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L^p-trace-free generalized Korn inequalities for incompatible tensor fields in three space dimensions

Peter Lewintan
Faculty of Mathematics, University of Duisburg-Essen,
Thea-Leymann-Str. 9, 45127 Essen, Germany
(peter.lewintan@uni-due.de)

Patrizio Neff
Faculty of Mathematics, University of Duisburg-Essen,
Thea-Leymann-Str. 9, 45127 Essen, Germany (patrizio.neff@uni-due.de)

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For $1 < p < \infty$ we prove an L^p-version of the generalized trace-free Korn inequality for incompatible tensor fields P in $W_0^{1,p}(\mathrm{Curl}; \Omega, \mathbb{R}^{3\times3})$. More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a constant $c > 0$ such that

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\left(\|\mathrm{dev\,sym}\, P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{dev\,Curl}\, P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\right)$$

holds for all tensor fields $P \in W_0^{1,p}(\mathrm{Curl}; \Omega, \mathbb{R}^{3\times3})$, i.e., for all $P \in W^{1,p}(\mathrm{Curl}; \Omega, \mathbb{R}^{3\times3})$ with vanishing tangential trace $P \times \nu = 0$ on $\partial\Omega$ where ν denotes the outward unit normal vector field to $\partial\Omega$ and $\mathrm{dev}\, P := P - \frac{1}{3}\mathrm{tr}(P)\cdot\mathbb{1}$ denotes the deviatoric (trace-free) part of P. We also show the norm equivalence

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{Curl}\, P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\left(\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{dev\,Curl}\, P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\right)$$

for tensor fields $P \in W^{1,p}(\mathrm{Curl}; \Omega, \mathbb{R}^{3\times3})$. These estimates also hold true for tensor fields with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary.

Keywords: $W^{1,p}$(Curl)-Korn's inequality; Poincaré's inequality; Lions lemma; Nečas estimate; incompatibility; Curl-spaces; Maxwell problems; gradient plasticity; dislocation density; relaxed micromorphic model; Cosserat elasticity; Kröner's incompatibility tensor; Saint-Venant compatibility; trace-free Korn's inequality; conformal mappings; conformal Killing vector field; Nye's formula

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In memoriam of Sergio Dain [1970-2016],
who gave the first proof of the trace-free Korn's inequality
on bounded Lipschitz domains.

1. Introduction

Korn-type inequalities are crucial for a priori estimates in linear elasticity and fluid mechanics. They allow to bound the L^p-norm of the gradient $\mathrm{D}u$ in terms of the symmetric gradient, i.e. Korn's first inequality states

$$\exists\, c>0\ \forall\, u\in W_0^{1,p}(\Omega,\mathbb{R}^n):\quad \|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}\leqslant c\,\|\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}. \tag{1.1}$$

Generalizations to many different settings have been obtained in the literature, including the geometrically nonlinear counterpart [23, 24, 39], mixed growth conditions [15], incompatible fields (also with dislocations) [6, 40–43, 48, 55–58], as well as the case of non-constant coefficients [37, 50, 59, 62] and on Riemannian manifolds [9]. In this paper we focus on their improvement towards the trace-free case:

$$\exists\, c>0\ \forall\, u\in W_0^{1,p}(\Omega,\mathbb{R}^n):\quad \|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}\leqslant c\,\|\operatorname{dev}_n\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}, \tag{1.2}$$

where $\operatorname{dev}_n X := X-\frac{1}{n}\operatorname{tr}(X)\cdot\mathbb{1}$ denotes the deviatoric (trace-free) part of the square matrix X. Note in passing that (1.2) implies (1.1).

There exist many different proofs and generalizations of the trace-free classical Korn's inequality in the literature, see [63, theorem 2] but also [6, 17, 27, 33, 64, 65] as well as [67] for trace-free Korn's inequalities in pseudo-Euclidean space and [17, 32] for trace-free Korn inequalities on manifolds, [8, 25] for trace-free Korn inequalities in Orlicz spaces and [18, 45] for weighted trace-free Korn inequalities in Hölder and John domains. Such coercive inequalities found application in micro-polar Cosserat-type models [27, 33, 34, 49] and general relativity [17]. On the other hand, corresponding trace-free coercive inequalities for incompatible tensor fields are useful in infinitesimal gradient plasticity as well as in linear relaxed micromorphic elasticity, see [31, 51] but also [6, sec. 7] and the references contained therein.

Notably, in case $n=2$, the condition $\operatorname{dev}_2\operatorname{sym}\mathrm{D}u\equiv 0$ becomes the system of Cauchy-Riemann equations, so that the corresponding kernel is infinite-dimensional and an adequate quantitative version of the trace-free classical Korn's inequality does not hold true. Nevertheless, in [27] it is proved that

$$\|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{2\times 2})}\leqslant c\,\|\operatorname{dev}_2\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{2\times 2})} \tag{1.3}$$

holds for each $u\in W_0^{1,p}(\Omega,\mathbb{R}^2)$,[1] but, again, this result ceases to be valid if the Dirichlet conditions are prescribed only on a part of the boundary, cf. the counterexample in [6, sec. 6.6].

Korn-type inequalities fail for the limiting cases $p=1$ and $p=\infty$. Indeed, from the counterexamples traced back in [16, 38, 47, 61] it follows that $\int_\Omega|\operatorname{sym}\mathrm{D}u|\mathrm{d}x$ does not dominate each quantity $\int_\Omega|\partial_i u_j|\mathrm{d}x$ for any vector field $u\in W_0^{1,1}(\Omega,\mathbb{R}^n)$. Hence, also trace-free versions fail for $p=1$ and $p=\infty$. On the other hand, Poincaré-type inequalities estimating certain integral norms of the deformation u in terms of the total variation of the symmetric strain tensor $\operatorname{sym}\mathrm{D}u$ are still valid.

[1] A simple proof using partial integration is given in the appendix for the case $p=2$ and all dimensions.

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In particular, for Poincaré-type inequalities for functions of bounded deformation involving the deviatoric part of the symmetric gradient we refer to [**26**].

The classical Korn's inequalities need compatibility, i.e. a gradient $\mathrm{D}u$; giving up the compatibility necessitates controlling the distance of P to a gradient by adding the incompatibility measure (the dislocation density tensor) $\operatorname{Curl} P$. We showed in [**43**] the following quantitative version of Korn's inequality for incompatible tensor fields $P \in W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$:

$$\inf_{\widetilde{A}\in\mathfrak{so}(3)} \|P - \widetilde{A}\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\, \left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\right). \quad (1.4)$$

Note that the constant skew-symmetric matrix fields (restricted to Ω) represent the elements from the kernel of the right-hand side of (1.4). For compatible $P = \mathrm{D}u$ recover from (1.4) the quantitative version of the classical Korn's inequality, namely for $u \in W^{1,p}(\Omega, \mathbb{R}^3)$:

$$\inf_{\widetilde{A}\in\mathfrak{so}(3)} \|\mathrm{D}u - \widetilde{A}\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\, \|\operatorname{sym} \mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \quad (1.5)$$

and for skew-symmetric matrix fields $P = A \in \mathfrak{so}(3)$ the corresponding Poincaré inequality for squared skew-symmetric matrix fields $A \in W^{1,p}(\Omega, \mathfrak{so}(3))$ (and thus for vectors in $\mathbb{R}^3$):

$$\inf_{\widetilde{A}\in\mathfrak{so}(3)} \|A - \widetilde{A}\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\, \|\operatorname{Curl} A\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant \tilde{c}\, \|\mathrm{D}A\|_{L^p(\Omega,\mathbb{R}^{3\times3^2})}, \quad (1.6)$$

where in the last step we have used that Curl consists of linear combinations from D. Interestingly, for skew-symmetric A also the converse is true, more precisely, the entries of $\mathrm{D}A$ are linear combinations of the entries from $\operatorname{Curl} A$, cf. e.g. [**43**, Cor. 2.3]:

$$\mathrm{D}A = L(\operatorname{Curl} A) \qquad \text{for skew-symmetric } A, \quad (1.7)$$

where $L(.)$ denotes a corresponding linear operator with constant coefficients, not necessarily the same in any two places in the present paper. In fact, the mentioned results also hold in higher dimensions $n > 3$, see [**42**] and the discussion contained therein. In our proof of (1.4) we were highly inspired by a proof of (1.5) advocated by P. G. Ciarlet and his collaborators [**10–14**, **19**, **29**], which uses the Lions lemma resp. Nečas estimate, the compact embedding $W^{1,p} \subset\subset L^p$ and the representation of the second distributional derivatives of the displacement u by a linear combination of the first derivatives of the symmetrized gradient $\mathrm{D}u$:

$$\mathrm{D}^2 u = L(\mathrm{D} \operatorname{sym} \mathrm{D}u). \quad (1.8)$$

It is worth mentioning that the role of the latter ingredient (1.8) was taken over by (1.7) in our proof of (1.4) in [**43**] resp. [**42**]. In $n = 3$ dimensions the relation (1.7)

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is an easy consequence of the so called *Nye's formula* [**60**, eq. (7)]:

$$\operatorname{Curl} A = \operatorname{tr}(\operatorname{Daxl} A)\cdot \mathbb{1} - (\operatorname{Daxl} A)^T, \tag{1.9a}$$

resp.

$$\operatorname{Daxl} A = \frac{1}{2}\operatorname{tr}(\operatorname{Curl} A)\cdot \mathbb{1} - (\operatorname{Curl} A)^T, \tag{1.9b}$$

where we identify the vectorspace of skew-symmetric matrices $\mathfrak{so}(3)$ and $\mathbb{R}^3$ via $\operatorname{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ which is defined by the cross product:

$$A\, b =: \operatorname{axl}(A) \times b \quad \forall\, b \in \mathbb{R}^3, \tag{1.10}$$

and associates with a skew-symmetric matrix $A \in \mathfrak{so}(3)$ the vector $\operatorname{axl} A := (-A_{23}, A_{13}, -A_{12})^T$. The relation (1.9a) admits moreover a counterpart on the group of orthogonal matrices O(3) and even in higher spatial dimensions, see [**54**]. In fact, Nye's formula is (formally) a consequence of the following algebraic identity:

$$(\operatorname{Anti} a) \times b = b \otimes a - \langle b, a\rangle \cdot \mathbb{1} \qquad \forall\, a, b \in \mathbb{R}^3, \tag{1.11}$$

where the vector product of a matrix and a vector is to be seen row-wise and $\operatorname{Anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the inverse of axl. Despite the absence of the simple algebraic relations in the higher dimensional case a corresponding relation to (1.7) also holds true in $n > 3$, see e.g. [**42**].

Moreover, the kernel in quantitative versions of Korn's inequalities is killed by corresponding boundary conditions, namely by a vanishing trace condition $u_{|\partial\Omega} = 0$ in the case of (1.5) and (1.6) and by a vanishing tangential trace condition $P \times \nu_{\ |\partial\Omega} = 0$ in the general case (1.4), cf. [**42**, **43**].

The objective of the present paper is to improve on inequality (1.4) by showing that it already suffices to consider the deviatoric (trace-free) parts on the right-hand side, hence, further contributing to the problems proposed in [**58**]. More precisely, the main results are

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p, \Omega) > 0$ such that for all $P \in L^p(\Omega, \mathbb{R}^{3\times3})$ we have*

$$\inf_{T \in K_{dS,dC}} \|P - T\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \leqslant c\left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})}\right), \tag{1.12}$$

where $\operatorname{dev} X := X - \frac{1}{3}\operatorname{tr}(X)\cdot\mathbb{1}$ *denotes the deviatoric part of a square tensor* $X \in \mathbb{R}^{3\times3}$ *and* $K_{dS,dC}$ *represent the kernel of the right-hand side and is given by*

$$K_{dS,dC} = \{T : \Omega \to \mathbb{R}^{3\times3} \mid T(x) = \operatorname{Anti}\left(\widetilde{A}\, x + \beta\, x + b\right) + \left(\langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma\right)\cdot\mathbb{1}, \quad \widetilde{A} \in \mathfrak{so}(3), b \in \mathbb{R}^3, \beta, \gamma \in \mathbb{R}\}. \tag{1.13}$$

By killing the kernel with tangential trace conditions (note that $\operatorname{dev}(P \times \nu) = 0$ iff $P \times \nu = 0$) we arrive at the following Korn's first type inequality

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THEOREM 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p, \Omega) > 0$ such that we have*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c \left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right) \tag{1.14}$$

for all

$$\begin{aligned} P \in W_0^{1,p}&(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3}) \\ &:= \{P \in L^p(\Omega,\mathbb{R}^{3\times 3}) \mid \operatorname{Curl} P \in L^p(\Omega,\mathbb{R}^{3\times 3}),\ P \times \nu \equiv 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The appearance of the term $\operatorname{dev}\operatorname{Curl} P$ on the right-hand side of (1.14) would suggest to consider p-integrable tensor fields P with 'only' p-integrable $\operatorname{dev}\operatorname{Curl} P$. However, this would not lead to a new Banach space, since we show that for all $m \in \mathbb{Z}$ it holds that

$$\operatorname{Curl} P \in W^{m,p}(\Omega,\mathbb{R}^{3\times 3}) \quad \Leftrightarrow \quad \operatorname{dev}\operatorname{Curl} P \in W^{m,p}(\Omega,\mathbb{R}^{3\times 3}). \tag{1.15}$$

The estimate (1.14) generalizes the corresponding result in [**6**] from the L^2-setting to the L^p-setting, whereas the trace-free second type inequality (1.12) is completely new. Generalizations to different right-hand sides and higher dimensions have been obtained in the recent papers [**40**, **41**]. Note however that the estimates (1.12) and (1.14) are restricted to the case of three dimensions since the deviatoric operator acts on square matrices and only in the three-dimensional setting the matrix Curl returns a square matrix.

