THE ASYMPTOTIC BEHAVIOUR OF THE HERMITE POLYNOMIALS

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Introduction. In a recent paper, Olver **(2)** obtains a set of formulae that completely determine the asymptotic behaviour of the Hermite polynomials, $H_n(z)$, as $n \to \infty$ and z is unrestricted. His proof depends on a technique that he has developed for discussing the asymptotics of solutions of second-order, linear, homogeneous differential equations satisfying certain conditions. We believe it fair to say that Olver's work follows the tradition of most of the major theorems of classical asymptotics. The results contained in theorems such as Watson's lemma and Perron's proof of the Method of Laplace are based on an acceptance, on an a priori basis, of the Poincaré type expansion. These theorems then establish classes of functions and domains of validity for which this type of expansion exists. In spite of the undisputed importance of theorems of this type, we believe that modern asymptotics will move in a different direction.

In order to illustrate the point of view that we hold, let us suppose that we are interested in the behaviour of a complex function, $F(n, z)$, where n and z are both complex variables, as $n \to \infty$ and z is unrestricted. The least amount of information that would retain some flavour of asymptotics would likely answer questions of the following type. What are the conditions for the existence of two functions, $G(n, z)$ and $H(n, z)$, with certain specified properties, such that

$$\lim_{n\to\infty} (F - G)/H = 1, \tag{1.1}$$

or

$$\lim_{n\to\infty} (F - G) = 0? \tag{1.2}$$

In order to avoid trivial answers such as $G = 0$, $H = F$, or $G = F$, we would expect G and H to possess properties that are not properties of F and in addition they should be simpler in some sense than is F. In such a situation the implications of (1.1) and (1.2) are

$$F = G + H(1 + o(1)) \qquad \text{or } F = G + o(1), \tag{1.3}$$

and (1.3) would contain asymptotic information of some value.

From the minimum information contained in (1.1) and (1.2), we might ask for more detailed information in a multitude of ways. In keeping with

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the traditional point of view, we might ask for the conditions under which, for each fixed z, a sequence $\{A_k(z)\}$ would exist such that

$$F = \sum_{k=0}^{N} A_k(z)n^{-k} + o(n^{-N}), \tag{1.4}$$

for every non-negative integer N. As an additional restriction, we might ask for the domain $D = D(z, n)$ for which the order relation of (1.3) is uniform in z. Such a procedure may of course produce only a partial answer to the problem of interest. For example, a necessary condition that a linear combination of the forms of the type (1.3) exist for the Hermite polynomials is that z be confined to a bounded domain. When a sequence of the type $A_k(z)$ exists, the formal series

$$\sum_{k=0}^{\infty} A_k(z)n^{-k}$$

is of Poincaré type and we write

$$F \sim \sum_{k=0}^{\infty} A_k(z)n^{-k}. \tag{1.5}$$

This type of asymptotic expansion has the property that it is unique.

Even though we may only obtain a partial answer to the problem that we have posed, a procedure exists by means of which we may obtain additional asymptotic information and still retain the framework of the Poincaré definition.

By a suitable transformation of variables $z = z(\xi, \tau)$, $n = n(\xi, \tau)$, we may be able to obtain a new asymptotic variable τ, with the property

$$\lim_{n\to\infty} \tau = \infty,$$

such that

$$F = F(z(\xi, n), n(\xi, n)) \sim \sum_{k=0}^{\infty} B_k(\xi)\tau^{-k}. \tag{1.6}$$

Further if we denote the domain of validity of (1.6) by $D' = D'(z, n)$, the union of D and D' might contain D as a proper subset. By a sequence of such asymptotic expansions, it is possible that we might obtain a definitive set of formulae that completely describe the behaviour of $F(z, n)$ as $n \to \infty$ and z is unrestricted. However, once we admit such a procedure, the individual members of a definitive set lose the property of uniqueness. For example, let $\phi(\xi)$ be any bounded function, and let τ_1 be defined by $\tau = \tau_1 + \phi$. Substitution into (1.5) and re-expansion in terms of powers of τ_1 will yield an asymptotic expansion whose domain of validity is still D'. Since the choice of the functions $z = z(\xi, \tau)$ and $n = n(\xi, \tau)$ will be dictated by the convenience of a procedure, it is reasonably certain that different procedures will produce different definitive sets of asymptotic formulae. We shall illustrate this point in our treatment of the asymptotics of Hermite polynomials.

