4 and Theorem B, it suffices to prove that if u is a character of D^A, then $u(f) \neq 0$ for every $f \in D^A$ satisfying $f(\alpha) \neq 0$ for all $\alpha \in A$. Define $p \in D^A$ such that for each $\alpha \in A$, $p(\alpha)$ is the highest power of π dividing $f(\alpha)$. Then $f = pq$ with $q \in D^A$ such that $(\pi, q(\alpha)) = 1$ for each $\alpha \in A$, and hence there exist $g_1, g_2 \in D^A$ such that $\pi g_1 + q g_2 = e$. This implies $\pi u(g_1) + u(q)u(g_2) = 1$ and therefore $u(q) \neq 0$ since π is not invertible. Similarly, there exist $h_1, h_2 \in D^A$ such that $\sigma h_1 + p h_2 = e$, yielding $u(p) \neq 0$. Therefore $u(f) = u(p)u(q) \neq 0$.

The author is indebted to the referee for the following theorem and remark.

THEOREM 6. *If R is a compact commutative ring with identity and if A is any infinite set, there exist characters of R^A which are not projections.*

Proof. Let $\mathfrak{U}$ be any ultrafilter on A which is not a point ultrafilter. For any $f \in R^A$, $f(\mathfrak{U})$ is an ultrafilter base on R and thus converges. Hence u: $f \to \lim f(\mathfrak{U})$ is clearly a character of R^A which is not a projection.

If D is the ring of p-adic integers for some prime p, D is a compact principal domain; if A is a countably infinite set, A admits no Ulam measure but there exist characters of D^A which are not projections by Theorem 6. Thus the condition in Theorem 5 that D have at least two non-associated extremal elements cannot be omitted without other restrictions on D.

6. Compact principal domains. Let K be a field with a discrete valuation v of rank 1. Then the valuation ring $D = [x \in K : v(x) \geqslant 0]$ is a principal domain whose field of quotients is K, and $P = [x \in K : v(x) > 0]$ is the unique maximal ideal of D. If the topology on K defined by v is locally compact (equivalently, if K is complete and if the residue class field D/P is finite (**3**, Exercise 24, p. 59)), D is compact.

Thus finite fields and valuation rings of locally compact fields whose topology is given by a discrete valuation of rank 1 are compact principal domains. We now show these are the only compact principal domains.

THEOREM 7. *If D is an infinite compact principal domain, then there exists a non-trivial discrete valuation v of rank 1 on the field of quotients K of D such that:*

(1) *The topology of K defined by v is locally compact and induces on D its given topology.*

(2) *D is the valuation ring of K with respect to v.*

Proof. As in the example following Theorem 6, if A is a countably infinite set, A admits no Ulam measure but by Theorem 6 there exist characters of D^A which are not projections; hence as D is infinite, D is not a field by Theorem B, so by Theorem 5 there exists an extremal element $p \in D$ such that $\{p\}$ is a representative system of extremal elements. For each non-zero element x of K there exist a unique unit u of D and a unique integer n such that $x = up^n$; if $x = up^n$, let $v(x) = n$, and let $v(0) = +\infty$. Clearly v is a discrete valuation of rank 1 on K and its valuation ring is D. The topology $\mathfrak{T}_v$ induced on D by the topology of K defined by v is separated, and the given topology $\mathfrak{T}$ of D is compact. Hence to prove $\mathfrak{T} = \mathfrak{T}_v$ it suffices to show $\mathfrak{T}_v$ is weaker than $\mathfrak{T}$, that is, for all positive integers n, $U_n = [x \in D : v(x) > n]$ is a neighbourhood of 0 for $\mathfrak{T}$. Let V be a neighbourhood of 0 for $\mathfrak{T}$ not containing $1, p, p^2, \ldots, p^n$. By **(3**, Exercise 7, p. 56**)** there exists a neighbourhood W of 0 for $\mathfrak{T}$ satisfying $DW \subseteq V$. Let $x = up^k \in W$. If $k \leqslant n$, $p^k = u^{-1}(up^k) \in DW \subseteq V$, a contradiction. Hence $k > n$, that is, $x \in U_n$. Thus $W \subseteq U_n$, so $\mathfrak{T}_v = \mathfrak{T}$. But then D is a compact neighbourhood of 0 for the topology of K, so K is locally compact.