Again, for compatible $P = \mathrm{D}u$ we get back a tangential trace-free classical Korn inequality for the displacement gradient, namely

$$\|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\|\operatorname{dev}\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \quad \text{with } \mathrm{D}u \times \nu = 0 \text{ on } \partial\Omega \tag{1.16}$$

as well as

$$\inf_{T \in K_{dS,C}} \|\mathrm{D}u - T\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\|\operatorname{dev}\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \tag{1.17}$$

respectively

$$\|u - \Pi u\|_{W^{1,p}(\Omega,\mathbb{R}^3)} \leqslant c\,\|\operatorname{dev}\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}, \tag{1.18}$$

where Π denotes an arbitrary projection operator from $W^{1,p}(\Omega,\mathbb{R}^3)$ onto the space of *conformal Killing vectors*, here the finite dimensional kernel of $\operatorname{dev}\operatorname{sym}\mathrm{D}$, which is given by quadratic polynomials of the form

$$\begin{aligned} \varphi_c(x) = \langle a, x\rangle\, x - \frac{1}{2} a\,\|x\|^2 + \operatorname{Anti}(b)\,x + \beta\,x + c, \\ \text{with } a := \operatorname{axl}\widetilde{A}, b, c \in \mathbb{R}^3 \text{and } \beta \in \mathbb{R}, \end{aligned}$$

namely the *infinitesimal conformal mappings*, cf. [**17**, **33**, **49**, **63**–**65**], see figure 1 for an illustration in 2D.

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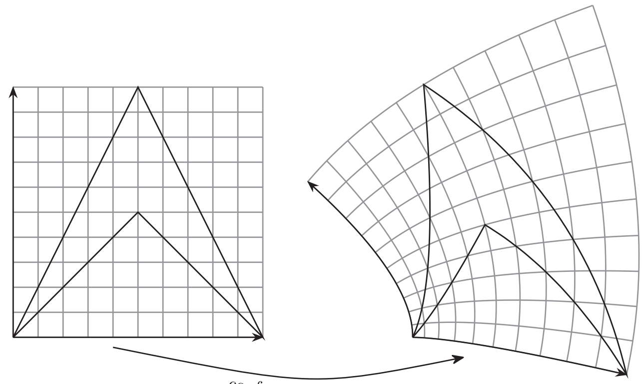

Figure 1. In the planar case, the condition $\operatorname{dev}_2 \operatorname{sym} \mathrm{D}u = 0$ coincides with the Cauchy-Riemann equations for the function u (see appendix). Therefore, infinitesimal conformal mappings in 2D are holomorphic functions which preserve angles exactly. This ceases to be the case for 3D infinitesimal conformal mappings defined by $\operatorname{dev}_3 \operatorname{sym} \mathrm{D}u = 0$.

A first proof of (1.18), even in all dimensions $n \geqslant 3$, was given by Reshetnyak [**63**] over domains which are star-like with respect to a ball. Over bounded Lipschitz domains the trace-free Korn's second inequality in all dimensions $n \geqslant 3$, namely

$$\exists c > 0 \ \forall u \in W^{1,p}(\Omega, \mathbb{R}^n): \\ \|u\|_{W^{1,p}(\Omega,\mathbb{R}^n)} \leqslant c \left(\|u\|_{L^p(\Omega,\mathbb{R}^n)} + \|\operatorname{dev}_n \operatorname{sym} \mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}\right), \tag{1.19}$$

was justified by Dain [**17**] in the case $p = 2$ and by Schirra [**65**] for all $p > 1$. Their proofs use again the Lions lemma and the 'higher order' analogues of the differential relation (1.8):

$$\mathrm{D}\Delta u = L(\mathrm{D}^2 \operatorname{dev}_n \operatorname{sym} \mathrm{D}u). \tag{1.20}$$

However, the differential operators $\operatorname{sym} \mathrm{D}$ and $\operatorname{dev}_n \operatorname{sym} \mathrm{D}$ are particular cases of the so-called *coercive elliptic operators* whose study began with Aronszajn [**5**].

Let us go back to

$$\|P\|_{L^2(\Omega,\mathbb{R}^{3\times3})} \leqslant c \left(\|\operatorname{dev} \operatorname{sym} P\|_{L^2(\Omega,\mathbb{R}^{3\times3})} + \|\operatorname{dev} \operatorname{Curl} P\|_{L^2(\Omega,\mathbb{R}^{3\times3})}\right) \tag{1.21}$$

whose first proof for $P \in W_0^{1,2}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3})$ was given in [**6**] via the trace-free classical Korn's inequality, a Maxwell estimate and a Helmholtz decomposition and is not directly amenable to the L^p-case. Here, we catch up with the latter.

In the following section we start by summarizing the notations and collect some preliminary results from algebraic calculations which are needed in the subsequent vector calculus to establish relations of the type:

$$\mathrm{D}^3(A + \zeta \cdot \mathbb{1}) = L(\mathrm{D}^2 \operatorname{dev} \operatorname{Curl}(A + \zeta \cdot \mathbb{1})) \tag{1.22}$$

for skew-symmetric tensor fields A and scalar functions ζ, where L denotes a corresponding constant coefficients linear operator. Based on this 'higher order' analogue

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of the differential relation (1.7) we prove our main results in the last section using a similar argumentation as in [17, 65] which argue by the Lions lemma resp. Nečas estimate and the compact embedding $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

2. Notations and preliminaries

Let $n \geqslant 2$. We consider for vectors $a, b \in \mathbb{R}^n$ the scalar product $\langle a, b\rangle := \sum_{i=1}^n a_i\, b_i \in \mathbb{R}$, the (squared) norm $\|a\|^2 := \langle a, a\rangle$ and the dyadic product $a \otimes b := (a_i\, b_j)_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$. Similarly, we define the scalar product for matrices $P, Q \in \mathbb{R}^{n\times n}$ by $\langle P, Q\rangle := \sum_{i,j=1}^n P_{ij}\, Q_{ij} \in \mathbb{R}$ and the (squared) Frobenius-norm by $\|P\|^2 := \langle P, P\rangle$. We highlight by $\cdot$ the scalar multiplication of a scalar with a matrix, whereas matrix multiplication is denoted only by juxtaposition.

Moreover, $P^T := (P_{ji})_{i,j=1,\dots,n}$ denotes the transposition of the matrix $P = (P_{ij})_{i,j=1,\dots,n}$. The latter decomposes orthogonally into the symmetric part $\operatorname{sym} P := \frac{1}{2}(P + P^T)$ and the skew-symmetric part $\operatorname{skew} P := \frac{1}{2}(P - P^T)$. We will denote by $\mathfrak{so}(n) := \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}$ the Lie-Algebra of skew-symmetric matrices.

For the identity matrix we will write $\mathbb{1}$, so that the trace of a squared matrix P is given by $\operatorname{tr} P := \langle P, \mathbb{1}\rangle$. The deviatoric (trace-free) part of P is given by $\operatorname{dev}_n P := P - \frac{1}{n}\operatorname{tr}(P)\cdot\mathbb{1}$ and in three dimensions its index will be suppressed, i.e. we write dev instead of dev_3.

We will denote by $\mathscr{D}'(\Omega)$ the space of distributions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and by $W^{-k,p}(\Omega)$ the dual space of $W_0^{k,p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the Hölder dual exponent to p.

Throughout the paper we use c as a generic positive constant, which is not necessarily the same in any two places, and we use $L(.)$ as a generic linear operator with constant coefficients, which also may differ in any two places within the paper.

In 3-dimensions we make use of the vector product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Since the vector product $a \times .$ with a fixed vector $a \in \mathbb{R}^3$ is linear in the second component, there exists a unique matrix $\operatorname{Anti}(a)$ such that

$$a \times b =: \operatorname{Anti}(a) b \quad \forall\, b \in \mathbb{R}^3, \tag{2.1}$$

and direct calculations show that for $a = (a_1, a_2, a_3)^T$ the matrix $\operatorname{Anti}(a)$ has the form

$$\operatorname{Anti}(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \tag{2.2}$$

The inverse of $\operatorname{Anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is denoted by $\operatorname{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ and fulfills $\operatorname{axl}(A) \times b = A\, b$ for all skew-symmetric (3×3)-matrices A and vectors $b \in \mathbb{R}^3$. The matrix representation of the cross product allows for a generalization towards a cross product of a matrix $P \in \mathbb{R}^{3\times 3}$ and a vector $b \in \mathbb{R}^3$ via

$$P \times b := P\ \operatorname{Anti}(b), \tag{2.3}$$

so, especially, for $P = \mathbb{1}$ it holds

$$\mathbb{1} \times b = \mathbb{1}\operatorname{Anti}(b) = \operatorname{Anti}(b) \qquad \forall\ b \in \mathbb{R}^3. \tag{2.4}$$

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We repeat the following crucial algebraic identity:

$$(\operatorname{Anti} a) \times b = b \otimes a - \langle b, a\rangle \cdot \mathbb{1} \quad \forall\ a, b \in \mathbb{R}^3. \tag{2.5}$$

Observation 3. *For $P \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$ we have*

$$\operatorname{dev}(P \times b) = 0 \;\Leftrightarrow\; P \times b = 0. \tag{2.6}$$

Proof. We decompose P into its symmetric and skew-symmetric part, i.e.,

$P = S + A = S + \operatorname{Anti}(a)$, for some $S \in \operatorname{Sym}(3)$, $A \in \mathfrak{so}(3)$ and with $a = \operatorname{axl}(A)$.

For a symmetric matrix S it holds $\operatorname{tr}(S \times b) = 0$ for any $b \in \mathbb{R}^3$, since[2]

$$\operatorname{tr}(S \times b) = \langle S \times b, \mathbb{1}\rangle_{\mathbb{R}^{3\times 3}} = \langle S \operatorname{Anti}(b), \mathbb{1}\rangle_{\mathbb{R}^{3\times 3}} = -\langle S, \operatorname{Anti}(b)\rangle_{\mathbb{R}^{3\times 3}} \overset{S\in\operatorname{Sym}(3)}{=} 0. \tag{2.7}$$

Thus, using the decomposition $P = S + \operatorname{Anti}(a)$, we have:

$$\begin{aligned}\operatorname{dev}(P \times b) &= P \times b - \frac{1}{3}\operatorname{tr}(P \times b)\cdot \mathbb{1} \overset{(2.7)}{=} P \times b - \frac{1}{3}\operatorname{tr}((\operatorname{Anti} a) \times b)\cdot \mathbb{1} \\ &\overset{(2.5)}{=} P \times b - \frac{1}{3}\operatorname{tr}(b \otimes a - \langle b, a\rangle \cdot \mathbb{1})\cdot \mathbb{1} = P \times b + \frac{2}{3}\langle a, b\rangle \cdot \mathbb{1}.\end{aligned} \tag{2.8}$$

Moreover, for any matrix $P \in \mathbb{R}^{3\times 3}$ we note that

$$(P \times b)\, b = (P \operatorname{Anti}(b))b = P(\operatorname{Anti}(b)\, b) = P(b \times b) = 0. \tag{2.9}$$

Thus, we obtain

$$\langle b, \operatorname{dev}(P \times b)\, b\rangle \overset{(2.8)}{=} \left\langle b, \left(P \times b + \frac{2}{3}\langle a, b\rangle \cdot \mathbb{1}\right) b \right\rangle \overset{(2.9)}{=} \frac{2}{3}\langle a, b\rangle\, \|b\|^2, \tag{2.10}$$

and the conclusion follows from the identity

$$\begin{aligned}\|b\|^2 \cdot P \times b &\overset{(2.8)}{=} \|b\|^2 \cdot \operatorname{dev}(P \times b) - \frac{2}{3}\|b\|^2 \langle a, b\rangle \cdot \mathbb{1} \\ &\overset{(2.10)}{=} \|b\|^2 \cdot \operatorname{dev}(P \times b) - \langle b, \operatorname{dev}(P \times b)\, b\rangle \cdot \mathbb{1}.\end{aligned} \tag{2.11}$$

An application of the Cauchy-Bunyakovsky-Schwarz inequality on the right-hand side of (2.11) shows that

$$\|\operatorname{dev}(P \times b)\| \leqslant \|P \times b\| \leqslant \left(1 + \sqrt{3}\right) \|\operatorname{dev}(P \times b)\|. \tag{2.12}$$

□

Observation 4. *Let $a \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, then*

$$(\operatorname{Anti}(a) + \alpha \cdot \mathbb{1}) \times b = 0 \textit{ for } b \in \mathbb{R}^3 \backslash \{0\} \quad \Rightarrow \quad a = 0 \textit{ and } \alpha = 0.$$

[2] Cf. the appendix for component-wise calculations.

