SOME RELATIONSHIPS BETWEEN FILTERS*

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A filter is a set theoretical concept and as such, its structure is independent of any topology which can be put on the given space. However, an O-filter, whose counterpart in the theory of nets is the O-nets of Robertson and Franklin [2], is defined with respect to the topology on the given space. The purpose of this paper is to give necessary and sufficient conditions for every O-filter to be an ultrafilter and for every Cauchy filter to be an O-filter.

1. Definition. A filter $\mathfrak{F}$ on a topological space (x, τ) is an O-filter if and only if for every open set $G \in \tau$, either G or $G^c \in \mathfrak{F}$.

It is clear that every ultrafilter is an O-filter. Robertson and Franklin [2] have given an example to show that an O-filter is not necessarily an ultrafilter.

2. LEMMA. If every O-filter on a topological space (X, τ) is an ultrafilter then (X, τ) is T_o.

Proof. Assume (X, τ) is not T_o. This implies for some $x, y \in X$ and $x \neq y$, each is a limit point of the other. Under this assumption we will construct an O-filter $\mathfrak{F}$ which is not an ultrafilter as follows: let (i) $\{x, y\} \in \mathfrak{F}$; (ii) for every open set $G \in \tau$, if $\{x, y\} \subset G$ then $G \in \mathfrak{F}$, otherwise $G^c \in \mathfrak{F}$. Clearly $\mathfrak{F}$ is an O-filter. But since neither $\{x\}$ nor $\{x\}^c$

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belongs to $\mathcal{F}$, $\mathcal{F}$ is not an ultrafilter.

3. LEMMA. Let (X, τ) be a topological space with at least three elements and such that $\{z\}$ is open for all $z \in X$ except $z = x$ and $z = y$. If at most one of x and y is a limit point of the other then every O-filter is an ultrafilter.

Proof. Without loss of generality we may assume that every open set containing y also contains x, but there exists an open set N_x, containing x, which does not contain y. Let $\mathcal{F}$ be an O-filter on X. If $\mathcal{F}$ is not an ultrafilter then there exists $A \subset X$ such that neither A nor $A^c \in \mathcal{F}$. Clearly one of x and y is in A and the other in A^c, since neither A nor A^c is open. Assume $x \in A$ and $y \in A^c$.

Let N_y be an arbitrary open neighbourhood of y. Then we prove that both N_y and N_x belong to $\mathcal{F}$. For, assume $N_y^c \in \mathcal{F}$. Then, since $(A \cup N_y)^c \subset A^c$ and $(A \cup N_y)^c$ is open, and since $A^c \notin \mathcal{F}$, clearly $A \cup N_y \in \mathcal{F}$. But then $(A \cup N_y) \cap N_y^c$, which is a subset of A, is also in $\mathcal{F}$. Hence $A \in \mathcal{F}$, a contradiction. Similarly $N_x \in \mathcal{F}$.

Hence $P = N_x \cap N_y \in \mathcal{F}$. If P were a subset of A or A^c then A or A^c would belong to $\mathcal{F}$. Hence R and S are non-empty where $R = A^c \cap P$ and $S = A \cap P - \{x\}$. Since R and S are open and contained in A^c and A respectively, R^c and S^c must both belong to $\mathcal{F}$. Hence $\{x\} = P \cap R^c \cap S^c \in \mathcal{F}$, implying $A \in \mathcal{F}$. Therefore $\mathcal{F}$ is an ultrafilter.

Combining Lemmas 2 and 3, we obtain the following:

4. THEOREM. Let (X, τ) be a topological space containing at least three elements all of which are open except $\{x\}$ and $\{y\}$. Every O-filter on X is an ultrafilter if and only if at most one of these points is a limit point of the other.

The following lemma follows from Definition 1.

5. LEMMA. A filter $\mathcal{F}$ on a topological space (X, τ)

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is an O-filter if and only if for each cover $\{H_i\}_{i=1}^{n}$ of X, where each H_i is either open or closed, then $H_i \in \mathcal{F}$ for some i.

It is easy to construct an example of a Cauchy filter which is not an O-filter (see e.g. Baggs [1]).

We now give a necessary and sufficient condition for a Cauchy filter on a complete uniform space to be an O-filter.

6. LEMMA. Every convergent filter on a topological space (X, τ) is an O-filter if and only if every open set is also closed.

Proof. Let every convergent filter be an O-filter. Let U be an arbitrary open set on X. We will show that U^c is open. By assumption every neighbourhood filter on X is an O-filter. If x is an arbitrary member of U^c, U is not a member of $N(x)$, the neighbourhood filter of x. Hence $U^c \in N(x)$. Therefore there exists an open set V such that $x \in V$ and $V \subseteq U^c$. Since this is true for every $x \in U^c$, U^c is open.

Conversely, let every open subset of X be closed. Let $A \in \tau$ and $\mathcal{F}$ be a convergent filter on X. For some $x \in X$, every neighbourhood of x is a member of $\mathcal{F}$. But since A is both open and closed, x has an open neighbourhood contained in either A or A^c. Hence A or $A^c \in \mathcal{F}$.

7. COROLLARY. In a complete uniform space every Cauchy filter is an O-filter if and only if every open set is closed.

Following the method of Sieber and Pervin [3], the following theorem can be easily shown:

8. THEOREM. Let (X, u) be a quasi-uniform space. The following are equivalent.

(i) (X, u) is precompact.

(ii) Every O-filter is a Cauchy filter.

(iii) Every ultrafilter is a Cauchy filter.

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