COROLLARY. *A compact principal domain is metrizable, totally disconnected, and has exactly one maximal ideal.*

7. A topological application. Let T be a topological space, $\mathfrak{C}(T)$ the algebra over the real numbers $\mathbf{R}$ of all continuous real-valued functions on T. We shall apply Theorem B to give necessary and sufficient conditions on $\mathfrak{C}(T)$ for every connected component of T to be open. Let us call an algebra *decomposable* if it is the direct sum of two proper ideals, *indecomposable* otherwise. The following theorem is well known and easy to prove:

THEOREM 8. *T is connected if and only if $\mathfrak{C}(T)$ is indecomposable.*

Let us call *Ulam's Axiom* the assertion that there exist no Ulam measures; it is known that Ulam's Axiom is consistent with the usual axioms of set theory **(9**, pp. 207–8**)**. Let us call an algebra *fully decomposable* if it is isomorphic with the Cartesian product of indecomposable algebras.

THEOREM 9. *If every connected component of T is open, $\mathfrak{C}(T)$ is fully decomposable. Conversely, Ulam's Axiom is equivalent to the following assertion: if T is any topological space such that $\mathfrak{C}(T)$ is fully decomposable, then every connected component of T is open.*

Proof. If $(T_\alpha)_{\alpha \in A}$ is the family of all connected components of T and if each T_α is open, clearly $\mathfrak{C}(T)$ is isomorphic with $\Pi_{\alpha \in A}\, \mathfrak{C}(T_\alpha)$, and by Theorem

8 each $\mathfrak{C}(T_\alpha)$ is indecomposable. To prove the second assertion, we shall first prove the following lemma:

Lemma. *Let $(E_\alpha)_{\alpha \in A}$ be a family of non-zero algebras over the real numbers* $\mathbf{R}$ *indexed by a set A admitting no Ulam measure, and let g be an isomorphism from $E = \Pi_{\alpha \in A} E_\alpha$ onto $\mathfrak{C}(T)$. Then T is the topological sum of a family $(T_\alpha)_{\alpha \in A}$ of subsets, also indexed by A, and for all $\alpha \in A$ $\rho_\alpha \circ g \circ i_\alpha$ is an isomorphism from E_α onto $\mathfrak{C}(T_\alpha)$, where i_α is the canonical injection isomorphism from E_α into E, ρ_α the restriction homomorphism from $\mathfrak{C}(T)$ into $\mathfrak{C}(T_\alpha)$.*

Proof. E has an identity e as it is isomorphic with $\mathfrak{C}(T)$, so for all $\alpha \in A$, E_α has $e_\alpha = pr_\alpha(e)$ as its identity. Then $h_\alpha = (g \circ i_\alpha)(e_\alpha) \in \mathfrak{C}(T)$ is an idempotent, hence the characteristic function of an open-closed set $T_\alpha \subseteq T$. Now $\alpha \neq \beta$ implies $i_\alpha(e_\alpha) \cdot i_\beta(e_\beta) = 0$ and thus $h_\alpha \cdot h_\beta = 0$, that is, $T_\alpha \cap T_\beta = \phi$. Furthermore, for any $t \in T$, $t^\wedge \circ g$ is a character of E and hence, by Theorem B, $t^\wedge \circ g = v_\alpha \circ pr_\alpha$ for some $\alpha \in A$ and some character v_α of E. Then $h_\alpha(t) = t^\wedge(h_\alpha)$ $= t^\wedge(g(i_\alpha(e_\alpha))) = (v_\alpha \circ pr_\alpha)(i_\alpha(e_\alpha)) = v_\alpha(e_\alpha) = 1$ and therefore $t \in T_\alpha$; this shows T is the union and hence the topological sum of $(T_\alpha)_{\alpha \in A}$. Next, suppose $(\rho_\beta \circ g \circ i_\beta)(x) = 0$ for some $x \in E_\beta$. Then $(g \circ i_\beta(x))(t) = (\rho_\beta \circ g \circ i_\beta(x))(t)$ $= 0$ for $t \in T_\beta$, whereas for $t \in T_\alpha \neq T_\beta$, $h_\beta(t) = 0$ and hence