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Proof. By (2.5) and (2.4) we have:

$$0 = (\text{Anti}(a) + \alpha \cdot \mathbb{1}) \times b = b \otimes a - \langle b, a \rangle \cdot \mathbb{1} + \alpha \cdot \text{Anti}(b). \tag{2.13}$$

Taking the trace on both sides we obtain

$$0 = \text{tr}(b \otimes a - \langle b, a \rangle \cdot \mathbb{1} + \alpha \cdot \text{Anti}(b)) = \langle a, b \rangle - 3 \langle a, b \rangle = -2 \langle a, b \rangle.$$

Thus, reinserting $\langle b, a \rangle = 0$ in (2.13) and applying sym on both sides, this implies $\text{sym}(b \otimes a) = 0$. Since

$$\|\text{sym}(a \otimes b)\|^2 = \frac{1}{2}\|a\|^2\|b\|^2 + \frac{1}{2}\langle a, b \rangle^2 \tag{2.14}$$

and $b \neq 0$ we must have $a = 0$. Hence, by (2.13) also $\alpha = 0$. □

Formally the gradient and the curl of a vector field $a : \Omega \to \mathbb{R}^3$ can be seen as

$$\mathrm{D}a = a \otimes \nabla \quad \text{and} \quad \text{curl}\, a = a \times (-\nabla).$$

The latter also generalizes to (3×3)-matrix fields $P : \Omega \to \mathbb{R}^{3\times 3}$ row-wise:[3]

$$\text{Curl}\, P = P \times (-\nabla) = \begin{pmatrix} (P^T e_1)^T \\ (P^T e_2)^T \\ (P^T e_3)^T \end{pmatrix} \times (-\nabla) = \begin{pmatrix} (\text{curl}\,(P^T e_1))^T \\ (\text{curl}\,(P^T e_2))^T \\ (\text{curl}\,(P^T e_3))^T \end{pmatrix} \in \mathbb{R}^{3\times 3}. \tag{2.15}$$

Replacing b by ∇ in (2.5) we obtain Nye's formulas

$$\text{Curl}\, A = \text{tr}(\mathrm{D}\text{axl}\, A) \cdot \mathbb{1} - (\mathrm{D}\text{axl}\, A)^T, \tag{2.16a}$$

and

$$\mathrm{D}\text{axl}\, A = \frac{1}{2}\, \text{tr}(\text{Curl}\, A) \cdot \mathbb{1} - (\text{Curl}\, A)^T \tag{2.16b}$$

for all skew-symmetric (3×3)-matrix fields A.

Remark 5. Formal calculations (e.g. replacing b by ∇) have to be performed very carefully. Indeed, they are allowed in algebraic identities but fail, in general, for implications, e.g. for $A \in \mathfrak{so}(3)$ and $b \in \mathbb{R}^3$ we have $A \times b = 0$ if and only if $\text{dev}(A \times b) = 0$, since the following expression holds true, cf. Observation 3 and (2.11):

$$\|b\|^2 \cdot A \times b = \|b\|^2 \cdot \text{dev}(A \times b) - \langle b, \text{dev}(A \times b)\, b \rangle \cdot \mathbb{1}\, . \tag{2.17}$$

However, $\text{dev}(\text{Curl}\, A) = \text{dev}(A \times (-\nabla)) = 0$ does not imply already that $\text{Curl}\, A = A \times (-\nabla) = 0$, due to the counterexample $A = \text{Anti}(x)$, since by Nye's formula (2.16) we have $\text{Curl}(\text{Anti}(x)) = 2 \cdot \mathbb{1}$. Of course, we can interpret (2.17) also in the sense of vector calculus, which gives then an expression for $\Delta\, \text{Curl}\, A$ in terms of the second distributional derivatives of $\text{dev}(\text{Curl}\, A)$, but, the latter would have no meaning for the relation of $\text{Curl}\, A$ and $\text{dev}\, \text{Curl}\, A$.

[3]In the literature, the matrix Curl operator is sometimes defined as our transposed $(\text{Curl}\, P)^T$, cf. Ciarlet [**12**, problem 6.18-4].

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Lemma 6. *Let* $A \in \mathscr{D}'(\Omega, \mathfrak{so}(3))$ *and* $\zeta \in \mathscr{D}'(\Omega, \mathbb{R})$*. Then*

(a) *the entries of* $\mathrm{D}^2(A + \zeta \cdot \mathbb{1})$ *are linear combinations of the entries of* $\mathrm{D}\,\mathrm{Curl}(A + \zeta \cdot \mathbb{1})$*.*

(b) *the entries of* $\mathrm{D}^2 A$ *are linear combinations of the entries of* $\mathrm{D}\,\mathrm{dev}\,\mathrm{Curl}\,A$*.*

(c) *the entries of* $\mathrm{D}^3(A + \zeta \cdot \mathbb{1})$ *are linear combinations of the entries of* $\mathrm{D}^2\,\mathrm{dev}\,\mathrm{Curl}(A + \zeta \cdot \mathbb{1})$*.*

Proof. Observe that applying (2.4) to the vector field $\nabla\zeta$ we obtain:

$$\mathrm{Curl}(\zeta \cdot \mathbb{1}) \overset{(2.15)}{=} \mathbb{1} \times (-\nabla\zeta) \overset{(2.4)}{=} -\mathrm{Anti}(\nabla\zeta). \tag{2.18}$$

Let us first start by proving part (b). From Nye's formula (2.16a) we obtain

$$\mathrm{dev}\,\mathrm{Curl}\,A = \frac{1}{3}\,\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,A)\cdot\mathbb{1} - (\mathrm{D}\,\mathrm{axl}\,A)^T \tag{2.19}$$

so that taking the Curl of the transpositions on both sides gives

$$\mathrm{Curl}([\mathrm{dev}\,\mathrm{Curl}\,A]^T) \underset{(2.19)}{\overset{\mathrm{Curl}\circ\mathrm{D}\equiv 0}{=}} \frac{1}{3}\,\mathrm{Curl}(\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,A)\cdot\mathbb{1}) \overset{(2.18)}{=} -\frac{1}{3}\,\mathrm{Anti}(\nabla\,\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,A)). \tag{2.20}$$

In other words, we have that $\mathrm{Curl}([\mathrm{dev}\,\mathrm{Curl}\,A]^T) \in \mathfrak{so}(3)$, and applying axl on both sides of (2.20) we obtain

$$\nabla\,\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,A) = -3\,\mathrm{axl}(\mathrm{Curl}([\mathrm{dev}\,\mathrm{Curl}\,A]^T)) = L_0(\mathrm{D}\,\mathrm{dev}\,\mathrm{Curl}\,A). \tag{2.21}$$

Taking the ∂_j-derivative of (2.19) for $j = 1, 2, 3$ we conclude

$$\partial_j(\mathrm{D}\,\mathrm{axl}\,A)^T \overset{(2.19)}{=} \frac{1}{3}\partial_j\,\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,A) - \partial_j\,\mathrm{dev}\,\mathrm{Curl}\,A \overset{(2.21)}{=} \widetilde{L}_0(\mathrm{D}\,\mathrm{dev}\,\mathrm{Curl}\,A)\,, \tag{2.22}$$

which establishes part (b), namely $\mathrm{D}^2 A = L_2(\mathrm{D}(\mathrm{dev}\,\mathrm{Curl}\,A))$ for skew-symmetric tensor fields A.

The proof of part (a) is divided into the following two key observations:

$$\text{(a.i)}\ \ \mathrm{D}^2\,\zeta = \widetilde{L}_1(\mathrm{D}\,\mathrm{Curl}(A + \zeta \cdot \mathbb{1})), \quad \text{(a.ii)}\ \ \mathrm{D}^2\,A = \widetilde{L}_2(\mathrm{D}\,\mathrm{Curl}(A + \zeta \cdot \mathbb{1})).$$

To show that each entry of the Hessian matrix $\mathrm{D}^2\zeta$ is a linear combination of the entries of $\mathrm{D}\,\mathrm{Curl}(A + \zeta \cdot \mathbb{1})$ we make use of the second-order differential operator **inc** given for $B \in \mathscr{D}'(\Omega, \mathbb{R}^{3\times 3})$ via[4]

$$\mathbf{inc}\,B := \mathrm{Curl}([\mathrm{Curl}\,B]^T) \tag{2.23}$$

so that

$$\begin{aligned}\mathbf{inc}\,(\zeta \cdot \mathbb{1}) &= \mathrm{Curl}([\mathrm{Curl}(\zeta \cdot \mathbb{1})]^T) \overset{(2.18)}{=} \mathrm{Curl}(-[\mathrm{Anti}(\nabla\zeta)]^T) = \mathrm{Curl}(\mathrm{Anti}(\nabla\zeta)) \\ &\overset{(2.16a)}{=} \mathrm{tr}(\mathrm{D}\nabla\zeta)\cdot\mathbb{1} - (\mathrm{D}\nabla\zeta)^T = \Delta\zeta\cdot\mathbb{1} - \mathrm{D}^2\zeta \in \mathrm{Sym}(3)\end{aligned} \tag{2.24}$$

[4]See Kröner [**35**, §8] for a component-wise expression of the incompatibility operator **inc**.

is symmetric. On the other hand, for a skew-symmetric matrix field $A \in \mathscr{D}'(\Omega, \mathfrak{so}(3))$ we have that

$$\begin{aligned}\mathbf{inc}\, A &= \operatorname{Curl}([\operatorname{Curl} A]^T) \overset{(2.16)}{=} \operatorname{Curl}(\operatorname{tr}(\mathrm{D}\operatorname{axl} A)\cdot \mathbb{1} - \mathrm{D}\operatorname{axl} A)\\ &\overset{\operatorname{Curl}\circ \mathrm{D}\equiv 0}{=} \operatorname{Curl}(\operatorname{tr}(\mathrm{D}\operatorname{axl} A)\cdot \mathbb{1}) \overset{(2.18)}{=} -\operatorname{Anti}(\nabla \operatorname{tr}(\mathrm{D}\operatorname{axl} A)) \in \mathfrak{so}(3) \qquad (2.25)\end{aligned}$$

is skew-symmetric. Hence,

$$\operatorname{sym}(\mathbf{inc}\,(A+\zeta\cdot\mathbb{1})) = \Delta\zeta\cdot\mathbb{1} - \mathrm{D}^2\zeta \quad \text{and} \quad \operatorname{tr}(\mathbf{inc}\,(A+\zeta\cdot\mathbb{1})) = 2\,\Delta\zeta. \qquad (2.26)$$

In other words, the entries of the Hessian matrix of ζ are linear combinations of entries from $\mathbf{inc}\,(A+\zeta\cdot\mathbb{1})$:

$$\begin{aligned}\mathrm{D}^2\zeta &= \Delta\zeta\cdot\mathbb{1} - \operatorname{sym}(\mathbf{inc}\,(A+\zeta\cdot\mathbb{1}))\\ &= \frac{1}{2}\operatorname{tr}(\mathbf{inc}\,(A+\zeta\cdot\mathbb{1}))\cdot\mathbb{1} - \operatorname{sym}(\mathbf{inc}\,(A+\zeta\cdot\mathbb{1}))\\ &= \widetilde{L}_1(\mathrm{D}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})), \qquad (2.27)\end{aligned}$$

where we have used that the entries of $\mathbf{inc}\, B$ are, of course, linear combinations of entries of $\mathrm{D}\operatorname{Curl} B$.

To establish (a.ii) from (a.i), recall that for a skew-symmetric matrix field A the entries of $\mathrm{D}A$ are linear combinations of the entries from $\operatorname{Curl} A$:

$$\begin{aligned}\mathrm{D}A &\overset{(1.7)}{=} L(\operatorname{Curl} A) = L(\operatorname{Curl}(A+\zeta\cdot\mathbb{1})) - L(\operatorname{Curl}(\zeta\cdot\mathbb{1}))\\ &\overset{(2.18)}{=} L(\operatorname{Curl}(A+\zeta\cdot\mathbb{1})) + L(\operatorname{Anti}(\nabla\zeta)). \qquad (2.28)\end{aligned}$$

We conclude by taking the ∂_j-derivative of (2.28) for $j = 1, 2, 3$, namely

$$\partial_j\,\mathrm{D}A = L(\partial_j \operatorname{Curl}(A+\zeta\cdot\mathbb{1})) + L(\partial_j \operatorname{Anti}(\nabla\zeta)) \overset{(\mathrm{a.i})}{=} \widetilde{L}_3(\mathrm{D}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})).$$

Finally, we establish part (c) arguing in a similar way by showing the following linear combinations:

(1) $\mathrm{D}^2\zeta = \widetilde{L}_4(\mathrm{D}\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1}))$,

(2) $\mathrm{D}^3 A = \widetilde{L}_7(\mathrm{D}^2\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1}))$.