Let us suppose that such a procedure fails to produce a definitive set. The reason for failure may well be that we are demanding a detail of asymptotic information in our definition of an asymptotic expansion that does not exist. Moreover, we are demanding a detail of information that is difficult to justify from a utilitarian point of view. It is for this reason that we believe modern asymptotics will discard the acceptance, on an a priori basis, of Poincaré type expansions and adopt a definition that demands much less information. We shall give a specialized version of a definition **(1)** that is at least capable of asking for the minimum amount of asymptotic information that we have described.

DEFINITION 1.1. *Let* $F = F(z, n)$ *and* $\tau = \tau(z, n)$ *be as before. Thus*

$$\lim_{n\to\infty} \tau = \infty.$$

If a sequence of functions $f_k = f_k(z, n)$ $(k = 0, 1, 2, \ldots, M)$ *exists such that*

$$F = \sum_{k=0}^{N} f_k + o(\tau^{-N}), \tag{1.7}$$

for every fixed integer N *in the set,* $0 \leqslant N \leqslant M$, *then*

$$\sum_{k=0}^{M} f_k$$

is called an asymptotic expansion of F *to* M *terms. We write*

$$F \sim \sum_{k=0}^{M} f_k; \{\tau^{-k}\}. \tag{1.8}$$

If $M = 0$, we obtain $F = f_0 + o(1)$, which is one form of what we described as minimum asymptotic information.

In addition, we write

$$F \sim G\left[\sum_{k=0}^{M} f_k; \{\tau^{-k}\}\right], \tag{1.9}$$

when $G = G(z, n)$ is a function for which

$$F/G \sim \sum_{k=0}^{M} f_k; \{\tau^{-k}\}. \tag{1.10}$$

There is of course no point in introducing (1.9) unless

$$\overline{\lim_{n\to\infty}}\, G = \infty.$$

If G is bounded, we could introduce the sequence $\{Gf_k\}$. When $M = 0$ and f_0 is bounded away from zero, we have

$$F/G f_0 = 1 + o(1), \tag{1.11}$$

the second form of minimal asymptotic information.

When f and g are two functions such that

$$f = g + o(\tau^{-k}), \tag{1.12}$$

for every fixed integer k in $0 \leqslant k \leqslant M$, we write $f \approx g$ and say that the two functions are asymptotically equal. Sequences such as τ^{-k} are special cases of a general class called asymptotic sequences. In the above definitions M might be ∞.

It is our belief that modern asymptotic theory will re-examine many of the major classical results and replace the basic assumptions of these theorems by a set designed to yield asymptotic expansions of the type (1.9). A paper is already in progress that shows that many of the classical results are capable of extension in this sense.

The method we shall use to discuss the asymptotics of $H_n(z)$ is based on an integral representation of the type

$$h_n(\phi) = \int_{-\pi}^{\pi} \exp[nF(\theta, \phi)]d\theta. \tag{1.13}$$

If ϕ is considered fixed, the results can easily be obtained by an appeal to theorems obtained by Perron **(3)**. In fact a substitution exists by means of which we could apply Watson's lemma. The proofs of our present paper would be considerably shortened if we first proved that Perron's central theorems still apply as long as the conditions of these theorems hold uniformly in ϕ, when ϕ is restricted to a suitable domain. We shall not follow this course. The pattern of proof that we give differs from that of Perron in several significant details. In fact a very much simplified proof of the theorems of Perron can be obtained by following the pattern of the proof we shall give.

In comparing the definitive set of formulae that we shall obtain with Olver's set, we should like to state that we do not claim that our procedure or our results are better than his. We simply claim they are different. It is our desire to introduce an integral representation procedure, in a non-trivial setting, that is capable of generalization. It is extremely useful to us to have Olver's results as a basis of comparison.