$$(g \circ i_\beta(x))(t) = (g \circ i_\beta(xe_\beta))(t) = (g \circ i_\beta(x))(t) \cdot h_\beta(t) = 0;$$

this means $g \circ i_\beta(x) = 0$ and therefore, as g and i_β are isomorphisms, $x = 0$. Hence $\rho_\beta \circ g \circ i_\beta$ is one-to-one. Finally, $\rho_\beta \circ g \circ i_\beta$ is onto: for any $f_\beta \in \mathfrak{C}(T_\beta)$ let $f \in \mathfrak{C}(T)$ be the function defined by $f(t) = f_\beta(t)$ if $t \in T_\beta$, $f(t) = 0$ otherwise. Since $f \cdot h_\alpha = 0$ for any $\alpha \neq \beta$, we have

$$\begin{aligned} pr_\alpha(g^{-1}(f)) = pr_\alpha(g^{-1}(f)) \cdot pr_\alpha(i_\alpha(e_\alpha)) &= pr_\alpha(g^{-1}(f) \cdot i_\alpha(e_\alpha)) \\ &= pr_\alpha(g^{-1}(f) \cdot g^{-1}(h_\alpha)) = (pr_\alpha \circ g^{-1})(f \cdot h_\alpha) = 0 \end{aligned}$$

for $\alpha \neq \beta$. This implies $(i_\beta \circ pr_\beta)(g^{-1}(f)) = g^{-1}(f)$ and thus

$$(\rho_\beta \circ g \circ i_\beta)(pr_\beta(g^{-1}(f))) = \rho_\beta(f) = f_\beta,$$

where $pr_\beta(g^{-1}(f)) \in E_\beta$.

We return now to the proof of the theorem. Let us assume Ulam's Axiom and suppose $\mathfrak{C}(T)$ is fully decomposable. Then by the lemma, T is the topological sum of a family $(T_\alpha)_{\alpha \in A}$ of subsets, and for all $\alpha \in A$, $\mathfrak{C}(T_\alpha)$ is isomorphic with an indecomposable algebra. Hence by Theorem 8 each T_α is connected. $(T_\alpha)_{\alpha \in A}$ is therefore the set of all connected components of T, and each T_α is open. Finally, suppose Ulam's Axiom is false. Now Ulam's Axiom is equivalent to the assertion that every discrete space S is a Q-space (that is, the weakest uniform structure $\mathfrak{V}_{\mathbf{R}}(S)$ on S for which each $f \in \mathfrak{C}(S)$ is uniformly continuous is complete). (Q-spaces are defined and discussed in **(7)**; a summary of results about Q-spaces is contained in **(6**, pp. 351–2**)**, and their relation to Ulam's Axiom is discussed in **(9**, pp. 206–8**)**.) Therefore there exists a discrete space S which is not a Q-space. Let T be the completion of S for

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$\mathfrak{B}_{\mathbf{R}}(S)$. Then $\mathfrak{C}(T)$ is isomorphic with $\mathfrak{C}(S)$, and as S is discrete, $\mathfrak{C}(S) = \mathbf{R}^S$, a fully decomposable algebra. Therefore $\mathfrak{C}(T)$ is fully decomposable. For any $s \in S$, there exists an open set V in T such that $V \cap S = \{s\}$; then $s \in V = V \cap \bar{S} \subseteq (V \cap S)^- = \{s\}^- = \{s\}$, so $\{s\}$ is both open and closed in T. Let C be a connected component of T containing some point in $T - S$. Then by the preceding, $s \notin C$ for all $s \in S$. Hence $C \subseteq T - S$. But then C cannot be open, since S is dense in T. $\mathfrak{C}(T)$ is therefore fully decomposable, but not every connected component of T is open.

COROLLARY. *Assume Ulam's Axiom. A topological space T is locally connected if and only if for every open subset G of T, $\mathfrak{C}(G)$ is fully decomposable.*

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