Regarding (2.18) and (2.16) we have

$$\begin{aligned}\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1}) &\overset{(2.18)}{=} \operatorname{dev}[\operatorname{Curl} A - \operatorname{Anti}(\nabla\zeta)] = \operatorname{dev}\operatorname{Curl} A - \operatorname{Anti}(\nabla\zeta)\\ &\overset{(2.16)}{=} \frac{1}{3}\operatorname{tr}(\mathrm{D}\operatorname{axl} A)\cdot\mathbb{1} - (\mathrm{D}\operatorname{axl} A)^T - \operatorname{Anti}(\nabla\zeta). \qquad (2.29)\end{aligned}$$

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Transposing and taking the Curl on both sides yields

$$\operatorname{Curl}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T) \underset{\operatorname{Curl}\circ\mathrm{D}\equiv 0}{\overset{(2.18),(2.16)}{=}} -\frac{1}{3}\underbrace{\operatorname{Anti}(\nabla\operatorname{tr}(\mathrm{D}\operatorname{axl}A))}_{\in\mathfrak{so}(3)} + \underbrace{\Delta\zeta\cdot\mathbb{1}-\mathrm{D}^2\zeta}_{\in\operatorname{Sym}(3)} \tag{2.30}$$

and we obtain, similar to the decomposition in (2.27):

$$\begin{aligned}\mathrm{D}^2\zeta &= \frac{1}{2}\operatorname{tr}(\operatorname{Curl}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T))\cdot\mathbb{1} - \operatorname{sym}(\operatorname{Curl}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T))\\ &= \widetilde{L}_4(\mathrm{D}\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})).\end{aligned} \tag{2.31}$$

On the other hand, taking **inc** of the transpositions on both sides of (2.29) gives

$$\begin{aligned}\operatorname{\mathbf{inc}}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T) \underset{(2.25)}{\overset{(2.24)}{=}} \frac{1}{3}\Delta\operatorname{tr}(\mathrm{D}\operatorname{axl}A)\cdot\mathbb{1}\\ -\frac{1}{3}\mathrm{D}^2\operatorname{tr}(\mathrm{D}\operatorname{axl}A) - \operatorname{Anti}(\nabla\Delta\zeta),\end{aligned} \tag{2.32}$$

yielding the relation

$$\begin{aligned}\mathrm{D}^2\operatorname{tr}(\mathrm{D}\operatorname{axl}A) &= \frac{3}{2}\operatorname{tr}(\operatorname{\mathbf{inc}}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T))\cdot\mathbb{1}\\ &\quad - \operatorname{sym}(\operatorname{\mathbf{inc}}([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T))\\ &= \widetilde{L}_5(\mathrm{D}^2\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})).\end{aligned} \tag{2.33}$$

Considering the second distributional derivatives in (2.29) we conclude

$$\begin{aligned}\mathrm{D}^3\operatorname{axl}A &= \frac{1}{3}\mathrm{D}^2\operatorname{tr}(\mathrm{D}\operatorname{axl}A)\cdot\mathbb{1} - \mathrm{D}^2([\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})]^T) + \mathrm{D}^2\operatorname{Anti}(\nabla\zeta)\\ &\underset{(2.33)}{\overset{(2.31)}{=}} \widetilde{L}_6(\mathrm{D}^2\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})).\end{aligned}$$

□

Remark 7. In the above proof we have used that the second-order differential operator **inc** does not change the symmetry property after application on square matrix fields, cf. the appendix. Further properties are collected e.g. in [**52**, appendix], [**1**, sec. 2] and [**12**, sec. 6.18].

The incompatibility operator **inc** arises in dislocation models, e.g., in the modelling of elastic materials with dislocations or in the modelling of dislocated crystals, since the strain cannot be a symmetric gradient of a vector field as soon as dislocations are present and the notion of incompatibility is at the basis of a new paradigm to describe the inelastic effects, cf. [**3**, **4**, **20**, **46**], cf. the appendix for further comments.

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Moreover, the equation $\mathbf{inc}\,\operatorname{sym} e \equiv 0$ is equivalent to the *Saint-Venant compatibility condition*[5] defining the relation between the symmetric strain $\operatorname{sym} e$ and the displacement vector field u:

$$\mathbf{inc}\,\operatorname{sym} e \equiv 0 \quad \Leftrightarrow \quad \operatorname{sym} e = \operatorname{sym} \mathrm{D}u \tag{2.34}$$

over simply connected domains, cf. [**1**, **46**]. In the appendix we show that the operators $\mathbf{inc}$ and sym can be interchanged, so that

$$\mathbf{inc}\,\operatorname{sym} e = \operatorname{sym} \mathbf{inc}\, e = \operatorname{sym} \operatorname{Curl}([\operatorname{Curl} e]^T). \tag{2.35}$$

Investigations over multiply connected domains can be found e.g. in [**30**, **66**].

Returning to our proof, a crucial ingredient in our following argumentation is

THEOREM 8 (Lions lemma and Nečas estimate). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $m \in \mathbb{Z}$ and $p \in (1,\infty)$. Then $f \in \mathscr{D}'(\Omega,\mathbb{R}^d)$ and $\mathrm{D}f \in W^{m-1,p}(\Omega,\mathbb{R}^{d\times n})$ imply $f \in W^{m,p}(\Omega,\mathbb{R}^d)$. Moreover,*

$$\|f\|_{W^{m,\,p}(\Omega,\mathbb{R}^d)} \leqslant c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^d)} + \|\mathrm{D}f\|_{W^{m-1,p}(\Omega,\mathbb{R}^{d\times n})}\right), \tag{2.36}$$

with a constant $c = c(m,p,n,d,\Omega) > 0$.

For the proof we refer to [**2**, proposition 2.10 and theorem 2.3], [**7**]. However, since we are dealing with higher order derivatives we also need a 'higher order' version of the Lions lemma resp. Nečas estimate.

COROLLARY 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $m \in \mathbb{Z}$ and $p \in (1,\infty)$. Denote by $\mathrm{D}^k f$ the collection of all distributional derivatives of order k. Then $f \in \mathscr{D}'(\Omega,\mathbb{R}^d)$ and $\mathrm{D}^k f \in W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^k})$ imply $f \in W^{m,p}(\Omega,\mathbb{R}^d)$. Moreover,*

$$\|f\|_{W^{m,p}(\Omega,\mathbb{R}^d)} \leqslant c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^d)} + \|\mathrm{D}^k f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^k})}\right), \tag{2.37}$$

with a constant $c = c(m,p,n,d,\Omega) > 0$.

[5]Those compatibility conditions are contained in the third appendix §32 p. 597 et seq. of the third edition of the lecture notes *Résistance des corps solides* given by Navier and extended with several notes and appendices by Barré de Saint-Venant and published as *Résumé des Leçons données à l'École des Ponts et Chaussées sur l'Application de la Mécanique*, vol. I, Paris, 1864. Their coordinate-free version can be found in Lagally's monograph on vector calculus from 1928 [**36**, Ziff. 191] where it reads:

$$\nabla \times (\operatorname{sym} \mathrm{D}u) \times \nabla \equiv 0$$

and formally follows from the definitions of those operators, see [**36**, Ziff. 191], since

$$\begin{aligned}\nabla \times (\operatorname{sym} \mathrm{D}u) \times \nabla &= \frac{1}{2} \nabla \times (\nabla \otimes u + u \otimes \nabla) \times \nabla \\ &= \frac{1}{2}[(\nabla \times \nabla) \otimes u \times \nabla + \nabla \times u \otimes (\nabla \times \nabla)] \equiv 0.\end{aligned}$$

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Proof. The assertion $f \in W^{m,p}(\Omega, \mathbb{R}^d)$ and the estimate (2.37) follow by inductive application of theorem 8 to $\mathrm{D}^l f$ with $l = k-1, k-2, \dots, 0$. Indeed, starting by applying theorem 8 to $\mathrm{D}^{k-1} f$ gives $\mathrm{D}^{k-1} f \in W^{m-k+1,p}(\Omega, \mathbb{R}^{d\times n^{k-1}})$ as well as

$$\begin{aligned}
&\|\mathrm{D}^{k-1} f\|_{W^{m-k+1,p}(\Omega,\mathbb{R}^{d\times n^{k-1}})} \\
&\quad\leqslant c\, \left(\|\mathrm{D}^{k-1} f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^{k-1}})} + \|\mathrm{D}^{k} f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^{k}})}\right) \\
&\quad\leqslant c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^{d})} + \|\mathrm{D}^{k} f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^{k}})}\right). \qquad (2.38)
\end{aligned}$$

Now, we can apply theorem 8 to $\mathrm{D}^{k-2} f$ to deduce $\mathrm{D}^{k-2} f \in W^{m-k+2,p}(\Omega, \mathbb{R}^{d\times n^{k-2}})$ and moreover

$$\begin{aligned}
&\|\mathrm{D}^{k-2} f\|_{W^{m-k+2,p}(\Omega,\mathbb{R}^{d\times n^{k-2}})} \\
&\quad\leqslant c\, \left(\|\mathrm{D}^{k-2} f\|_{W^{m-k+1,p}(\Omega,\mathbb{R}^{d\times n^{k-1}})} + \|\mathrm{D}^{k-1} f\|_{W^{m-k+1,p}(\Omega,\mathbb{R}^{d\times n^{k-1}})}\right) \\
&\quad\leqslant c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^{d})} + \|\mathrm{D}^{k-1} f\|_{W^{m-k+1,p}(\Omega,\mathbb{R}^{d\times n^{k-1}})}\right) \\
&\quad\overset{(2.38)}{\leqslant} c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^{d})} + \|\mathrm{D}^{k} f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^{k}})}\right). \qquad (2.39)
\end{aligned}$$

Consequently, for all $l = k-1, k-2, \dots, 0$ we deduce $\mathrm{D}^l f \in W^{m-l,p}(\Omega, \mathbb{R}^{d\times n^l})$ as well as

$$\|\mathrm{D}^{l} f\|_{W^{m-l,p}(\Omega,\mathbb{R}^{d\times n^{l}})} \leqslant c\, \left(\|f\|_{W^{m-1,p}(\Omega,\mathbb{R}^{d})} + \|\mathrm{D}^{k} f\|_{W^{m-k,p}(\Omega,\mathbb{R}^{d\times n^{k}})}\right). \qquad (2.40)$$

□

Remark 10. The need to consider higher order derivatives is indicated by the appearance of linear terms in the kernel of Korn's quantitative versions, similar to the situation at the classical trace-free Korn inequalities [**17**, **65**]. In our case we have:

LEMMA 11. *Let $A \in L^p(\Omega, \mathfrak{so}(3))$ and $\zeta \in L^p(\Omega, \mathbb{R})$. Then we have in the distributional sense*

(a) $\mathrm{Curl}(A + \zeta\cdot\mathbb{1}) \equiv 0$ *if and only if* $A + \zeta\cdot\mathbb{1} = \mathrm{Anti}(\widetilde{A}\,x + b) + (\langle \operatorname{axl}\widetilde{A}, x\rangle + \beta)\cdot\mathbb{1}$ *a.e. on* Ω,

(b) $\operatorname{dev}\mathrm{Curl}\, A \equiv 0$ *if and only if* $A = \mathrm{Anti}(\beta\, x + b)$ *a.e. on* Ω,

(c) $\operatorname{dev}\mathrm{Curl}(A + \zeta\cdot\mathbb{1}) \equiv 0$ *if and only if* $A + \zeta\cdot\mathbb{1} = \mathrm{Anti}(\widetilde{A}\,x + \beta\,x + b) + (\langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma)\cdot\mathbb{1}$ *a.e. on* Ω,

with constant $\widetilde{A} \in \mathfrak{so}(3)$, $b \in \mathbb{R}^3$, $\beta, \gamma \in \mathbb{R}$.

Proof. Although the deductions have already been partially indicated in the literature, cf. e.g. [**53**, sec. 3.4] and [**6**, **17**, **63**, **64**], we include it here for the sake of

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completeness. The 'if'-parts are seen by direct calculations, cf. the relations (2.16) and (2.18):

(a) $\operatorname{Curl}(\operatorname{Anti}(\widetilde{A}\,x+b)+(\langle \operatorname{axl}\widetilde{A},x\rangle+\beta)\cdot\mathbb{1})=\widetilde{A}-\operatorname{Anti}(\operatorname{axl}\widetilde{A})\equiv 0$,

(b) $\operatorname{dev}\operatorname{Curl}(\operatorname{Anti}(\beta\,x+b))=\operatorname{dev}(\operatorname{tr}(\beta\cdot\mathbb{1})\cdot\mathbb{1}-\beta\cdot\mathbb{1})=\operatorname{dev}(2\,\beta\cdot\mathbb{1})\equiv 0$,

(c) $\operatorname{dev}\operatorname{Curl}(\operatorname{Anti}(\widetilde{A}\,x+\beta\,x+b)+(\langle \operatorname{axl}\widetilde{A},x\rangle+\gamma)\cdot\mathbb{1})$
$=\operatorname{dev}(\widetilde{A}+2\,\beta\cdot\mathbb{1}-\operatorname{Anti}(\operatorname{axl}\widetilde{A}))\equiv 0.$

Now, we focus on the 'only if'-directions, starting with

$$\operatorname{Curl}(A+\zeta\cdot\mathbb{1})\equiv 0 \quad \overset{(2.18)}{\Longleftrightarrow} \quad \operatorname{Anti}(\nabla\zeta)=\operatorname{Curl}A \overset{(2.16)}{=} \operatorname{tr}(\operatorname{D}\operatorname{axl}A)\cdot\mathbb{1}-(\operatorname{D}\operatorname{axl}A)^T.$$

Taking the trace on both sides we obtain $\operatorname{tr}(\operatorname{D}\operatorname{axl}A)=0$ and consequently

$$\operatorname{Anti}(\nabla\zeta)=-(\operatorname{D}\operatorname{axl}A)^T, \tag{2.41}$$

hence $\operatorname{sym}(\operatorname{D}\operatorname{axl}A)=0$. By the classical Korn's inequality (1.5) it follows that there exists a constant skew-symmetric matrix $\widetilde{A}\in\mathfrak{so}(3)$ so that $\operatorname{D}\operatorname{axl}A\equiv\widetilde{A}$, which implies $A=\operatorname{Anti}(\widetilde{A}x+b)$ with $b\in\mathbb{R}^3$. Furthermore, by (2.41) we obtain

$$\operatorname{Anti}(\nabla\zeta)=\widetilde{A} \quad \Rightarrow \quad \zeta=\langle \operatorname{axl}\widetilde{A},x\rangle+\beta \quad \text{with } \beta\in\mathbb{R},$$

which establishes (a).

For part (b) we start with the relation $\operatorname{dev}\operatorname{Curl}A\equiv 0$ in (2.20) and have

$$\operatorname{Anti}(\nabla\operatorname{tr}(\operatorname{D}\operatorname{axl}A))\equiv 0 \quad \Rightarrow \quad \nabla\operatorname{tr}(\operatorname{D}\operatorname{axl}A)\equiv 0, \tag{2.42}$$

so that

$$\frac{1}{3}\operatorname{tr}(\operatorname{D}\operatorname{axl}A)=\beta \tag{2.43}$$

for some $\beta\in\mathbb{R}$. Reinserting in the deviatoric counterpart of Nye's formula (2.19) gives

$$0=\beta\cdot\mathbb{1}-(\operatorname{D}\operatorname{axl}A)^T \quad \text{resp.} \quad \operatorname{D}\operatorname{axl}A=\beta\cdot\mathbb{1} \quad \Rightarrow \quad \operatorname{axl}A=\beta\,x+b \tag{2.44}$$

for some $b\in\mathbb{R}^3$ and thus $A=\operatorname{Anti}(\beta\,x+b)$.

Finally, for part (c), let now $\operatorname{dev}\operatorname{Curl}(A+\zeta\cdot\mathbb{1})\equiv 0$. Then considering the skew-symmetric parts of (2.30) we obtain

$$\operatorname{Anti}(\nabla\operatorname{tr}(\operatorname{D}\operatorname{axl}A))\equiv 0 \quad \Rightarrow \quad \nabla\operatorname{tr}(\operatorname{D}\operatorname{axl}A)\equiv 0.$$

Hence, again

$$\frac{1}{3}\operatorname{tr}(\operatorname{D}\operatorname{axl}A)=\beta \tag{2.45}$$

for some $\beta\in\mathbb{R}$, so that considering the symmetric parts of (2.29) we get

$$0=\frac{1}{3}\operatorname{tr}(\operatorname{D}\operatorname{axl}A)\cdot\mathbb{1}-\operatorname{sym}(\operatorname{D}\operatorname{axl}A) \overset{(2.45)}{=} \beta\cdot\mathbb{1}-\operatorname{sym}(\operatorname{D}\operatorname{axl}A). \tag{2.46}$$

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In other words, we have

$$\operatorname{sym}(\mathrm{D}(\operatorname{axl} A - \beta\, x)) \equiv 0$$

and by (1.5), it follows that $\mathrm{D}(\operatorname{axl} A - \beta\, x)$ must be a constant skew-symmetric matrix. Thus

$$\operatorname{axl} A = \widetilde{A}\, x + \beta\, x + b \tag{2.47}$$

for some $\widetilde{A} \in \mathfrak{so}(3)$, $b \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$. Furthermore, by (2.29) we have

$$\operatorname{Anti}(\nabla\zeta) \overset{(2.29)}{=} \operatorname{skew}(\mathrm{D} \operatorname{axl} A) \overset{(2.47)}{=} \widetilde{A}$$

so that ζ is of the form

$$\zeta = \langle \operatorname{axl} \widetilde{A}, x\rangle + \gamma \tag{2.48}$$

for some $\gamma \in \mathbb{R}$, and we arrive at (c):

$$A + \zeta \cdot \mathbb{1} \underset{(2.48)}{\overset{(2.47)}{=}} \operatorname{Anti}\left(\widetilde{A}\, x + \beta\, x + b\right) + \left(\langle \operatorname{axl} \widetilde{A}, x\rangle + \gamma\right)\cdot \mathbb{1}. \qquad \square$$

We are now prepared to proceed as in the proof of the generalized Korn inequality for incompatible tensor fields.

3. Main results

We will make use of the Banach space

$$W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3}) := \{P \in L^p(\Omega, \mathbb{R}^{3\times3}) \mid \operatorname{Curl} P \in L^p(\Omega, \mathbb{R}^{3\times3})\} \tag{3.1a}$$

equipped with the norm

$$\|P\|_{W^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times3})} := \left(\|P\|^p_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\operatorname{Curl} P\|^p_{L^p(\Omega,\mathbb{R}^{3\times3})}\right)^{\frac{1}{p}}, \tag{3.1b}$$

as well as its subspace

$$W^{1,p}_0(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3}) := \{P \in W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times3}) \mid P \times \nu = 0 \text{ on } \partial\Omega\},$$

where ν denotes the outward unit normal vector field to $\partial\Omega$, and the tangential trace $P \times \nu$ is understood in the sense of $W^{-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^{3\times3})$ which is justified by partial integration, so that its trace is defined by

$$\forall Q \in W^{1-\frac{1}{p'},p'}(\partial\Omega, \mathbb{R}^{3\times3}):$$
$$\langle P \times (-\nu), Q\rangle_{\partial\Omega} = \int_\Omega \langle \operatorname{Curl} P, \widetilde{Q}\rangle - \langle P, \operatorname{Curl} \widetilde{Q}\rangle \,\mathrm{d}x, \tag{3.2}$$

where $\widetilde{Q} \in W^{1,p'}(\Omega, \mathbb{R}^{3\times3})$ denotes any extension of Q in Ω. Here, $\langle ., .\rangle_{\partial\Omega}$ indicates the duality pairing between $W^{-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^{3\times3})$ and $W^{1-\frac{1}{p'},p'}(\partial\Omega, \mathbb{R}^{3\times3})$.

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However, the appearance of the operator dev Curl on the right-hand side of our designated results in this paper would suggest to work in

$$W^{1,p}(\operatorname{dev}\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3}) := \{P \in L^p(\Omega,\mathbb{R}^{3\times 3}) \mid \operatorname{dev}\operatorname{Curl} P \in L^p(\Omega,\mathbb{R}^{3\times 3})\} \quad (3.3)$$

but this is, surprisingly at first glance, not a new space:

LEMMA 12. $W^{1,p}(\operatorname{dev}\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3}) = W^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$.

It is sufficient to show that the p-integrability of $\operatorname{dev}\operatorname{Curl} P$ already implies the p-integrability of $\operatorname{Curl} P$, and follows from the general case:

LEMMA 13. *Let $P \in \mathscr{D}'(\Omega,\mathbb{R}^{3\times 3})$. Then we have for all $m \in \mathbb{Z}$ that*

$$\operatorname{Curl} P \in W^{m,p}(\Omega,\mathbb{R}^{3\times 3}) \quad \Leftrightarrow \quad \operatorname{dev}\operatorname{Curl} P \in W^{m,p}(\Omega,\mathbb{R}^{3\times 3}). \quad (3.4)$$

Proof. We again consider the decomposition of P into its symmetric and skew-symmetric part, i.e.

$$P = S + A = S + \operatorname{Anti}(a) \quad \text{for some } S \in \operatorname{Sym}(3), A \in \mathfrak{so}(3) \text{ and with } a = \operatorname{axl}(A).$$

Then by Nye's formula (2.16a) we have

$$\operatorname{Curl} P = \operatorname{Curl}(S + \operatorname{Anti}(a)) \overset{(3.5)}{=} \operatorname{Curl} S + \operatorname{div} a \cdot \mathbb{1} - (\mathrm{D}a)^T \quad (3.5)$$

and in view of $\operatorname{tr}(\operatorname{Curl} S) = 0$ we obtain

$$\operatorname{dev}\operatorname{Curl} P = \operatorname{Curl} S - (\mathrm{D}a)^T + \frac{1}{3}\operatorname{div} a \cdot \mathbb{1} \quad (3.6)$$

so that taking the Curl of the transpositions on both sides gives

$$\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) \underset{(2.18)}{\overset{\operatorname{Curl}\circ \mathrm{D}\equiv 0}{=}} \underbrace{\operatorname{\mathbf{inc}} S}_{\in \operatorname{Sym}(3)} - \frac{1}{3}\underbrace{\operatorname{Anti}(\nabla \operatorname{div} a)}_{\in \mathfrak{so}(3)}, \quad (3.7)$$

which gives

$$\operatorname{skew}\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) = -\frac{1}{3}\operatorname{Anti}(\nabla \operatorname{div} a). \quad (3.8)$$

Thus, $\operatorname{dev}\operatorname{Curl} P \in W^{m,p}(\Omega,\mathbb{R}^{3\times 3})$ implies $\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) \in W^{m-1,p}(\Omega, \mathbb{R}^{3\times 3})$ as well as

$$\begin{aligned}\operatorname{skew}\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) &= \frac{1}{2}(\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) - [\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T)]^T) \\ &\in W^{m-1,p}(\Omega,\mathbb{R}^{3\times 3}),\end{aligned} \quad (3.9)$$

so that we obtain

$$\nabla \operatorname{div} a \overset{(3.8)}{=} -3\operatorname{axl}\operatorname{skew}\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T) \in W^{m-1,p}(\Omega,\mathbb{R}^3). \quad (3.10)$$

Since $a = \operatorname{axl}\operatorname{skew} P \in \mathscr{D}'(\Omega,\mathbb{R}^3)$, we apply theorem 8 to $\operatorname{div} a \in \mathscr{D}'(\Omega,\mathbb{R})$ to conclude from (3.10) that $\operatorname{div} a \in W^{m,p}(\Omega,\mathbb{R})$. The statement of the lemma then follows

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from the decompositions (3.5) and (3.6) which give the expression

$$\operatorname{Curl} P = \operatorname{dev}\operatorname{Curl} P + \frac{2}{3}\operatorname{div} a \cdot \mathbb{1} \in W^{m,p}(\Omega, \mathbb{R}^{3\times 3}). \tag{3.11}$$

□

Corollary 14. *The classical Hilbert space* $H(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3})$ *coincides with the Hilbert space* $H(\operatorname{dev}\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}) := \{P \in L^2(\Omega, \mathbb{R}^{3\times 3}) \mid \operatorname{dev}\operatorname{Curl} P \in L^2(\Omega, \mathbb{R}^{3\times 3})\}$.

Remark 15 (Equivalence of norms)**.** In view of (3.10) an application of the Lions lemma to $\operatorname{div} a$, with $a = \operatorname{axl}\operatorname{skew} P$, gives us $\operatorname{div} a \in W^{m,p}(\Omega, \mathbb{R})$. Moreover, by the Nečas estimate we have

$$\begin{aligned} \|\operatorname{div} a\|_{W^{m,p}(\Omega,\mathbb{R})} &\leqslant c_1\left(\|\operatorname{div} a\|_{W^{m-1,p}(\Omega,\mathbb{R})} + \|\nabla \operatorname{div} a\|_{W^{m-1,p}(\Omega,\mathbb{R}^3)}\right) \\ &\overset{(3.10)}{\leqslant} c_2\left(\|\operatorname{div}\operatorname{axl}\operatorname{skew} P\|_{W^{m-1,p}(\Omega,\mathbb{R})}\right. \\ &\quad \left. + \|\operatorname{Curl}([\operatorname{dev}\operatorname{Curl} P]^T)\|_{W^{m-1,p}(\Omega,\mathbb{R}^{3\times 3})}\right) \\ &\leqslant c_3\left(\|P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})}\right), \end{aligned}$$

provided that $P \in W^{m,p}(\Omega, \mathbb{R}^{3\times 3})$. Together with (3.11) we conclude:

$$\|\operatorname{Curl} P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} \leqslant c_4\left(\|P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})}\right) \tag{3.12}$$

as well as

$$\begin{aligned} &\|P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} \\ &\qquad \leqslant c_5\left(\|P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{m,p}(\Omega,\mathbb{R}^{3\times 3})}\right) \end{aligned} \tag{3.13}$$

and especially for $m = 0$:

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c_5\left(\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right) \tag{3.14}$$

for all $P \in W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3})$.[6]

[6] This result also follows from the open mapping theorem (also known as Banach-Schauder theorem [**12**, Thm 5.6-1]) in functional analysis. More precisely, the latter provides the following sufficient condition for two norms to be equivalent in an infinite-dimensional space, see [**12**, Thm 5.6-4]:

Corollary 16. *Let* $\|.\|$ *and* $\|.\|'$ *be two norms on the same vector space* X*, with the following properties: both spaces* $(X, \|.\|)$ *and* $(X, \|.\|')$ *are complete, and there exists a constant* C *such that*

$$\|x\|' \leqslant C\,\|x\| \quad \textit{for all } x \in X.$$

Then the two norms $\|.\|$ *and* $\|.\|'$ *are equivalent.*

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Remark 17. The last identity in (3.11), which could also be formally obtained from (2.8) with $b = -\nabla$, together with the expression (3.10) gives for general matrix field $P \in \mathscr{D}'(\Omega, \mathbb{R}^{3\times3})$:

$$\mathrm{D}\,\mathrm{Curl}\,P = L(\mathrm{D}\,\mathrm{dev}\,\mathrm{Curl}\,P). \tag{3.15}$$

Thus, recalling (1.7), we arrive directly at the case (b) of lemma 6.

COROLLARY 18. *Notably, the trace condition in* $W_0^{1,p}(\mathrm{dev}\,\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$ *would read* $\mathrm{dev}(P\times\nu)=0$ *on* $\partial\Omega$*, to be understood by partial integration via*

$$\begin{aligned}
&\forall\ Q \in W^{1-\frac{1}{p'},p'}(\partial\Omega,\mathbb{R}^{3\times3}):\\
\langle \mathrm{dev}(P\times(-\nu)), Q\rangle_{\partial\Omega} &= \int_\Omega \langle \mathrm{dev}\,\mathrm{Curl}\,P, \widetilde{Q}\rangle - \langle P, \mathrm{Curl}\,\mathrm{dev}\,\widetilde{Q}\rangle\,\mathrm{d}x\\
&= \int_\Omega \langle \mathrm{Curl}\,P, \mathrm{dev}\,\widetilde{Q}\rangle - \langle P, \mathrm{Curl}\,\mathrm{dev}\,\widetilde{Q}\rangle\,\mathrm{d}x\\
&\overset{(3.2)}{=} \langle P\times(-\nu), \mathrm{dev}\,Q\rangle_{\partial\Omega},
\end{aligned} \tag{3.16}$$

where $\widetilde{Q}\in W^{1,p'}(\Omega,\mathbb{R}^{3\times3})$ *denotes any extension of* Q *in* Ω*. However, it follows from observation 3 that the boundary conditions* $P\times\nu=0$ *and* $\mathrm{dev}(P\times\nu)=0$ *on* $\partial\Omega$ *are the same.*

LEMMA 19. *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded Lipschitz domain,* $1<p<\infty$ *and* $P\in\mathscr{D}'(\Omega,\mathbb{R}^{3\times3})$*. Then either of the conditions*

(a) $\mathrm{dev}\,\mathrm{sym}\,P\in L^p(\Omega,\mathbb{R}^{3\times3})$ *and* $\mathrm{Curl}\,P\in W^{-1,\,p}(\Omega,\mathbb{R}^{3\times3})$,

(b) $\mathrm{sym}\,P\in L^p(\Omega,\mathbb{R}^{3\times3})$ *and* $\mathrm{dev}\,\mathrm{Curl}\,P\in W^{-1,\,p}(\Omega,\mathbb{R}^{3\times3})$,

(c) $\mathrm{dev}\,\mathrm{sym}\,P\in L^p(\Omega,\mathbb{R}^{3\times3})$ *and* $\mathrm{dev}\,\mathrm{Curl}\,P\in W^{-1,p}(\Omega,\mathbb{R}^{3\times3})$,

implies $P\in L^p(\Omega,\mathbb{R}^{3\times3})$*. Moreover, we have the corresponding estimates*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\leqslant c\Big(\|\mathrm{skew}\,P+\tfrac{1}{3}\,\mathrm{tr}\,P\cdot\mathbb{1}\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{dev}\,\mathrm{sym}\,P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}+\|\mathrm{Curl}\,P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})}\Big), \tag{3.17a}$$

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\leqslant c\Big(\|\mathrm{skew}\,P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{sym}\,P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}+\|\mathrm{dev}\,\mathrm{Curl}\,P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})}\Big), \tag{3.17b}$$

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\leqslant c\Big(\|\mathrm{skew}\,P+\tfrac{1}{3}\,\mathrm{tr}\,P\cdot\mathbb{1}\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})} + \|\mathrm{dev}\,\mathrm{sym}\,P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}+\|\mathrm{dev}\,\mathrm{Curl}\,P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times3})}\Big), \tag{3.17c}$$

each with a constant $c=c(p,\Omega)>0$.

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Proof. We start by proving part (b). For that purpose we will follow the proof of [**43**, lemma 3.1]. Thus, for part (b) it remains to deduce that $\operatorname{skew} P \in L^p(\Omega, \mathbb{R}^{3\times 3})$. We have

$$\begin{aligned}\|\mathrm{D}^2 \operatorname{skew} P\|_{W^{-2,p}(\Omega,\mathbb{R}^{3\times 3^3})} &\overset{\text{Lem. 6(b)}}{\leqslant} c\,\|\mathrm{D} \operatorname{dev}\operatorname{Curl}\operatorname{skew} P\|_{W^{-2,p}(\Omega,\mathbb{R}^{3\times 3^2})}\\ &\leqslant c\,\|\operatorname{dev}\operatorname{Curl}(P-\operatorname{sym} P)\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\\ &\leqslant c\,(\|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{Curl}\operatorname{sym} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})})\\ &\leqslant c\,(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}). \end{aligned}\tag{3.18}$$

Hence, the assumptions of part (b) yield $\mathrm{D}^2 \operatorname{skew} P \in W^{-2,p}(\Omega, \mathbb{R}^{3\times 3^3})$, so that, by corollary 9, we obtain $\operatorname{skew} P \in L^p(\Omega, \mathbb{R}^{3\times 3})$ and moreover the estimate

$$\begin{aligned}\|\operatorname{skew} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} &\leqslant c\,(\|\operatorname{skew} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}+\|\mathrm{D}^2\operatorname{skew} P\|_{W^{-2,p}(\Omega,\mathbb{R}^{3\times 3^3})})\\ &\overset{(3.18)}{\leqslant} c\Big(\|\operatorname{skew} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\\ &\quad+\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\Big). \end{aligned}\tag{3.19}$$

Then by adding $\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}$ on both sides we obtain (3.17b).

Clearly, the conclusion of (a) as well as the estimate (3.17a) follow from (c) and (3.17c), respectively. To establish (c), we make use of the orthogonal decomposition $P = \operatorname{dev}\operatorname{sym} P + (\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P\cdot \mathbb{1})$. Then, to obtain $\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P\cdot\mathbb{1} \in L^p(\Omega, \mathbb{R}^{3\times 3})$ for (c), we consider

$$\begin{aligned}&\|\mathrm{D}^2\operatorname{dev}\operatorname{Curl}(\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1})\|_{W^{-3,p}(\Omega,\mathbb{R}^{3\times 3^3})}\\ &\quad\leqslant c\,\|\operatorname{dev}\operatorname{Curl}(P-\operatorname{dev}\operatorname{sym} P)\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\\ &\quad\leqslant c\,(\|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{Curl}\operatorname{dev}\operatorname{sym} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})})\\ &\quad\leqslant c\,(\|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}). \end{aligned}\tag{3.20}$$

Therefore, $\mathrm{D}^2\operatorname{dev}\operatorname{Curl}(\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P\cdot\mathbb{1}) \in W^{-3,p}(\Omega, \mathbb{R}^{3\times 3^3})$ follows from the assumptions of (c) and lemma 6(c) implies

$$\mathrm{D}^3(\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1}) \in W^{-3,p}(\Omega,\mathbb{R}^{3\times 3^4}).\tag{3.21}$$

Applying corollary 9 again, this time to $\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P\cdot\mathbb{1}$, we arrive at $\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P\cdot\mathbb{1} \in L^p(\Omega, \mathbb{R}^{3\times 3})$ and, moreover,

$$\begin{aligned}&\|\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1}\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\\ &\quad\leqslant c\Big(\|\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1}\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\\ &\qquad+\|\mathrm{D}^3(\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1})\|_{W^{-3,p}(\Omega,\mathbb{R}^{3\times 3^4})}\Big)\\ &\quad\overset{\text{Lem. 6(c)}}{\leqslant} c\Big(\|\operatorname{skew} P+\tfrac{1}{3}\operatorname{tr} P\cdot\mathbb{1}\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} \end{aligned}\tag{3.22}$$

$$+ \|\mathrm{D}^2 \operatorname{dev}\operatorname{Curl}(\operatorname{skew} P + \tfrac{1}{3}\operatorname{tr} P \cdot \mathbb{1})\|_{W^{-3,p}(\Omega,\mathbb{R}^{3\times 3^3})}\Big)$$

$$\overset{(3.20)}{\leqslant} c\Big(\|\operatorname{skew} P + \tfrac{1}{3}\operatorname{tr} P \cdot \mathbb{1}\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}$$

$$+ \|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\Big). \qquad \square$$

Remark 20. Of course, part (a) can also be proven independently of part (c). Indeed, using lemma 6(a) we obtain

$$\begin{aligned}
&\|\mathrm{D}^2(\operatorname{skew} P + \tfrac{1}{3}\operatorname{tr} P \cdot \mathbb{1})\|_{W^{-2,p}(\Omega,\mathbb{R}^{3\times 3^3})} \\
&\overset{\text{Lem. 6(a)}}{\leqslant} c\,\|\mathrm{D}\operatorname{Curl}(\operatorname{skew} P + \tfrac{1}{3}\operatorname{tr} P \cdot \mathbb{1})\|_{W^{-2,p}(\Omega,\mathbb{R}^{3\times 3^2})} \\
&\leqslant c\,\|\operatorname{Curl}(P - \operatorname{dev}\operatorname{sym} P)\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} \\
&\leqslant c\,(\|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}) \qquad (3.23)
\end{aligned}$$

and the conclusion follows from an application of corollary 9 to $\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P \cdot \mathbb{1}$.

The rigidity results now follow by elimination of the corresponding first term on the right-hand side.

Theorem 21. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p,\Omega) > 0$ such that for all $P \in L^p(\Omega,\mathbb{R}^{3\times 3})$ we have*

$$\begin{aligned}
&\inf_{T\in K_{dS,C}} \|P - T\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \\
&\quad\leqslant c\,\left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\right), \qquad (3.24a)\\
&\inf_{T\in K_{S,dC}} \|P - T\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \\
&\quad\leqslant c\,\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\right), \qquad (3.24b)\\
&\inf_{T\in K_{dS,dC}} \|P - T\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \\
&\quad\leqslant c\,\left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\right), \qquad (3.24c)
\end{aligned}$$

where the kernels are given, respectively, by

$$\begin{aligned}
K_{dS,C} &= \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}(\widetilde{A}\,x + b) + (\langle \operatorname{axl}\widetilde{A}, x\rangle + \beta)\cdot\mathbb{1}, \\
&\qquad \widetilde{A} \in \mathfrak{so}(3), b \in \mathbb{R}^3, \beta \in \mathbb{R}\}, \qquad (3.25a)\\
K_{S,dC} &= \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}(\beta\,x + b),\ b \in \mathbb{R}^3, \beta \in \mathbb{R}\}, \qquad (3.25b)\\
K_{dS,dC} &= \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}\left(\widetilde{A}\,x + \beta\,x + b\right) + \left(\langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma\right)\cdot\mathbb{1}, \\
&\qquad \widetilde{A} \in \mathfrak{so}(3), b \in \mathbb{R}^3, \beta, \gamma \in \mathbb{R}\}. \qquad (3.25c)
\end{aligned}$$

Proof. We proceed as in the proof of Korn's inequalities (1.4) resp. (1.5), see [**43**, theorem 3.3] resp. [**12**, theorem 6.15-3], and start by characterizing the kernel of

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the right-hand side,

$$K_{dS,C} \coloneqq \{P \in L^p(\Omega, \mathbb{R}^{3\times 3}) \mid \operatorname{dev}\operatorname{sym} P = 0 \text{ a.e. and } \operatorname{Curl} P = 0 \text{ in the distributional sense}\},$$

so that $P \in K_{dS,C}$ if and only if $P = \operatorname{skew} P + \frac{1}{3}\operatorname{tr} P \cdot \mathbb{1}$ and $\operatorname{Curl}(\operatorname{skew} P + \frac{1}{3}\operatorname{tr} P \cdot \mathbb{1}) \equiv 0$. Hence, (3.25a) follows by virtue of Lemma 11(a).

Let us denote by $e_1, \ldots, e_M$ a basis of $K_{dS,C}$, where $M \coloneqq \dim K_{dS,C} = 7$, and by $\ell_1, \ldots, \ell_M$ the corresponding continuous linear forms on $K_{dS,C}$ given by

$$\ell_\alpha(e_j) \coloneqq \delta_{\alpha j}. \tag{3.26}$$

By the Hahn-Banach theorem in a normed vector space (see e.g. [**12**, theorem 5.9-1]), we extend ℓ_α to continuous linear forms—again denoted by ℓ_α—on the Banach space $L^p(\Omega, \mathbb{R}^{3\times 3})$, $1 \leqslant \alpha \leqslant M$. Notably,

$$T \in K_{dS,C} \text{ is equal to } 0 \quad \Leftrightarrow \quad \ell_\alpha(T) = 0 \ \forall\ \alpha \in \{1, \ldots, M\}.$$

Following the proof of [**43**, theorem 3.4] we eliminate the first term on the right-hand side of (3.17a) by exploiting the compactness $L^p(\Omega, \mathbb{R}^{3\times 3}) \subset\subset W^{-1,p}(\Omega, \mathbb{R}^{3\times 3})$ and arrive at

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c \left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \sum_{\alpha=1}^{M} |\ell_\alpha(P)| \right). \tag{3.27}$$

Indeed, if (3.27) were false, there would exist a sequence $P_k \in L^p(\Omega, \mathbb{R}^{3\times 3})$ such that

$$\|P_k\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} = 1 \quad \text{and} \quad \left(\|\operatorname{dev}\operatorname{sym} P_k\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P_k\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \sum_{\alpha=1}^{M} |\ell_\alpha(P_k)| \right) < \frac{1}{k}.$$

Thus, for a subsequence $P_k \rightharpoonup P^*$ in $L^p(\Omega, \mathbb{R}^{3\times 3}$ with $\operatorname{dev}\operatorname{sym} P^* = 0$ a.e., $\operatorname{sym}\operatorname{Curl} P^* = 0$ in the distributional sense and $\ell_\alpha(P_k) = 0$ for all $\alpha = 1, \ldots, M$, so that $P^* = 0$ a.e.. By the compact embedding $L^p(\Omega, \mathbb{R}^{3\times 3}) \subset\subset W^{-1,p}(\Omega, \mathbb{R}^{3\times 3})$ there exists a subsequence P_k, so that $\operatorname{skew} P_k + \frac{1}{3}\operatorname{tr} P_k \cdot \mathbb{1} \to 0$ in $W^{-1,p}(\Omega, \mathbb{R}^{3\times 3})$. This is a contradiction to (3.17a).

Considering now the projection $\pi_a : L^p(\Omega, \mathbb{R}^{3\times 3}) \to K_{dS,C}$ given by

$$\pi_a(P) \coloneqq \sum_{j=1}^{M} \ell_j(P)\, e_j \tag{3.28}$$

we obtain $\ell_\alpha(P - \pi_a(P)) \overset{(3.26)}{=} 0$ for all $1 \leqslant \alpha \leqslant M$, so that (3.24a) follows after applying (3.27) to $P - \pi_a(P)$.

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Furthermore, we obtain the characterizations (3.25b) and (3.25c) by lemma 11 (b) and (c), respectively, since

$$\begin{aligned} K_{S,dC} &\coloneqq \{P \in L^p(\Omega,\mathbb{R}^{3\times 3}) \mid \operatorname{sym} P = 0 \text{ a.e. and} \\ &\qquad \operatorname{dev}\operatorname{Curl} P = 0 \text{ in the distributional sense}\} \qquad (3.29) \\ &\overset{\text{Lemma 11(b)}}{=} \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}(\beta\, x + b),\ b \in \mathbb{R}^3, \beta \in \mathbb{R}\} \qquad (3.30) \end{aligned}$$

and

$$\begin{aligned} K_{dS,dC} &\coloneqq \{P \in L^p(\Omega,\mathbb{R}^{3\times 3}) \mid \operatorname{dev}\operatorname{sym} P = 0 \text{ a.e. and} \\ &\qquad \operatorname{dev}\operatorname{Curl} P = 0 \text{ in the distributional sense}\} \qquad (3.31) \\ &\overset{\text{Lemma 11(c)}}{=} \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}\big(\widetilde{A}\, x + \beta\, x + b\big) \\ &\qquad + \big(\langle \operatorname{axl} \widetilde{A}, x\rangle + \gamma\big)\cdot\mathbb{1}, \widetilde{A} \in \mathfrak{so}(3), b \in \mathbb{R}^3, \beta, \gamma \in \mathbb{R}\} \end{aligned}$$

with $\dim K_{S,dC} = 4$ and $\dim K_{dS,dC} = 8$. Hence, we can argue as above to deduce (3.24b) and (3.24c) from (3.17b) and (3.17c), respectively, since we end up with

$$\|P - \pi_b(P)\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\ \big(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\big) \qquad (3.32)$$

and

$$\|P - \pi_c(P)\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\ \big(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\big) \qquad (3.33)$$

respectively, with projections $\pi_b : L^p(\Omega,\mathbb{R}^{3\times 3}) \to K_{S,dC}$ and $\pi_c : L^p(\Omega,\mathbb{R}^{3\times 3}) \to K_{dS,dC}$. □

Finally, the kernel is killed by the tangential trace condition $P \times \nu \equiv 0$ ($\Leftrightarrow$ $\operatorname{dev}(P \times \nu) = 0$, cf. Obs. 3):

THEOREM 22. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p,\Omega) > 0$ such that for all $P \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ we have*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\ \big(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\big)\,. \qquad (3.34)$$

Proof. We argue as in the proof of [**43**, theorem 3.5] and consider a sequence $\{P_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ which converges weakly in $L^p(\Omega,\mathbb{R}^{3\times 3})$ to P^* so that $\operatorname{dev}\operatorname{sym} P^* = 0$ a.e. and $\operatorname{dev}\operatorname{Curl} P^* = 0$ in the distributional sense, i.e. $P^* \in K_{dS,dC}$, where

$$\begin{aligned} K_{dS,dC} &\overset{(3.25c)}{=} \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}\big(\widetilde{A}\, x + \beta\, x + b\big) + \big(\langle \operatorname{axl} \widetilde{A}, x\rangle + \gamma\big)\cdot\mathbb{1}, \\ &\qquad \widetilde{A} \in \mathfrak{so}(3), b \in \mathbb{R}^3, \beta, \gamma \in \mathbb{R}\}. \end{aligned}$$

By (3.16) it further follows that $\big\langle \operatorname{dev}(P^* \times (-\nu)), Q\big\rangle_{\partial\Omega} = 0$ for all $Q \in W^{1-\frac{1}{p'},p'}(\partial\Omega,\mathbb{R}^{3\times 3})$. However, since $P^* \in K_{dS,dC}$ also has an explicit representation, the boundary condition $\operatorname{dev}(P^* \times \nu) = 0$ is also valid in the classical sense.

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Furthermore, we deduce by observation 3 that $P^* \times \nu = 0$ on $\partial\Omega$, so that $P^* \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$. Again, using the explicit representation of $P^* = \operatorname{Anti}(\widetilde{A}\,x + \beta\,x + b) + (\langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma)\cdot\mathbb{1}$, we conclude with Observation 4 that, in fact, $P^* \equiv 0$:

$$
\begin{aligned}
&[\operatorname{Anti}\left(\widetilde{A}\,x + \beta\,x + b\right) + \left(\langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma\right)\cdot\mathbb{1}] \times \nu = 0\\
&\overset{\text{Obs. 4}}{\Rightarrow} \quad \widetilde{A}\,x + \beta\,x + b = 0 \quad \text{and} \quad \langle \operatorname{axl}\widetilde{A}, x\rangle + \gamma = 0 \quad \text{for all } x \in \partial\Omega\\
&\Rightarrow \quad \gamma = 0,\ \widetilde{A} = 0 \quad \Rightarrow \quad b = 0, \beta = 0.
\end{aligned}
$$
□

Remark 23. Similarly, the following estimates can also be deduced, even independently of (3.34), for $P \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$:

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right), \tag{3.35}$$

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right). \tag{3.36}$$

Since by [6, theorem 3.1 (ii)] it holds

$$\|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \qquad \text{for } P \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3}), \tag{3.37}$$

we can recover (3.34) from (3.35) and (3.37).

However, without boundary conditions the Nečas estimate provides for $P \in W^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$:

$$
\begin{aligned}
\|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} &\overset{(2.36)}{\leqslant} c\,(\|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\mathrm{D}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3^2})})\\
&\overset{(3.15)}{\leqslant} c\,(\|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\mathrm{D}\operatorname{dev}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3^2})})\\
&\leqslant c\,(\|\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}).
\end{aligned} \tag{3.38}
$$

Remark 24. Among the inequalities (3.34), (3.35) and (3.36) we expect (3.35) also to hold true in higher space dimensions $n > 3$, see the discussion in our Introduction.

Remark 25. Regarding (3.14) and (3.34) or (3.37) and (3.34) we obtain the norm equivalence

$$
\begin{aligned}
\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} &+ \|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\\
&\leqslant c\,\left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right)
\end{aligned}
$$

for tensor fields $P \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$.

For $P = \mathrm{D}u$ in (3.34) we recover the following tangential trace-free Korn inequality:

Corollary 26. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p,\Omega) > 0$ such that for all $u \in W^{1,p}(\Omega,\mathbb{R}^3)$ with $\mathrm{D}u \times \nu = 0$ on $\partial\Omega$ we have*

$$\|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\,\|\operatorname{dev}\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}. \tag{3.39}$$

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For skew-symmetric $P = \operatorname{Anti}(a)$ we recover from (3.34) a Poincaré inequality involving only the deviatoric (trace-free) part of the gradient:

COROLLARY 27. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(p, \Omega) > 0$ such that for all $a \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ we have*

$$\|a\|_{L^p(\Omega,\mathbb{R}^3)} \leqslant c\, \|\operatorname{dev} \mathrm{D}a\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}. \tag{3.40}$$

Proof. This follows from theorem 22 by setting $P = \operatorname{Anti}(a)$ and the following observations:

$\operatorname{Anti}(a) \times \nu = 0 \;\Leftrightarrow\; a = 0$ on $\partial\Omega$, see observation 4, $\operatorname{Curl}(\operatorname{Anti}(a)) = L(\mathrm{D}a)$, see (2.16a) and the form of $\operatorname{Anti}(a)$, see (2.2). □

Remark 28. The previous results also hold true for functions with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary. So, e.g., we have

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\, \left(\|\operatorname{dev}\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right) \tag{3.41}$$

for all $P \in W^{1,p}_{\Gamma,0}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3})$, which is the completion of $C^\infty_{\Gamma,0}(\Omega, \mathbb{R}^{3\times 3})$ with respect to the $W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3})$-norm.

Remark 29. In [**28**] the authors proved that in $n = 2$ dimensions, for $p = 2$ a Korn inequality for incompatibile fields also holds true when $\operatorname{Curl} P$ is only in L^1 (actually when it is a measure with bounded total variation) under the normalization condition $\int_\Omega \operatorname{skew} P \,\mathrm{d}x = 0$. In terms of scaling, it is interesting to involve in (3.34) the Sobolev exponent. So, we will show in a forthcoming paper that for $1 < p < 3$ the following estimate holds true on an arbitrary open set $\Omega \subseteq \mathbb{R}^3$:

$$\|P\|_{L^{p^*}(\Omega,\mathbb{R}^{3\times 3})} \leqslant c\, (\|\operatorname{dev}\operatorname{sym} P\|_{L^{p^*}(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}) \tag{3.42}$$

for all $P \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$, where $p^* = \frac{3p}{3-p}$. However, we do not know if such a result still holds in the borderline case $p = 1$.

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Appendix A. Appendix

Appendix A.1. On the trace-free Korn's first inequality in L^2

Using partial integration (see also [58, appendix A.1]) we catch up with a simple proof of

LEMMA 30. *Let $n \geqslant 2$, Ω (open) $\subset \mathbb{R}^n$, $u \in W_0^{1,2}(\Omega, \mathbb{R}^n)$. Then*

$$\int_\Omega \|\mathrm{D}u\|^2 \mathrm{d}x \leqslant 2 \int_\Omega \|\mathrm{dev}_n \operatorname{sym} \mathrm{D}u\|^2 \mathrm{d}x. \tag{A.1}$$

Proof. For $u \in C_c^\infty(\Omega, \mathbb{R}^n)$ we have

$$\begin{aligned} 2\int_\Omega \|\operatorname{sym} \mathrm{D}u\|^2 \mathrm{d}x &= \int_\Omega \|\mathrm{D}u\|^2 + \sum_{i,j=1}^n (\partial_i u_j)(\partial_j u_i) \mathrm{d}x \\ &\overset{\text{part. int.}}{=} \int_\Omega \|\mathrm{D}u\|^2 + \sum_{i,j=1}^n (\partial_j u_j)(\partial_i u_i) \mathrm{d}x \\ &= \int_\Omega \|\mathrm{D}u\|^2 + (\operatorname{div} u)^2 \mathrm{d}x, \end{aligned} \tag{A.2}$$

from where the 'baby' Korn inequality $\int_\Omega \|\mathrm{D}u\|^2 \mathrm{d}x \leqslant 2\int_\Omega \|\operatorname{sym} \mathrm{D}u\|^2 \mathrm{d}x$ for $u \in W_0^{1,2}(\Omega, \mathbb{R}^n)$ follows. Its improvement is obtained in regard with the decomposition

$$\|\mathrm{dev}_n \operatorname{sym} \mathrm{D}u\|^2 = \|\operatorname{sym} \mathrm{D}u - \frac{1}{n} \underbrace{\operatorname{tr}(\operatorname{sym} \mathrm{D}u)}_{=\operatorname{div} u} \cdot \mathbb{1}\|^2 = \|\operatorname{sym} \mathrm{D}u\|^2 - \frac{1}{n} (\operatorname{div} u)^2, \tag{A.3}$$

since we obtain

$$\begin{aligned} 2\int_\Omega \|\mathrm{dev}_n \operatorname{sym} \mathrm{D}u\|^2 \mathrm{d}x &\overset{(A.3)}{=} 2\int_\Omega \|\operatorname{sym} \mathrm{D}u\|^2 \mathrm{d}x - \frac{2}{n} \int_\Omega (\operatorname{div} u)^2 \mathrm{d}x \\ &\overset{(A.2)}{=} \int_\Omega \|\mathrm{D}u\|^2 \mathrm{d}x + \frac{n-2}{n} \int_\Omega (\operatorname{div} u)^2 \mathrm{d}x \overset{n \geqslant 2}{\geqslant} \int_\Omega \|\mathrm{D}u\|^2 \mathrm{d}x. \ \square \end{aligned}$$

Remark 31. The trace-free Korn's first inequality (A.1) is also valid in L^p, $p > 1$, see [27, Prop. 1] for the $n = 2$ case and [65, Thm. 2.3] for all $n \geqslant 2$ where again the justification was based on the Lions lemma.

Appendix A.2. Infinitesimal planar conformal mappings

Infinitesimal conformal mappings are defined by $\mathrm{dev}_n \operatorname{sym} \mathrm{D}u \equiv 0$ and in $n > 2$ they have the representation

$$\langle a, x \rangle\, x - \frac{1}{2} a \|x\|^2 + A\, x + \beta\, x + c, \quad \text{with } A \in \mathfrak{so}(n), a, c \in \mathbb{R}^n \text{ and } \beta \in \mathbb{R},$$

cf. [17, 33, 49, 63–65].

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In the planar case, the situation is quite different. Indeed, the condition $\operatorname{dev}_2 \operatorname{sym} \mathrm{D}u \equiv 0$ reads

$$\begin{aligned} &\begin{pmatrix} u_{1,x} & \frac{1}{2}(u_{1,y}+u_{2,x}) \\ \frac{1}{2}(u_{1,y}+u_{2,x}) & u_{2,y} \end{pmatrix} - \frac{1}{2}(u_{1,x}+u_{2,y}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \\ \Leftrightarrow \quad &\begin{pmatrix} \frac{1}{2}(u_{1,x}-u_{2,y}) & \frac{1}{2}(u_{1,y}+u_{2,x}) \\ \frac{1}{2}(u_{1,y}+u_{2,x}) & \frac{1}{2}(u_{2,y}-u_{1,x}) \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases} u_{1,x} &= u_{2,y} \\ u_{1,y} &= -u_{2,x} \end{cases} \end{aligned}$$

and corresponds to the validity of the Cauchy-Riemann-equations. Thus, in the planar case, infinitesimal conformal mappings are conformal mappings.

Appendix A.3. Kröner's relation in infinitesimal elasto-plasticity

At the macroscopic scale, in infinitesimal elasto-plastic theory, see e.g. [**3**, **4**, **20**–**22**, **44**, **46**], the incompatibility of the elastic strain is related to the Curl of the *contortion tensor* $\kappa := \alpha^T - \frac{1}{2}\operatorname{tr}(\alpha)\cdot \mathbb{1}$, where $\alpha := \operatorname{Curl} P$ is the dislocation density tensor, by Kröner's relation [**35**]:

$$\mathbf{inc}\,(\operatorname{sym} e) = -\operatorname{Curl}\kappa, \tag{A.4}$$

where the additive decomposition of the displacement gradient into non-symmetric elastic and plastic distortions is assumed:

$$\mathrm{D}u = e + P. \tag{A.5}$$

Indeed, (A.4) follows from Nye's formula (2.16) and the identities

$$\operatorname{tr}\operatorname{Curl}\operatorname{sym} e = 0 \quad \text{as well as} \quad \alpha := \operatorname{Curl} P \overset{(A.5)}{=} -\operatorname{Curl} e,$$

since we have

$$\begin{aligned} \mathrm{D}\operatorname{axl}\operatorname{skew} e &\overset{(2.16b)}{=} \frac{1}{2}\operatorname{tr}(\operatorname{Curl}\operatorname{skew} e)\cdot \mathbb{1} - (\operatorname{Curl}\operatorname{skew} e)^T \\ &\overset{\operatorname{tr}\operatorname{Curl}\operatorname{sym} e=0}{=} \frac{1}{2}\operatorname{tr}(\operatorname{Curl}\operatorname{skew} e + \operatorname{Curl}\operatorname{sym} e)\cdot \mathbb{1} - (\operatorname{Curl}\operatorname{skew} e)^T \\ &= \frac{1}{2}\operatorname{tr}(\operatorname{Curl} e)\cdot \mathbb{1} - (\operatorname{Curl} e)^T + (\operatorname{Curl}\operatorname{sym} e)^T \\ &\overset{\alpha=-\operatorname{Curl} e}{=} -\frac{1}{2}\operatorname{tr}(\alpha)\cdot \mathbb{1} + \alpha^T + (\operatorname{Curl}\operatorname{sym} e)^T = \kappa + (\operatorname{Curl}\operatorname{sym} e)^T. \end{aligned} \tag{A.6}$$

Thus, applying Curl on both sides of (A.6) establishes (A.4), since $\operatorname{Curl}\circ \mathrm{D} \equiv 0$:

$$0 = \operatorname{Curl}\mathrm{D}\operatorname{axl}\operatorname{skew} e \overset{(A.6)}{=} \operatorname{Curl}\kappa + \operatorname{Curl}([\operatorname{Curl}\operatorname{sym} e]^T) = \operatorname{Curl}\kappa + \mathbf{inc}\,(\operatorname{sym} e). \tag{A.7}$$

From the decomposition $\operatorname{sym}\mathrm{D}u = \operatorname{sym} e + \operatorname{sym} P$ it follows moreover $\mathbf{inc}\,(\operatorname{sym} e) = -\mathbf{inc}\,(\operatorname{sym} P)$, see also the last calculation in footnote 5.

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In finite strain elasticity [10], the Riemann-Christoffel tensor $\mathcal{R}$ expresses the compatibility of strain tensors in the sense of

$$\begin{aligned} C \in C^2(\Omega, \operatorname{Sym}^+(3)) : \quad \mathcal{R}(C) = 0 \quad \Leftrightarrow \\ C = (D\varphi)^T D\varphi \quad \text{in simply connected domains.} \end{aligned} \tag{A.8}$$

Writing $C = (\mathbb{1}+P)^T(\mathbb{1}+P) = 1 + 2\cdot \operatorname{sym} P + P^T P$ for $P \in C^2(\Omega, \mathbb{R}^{3\times 3})$, the incompatibility operator is the linearization of the Riemann-Christoffel tensor at the identity, since

$$\begin{aligned} \mathcal{R}(\mathbb{1} + 2\cdot \operatorname{sym} P + P^T P) &= \mathcal{R}(\mathbb{1}) + 2\cdot \mathrm{D}\mathcal{R}(\mathbb{1}) \operatorname{sym} P + \text{h.o.t.} \\ &= 0 + 2\cdot \mathbf{inc}\,(\operatorname{sym} P) + \text{h.o.t.} \end{aligned} \tag{A.9}$$

see also [20] and the references contained therein.

Appendix A.4. Further identities

Symmetric tensors play an important role in the above considerations. We mention here the full expression of $S \times b$ for $S \in \operatorname{Sym}(3)$ and $b \in \mathbb{R}^3$:

$$S \times b = \begin{pmatrix} S_{12}\, b_3 - S_{13}\, b_2 & S_{13}\, b_1 - S_{11}\, b_3 & S_{11}\, b_2 - S_{12}\, b_1 \\ S_{22}\, b_3 - S_{23}\, b_2 & S_{23}\, b_1 - S_{12}\, b_3 & S_{12}\, b_2 - S_{22}\, b_1 \\ S_{23}\, b_3 - S_{33}\, b_2 & S_{33}\, b_1 - S_{13}\, b_3 & S_{13}\, b_2 - S_{23}\, b_1 \end{pmatrix} \tag{A.10}$$

which is an example of a trace-free matrix with non-zero entries on the diagonal:

$$\operatorname{tr}(S \times b) = S_{12}\, b_3 - S_{13}\, b_2 + S_{23}\, b_1 - S_{12}\, b_3 + S_{13}\, b_2 - S_{23}\, b_1 = 0.$$

Moreover, we outline some basic identities which played useful roles in our considerations:

1. from linear algebra:
   (a) $P \times b$ row-wise,
   (b) $\mathbb{1} \times b = \operatorname{Anti}(b) \in \mathfrak{so}(3)$,
   (c) $(\operatorname{Anti} a) \times b = b \otimes a - \langle b, a\rangle\cdot\mathbb{1}$,
   (d) $\operatorname{tr}(S \times b) = 0$,
   (e) $(\mathbb{1} \times b)^T \times b = \|b\|^2 \cdot \mathbb{1} - b \otimes b \in \operatorname{Sym}(3)$,
   (f) $((\operatorname{Anti} a) \times b)^T \times b = -\langle b, a\rangle\cdot \operatorname{Anti}(b) \in \mathfrak{so}(3)$,
   (g) $(S \times b)^T \times b \in \operatorname{Sym}(3)$,
   (h) $\operatorname{dev}(P \times b) = P \times b + \frac{2}{3}\langle \operatorname{axl}\operatorname{skew} P, b\rangle \cdot \mathbb{1}$,
   (i) $\operatorname{sym}[(P \times b)^T \times b] = ((\operatorname{sym} P) \times b)^T \times b$,
   (j) $\operatorname{skew}[(P \times b)^T \times b] = ((\operatorname{skew} P) \times b)^T \times b$,
   (k) $a \otimes b = 0 \;\Leftrightarrow\; \operatorname{sym}(a \otimes b) = 0 \;\Leftrightarrow\; \operatorname{dev}(a \otimes b) = 0 \;\Leftrightarrow\; \operatorname{dev}\operatorname{sym}(a \otimes b) = 0$,
   (l) $\operatorname{dev}(P \times b) = 0 \;\Leftrightarrow\; P \times b = 0$,

   for $a, b \in \mathbb{R}^3$, $S \in \operatorname{Sym}(3)$ and $P \in \mathbb{R}^{3\times 3}$,

2. and their formal equivalents from calculus:
   (a) $\operatorname{Curl} P = P \times (-\nabla)$,
   (b) $\operatorname{Curl}(\zeta \cdot \mathbb{1}) = -\operatorname{Anti}(\nabla\zeta) \in \mathfrak{so}(3)$,
   (c) $\operatorname{Curl} A = \operatorname{tr}(\operatorname{Daxl} A)\cdot\mathbb{1} - (\operatorname{Daxl} A)^T$,
   (d) $\operatorname{tr}(\operatorname{Curl} S) = 0$,
   (e) $\mathbf{inc}\,(\zeta \cdot \mathbb{1}) = \Delta\zeta \cdot \mathbb{1} - \mathrm{D}^2\zeta \in \operatorname{Sym}(3)$,
   (f) $\mathbf{inc}\, A = -\operatorname{Anti}(\nabla \operatorname{tr}(\operatorname{Daxl} A)) \in \mathfrak{so}(3)$,
   (g) $\mathbf{inc}\, S \in \operatorname{Sym}(3)$,
   (h) $\operatorname{dev}\operatorname{Curl} P = \operatorname{Curl} P - \frac{2}{3}\operatorname{div}\operatorname{axl}\operatorname{skew} P \cdot \mathbb{1}$,
   (i) $\operatorname{sym}\mathbf{inc}\, P = \mathbf{inc}\operatorname{sym} P$,
   (j) $\operatorname{skew}\mathbf{inc}\, P = \mathbf{inc}\operatorname{skew} P$,

   for $\zeta \in \mathscr{D}'(\Omega, \mathbb{R})$, $A \in \mathscr{D}'(\Omega, \mathfrak{so}(3))$, $S \in \mathscr{D}'(\Omega, \operatorname{Sym}(3))$ and $P \in \mathscr{D}'(\Omega, \mathbb{R}^{3\times 3})$.

We catch up with the verification of the identities not contained in our considerations explicitly:

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- $(\mathbb{1} \times b)^T \times b \overset{1.(a)}{=} (\mathrm{Anti}(b))^T \times b = -(\mathrm{Anti}\, b) \times b \overset{1.(b)}{=} -b \otimes b + \langle b, b\rangle \cdot \mathbb{1} \;\Rightarrow\; 1.(d),$
- we have the decompositions:

$$(P \times b)^T \times b = (\mathrm{sym}\, P \times b + \mathrm{skew}\, P \times b)^T \times b$$
$$= \underbrace{((\mathrm{sym}\, P) \times b)^T \times b}_{\in \mathrm{Sym}(3)} + \underbrace{((\mathrm{skew}\, P) \times b)^T \times b}_{\in \mathfrak{so}(3)}$$

but also

$$\mathbf{inc}\, P = \mathbf{inc}\,(\mathrm{sym}\, P + \mathrm{skew}\, P) = \underbrace{\mathbf{inc}\ \mathrm{sym}\, P}_{\in \mathrm{Sym}(3)} + \underbrace{\mathbf{inc}\ \mathrm{skew}\, P}_{\in \mathfrak{so}(3)}$$

where we have used (e) and (f), so that (h) and (i) follow,

- the equivalence $a \otimes b = 0 \Leftrightarrow \mathrm{dev}\,\mathrm{sym}(a \otimes b) = 0$ follows from the expression:

$$\frac{\|b\|^4}{2}\|a \otimes b\|^2 = \|b\|^4 \|\mathrm{dev}\,\mathrm{sym}(a \otimes b)\|^2 + \frac{1}{2}\left(\frac{n}{n-1}\right)^2 \big\langle b, \mathrm{dev}\,\mathrm{sym}(a \otimes b) b\big\rangle^2.$$

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