2. Asymptotic formulae in terms of elementary functions. The conditions under which we investigate the asymptotic behaviour of $H_n(z)$ are one or both of n and $z \to \infty$. When n is bounded and $z \to \infty$, the explicit expression

$$H_n(z) = n!(2z)^n \sum_{m=0}^{[n/2]} (-1)^m (2z)^{-2m}/m!(n-2m)! \tag{2.1}$$

is an Erdélyi type of asymptotic expansion. To complete our discussion, we consider $n \to \infty$ and z unrestricted.

Since $H_n(z) = (-1)^n H_n(-z)$ and $H_n(z) = \overline{H_n(\bar{z})}$, we can restrict the values of z to lie in any convenient quadrant of the complex plane. For convenience, we make the substitution

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$$(2.2)\qquad z = \sqrt{2n}\cosh\phi, \qquad \phi = \alpha + i\beta,$$

and note that the conditions

$$(2.3)\qquad \alpha \geqslant 0, \qquad 0 \leqslant \beta \leqslant \pi/2$$

are equivalent to the requirement that z be confined to a specified quadrant of the complex plane. We note that z real implies that either $\alpha = 0$ or $\beta = 0$. The Hermite polynomials have the well-known integral representation

$$(2.4)\qquad H_n(\sqrt{2n}\cosh\phi) = \frac{n!}{2\pi i}\int_C t^{-(n+1)}\exp(2\sqrt{2n}\cosh\phi t - t^2)dt,$$

where we choose C to be the circle $t = \sqrt{n/2}\exp(i\theta - \phi)$ in order to locate the critical points of the integrand at convenient positions.

Let us define $A_n(\phi)$, $h_n(\phi)$, and $F(\theta, \phi)$ by

$$(2.5)\qquad A_n = (2\pi)^{-1}n!(2/n)^{\frac{1}{2}n}\exp[n(\phi + 1 + \tfrac{1}{2}e^{-2\phi})],$$

$$(2.6)\qquad h_n = H_n(\sqrt{2n}\cosh\phi)/A_n,$$

$$(2.7)\qquad F = (1 + e^{-2\phi})(e^{i\theta} - 1) - \tfrac{1}{2}e^{-2\phi}(e^{2i\theta} - 1) - i\theta.$$

In this notation, the substitution $t = \sqrt{n/2}\exp(i\theta - \phi)$ into (2.4) gives

$$(2.8)\qquad h_n = \int_{-\pi}^{\pi}\exp(nF(\theta, \phi))d\theta.$$

The derivative, $dF/d\theta$, is zero at $\theta = 0$ and $\theta = -2i\phi$. These are the only critical points that need be considered. From these two, we can obtain the asymptotic behaviour of h_n by means of linear combinations of asymptotic series of Poincaré type. These can be combined into single asymptotic expansions of Erdélyi type.

Although it is possible to produce a definitive set in terms of two formulae, we shall not attempt a proof along these lines. Since the argument of the modified Bessel function $K_{1/3}(\psi)$ involved in these formulae is a multi-valued function, it is even debatable as to whether only two formulae are involved. In order to use these formulae, accurate descriptions of the proper branch to use must be given. The proper branch is different in different domains.

For reasons of simplicity, we arbitrarily divide the discussion into three parts.

Case I. There exists a positive real number, λ, such that

$$n^{\lambda} = O(n^{1/3}(1 - e^{-2\alpha})).$$

Obviously λ must be in $0 < \lambda \leqslant \frac{1}{3}$.

Case II. Let $\beta_0 = n^{-\frac{1}{3}+\lambda}$ for some arbitrarily small but fixed $\lambda > 0$. For this case, we assume $\alpha = o(\beta_0)$ and $\beta_0 = O(\beta^2)$. Thus $\alpha = o(\beta^2)$.

Case III. $\phi = \alpha + i\beta \to 0$.

A discussion of these three possibilities will produce a definitive set. In addition the domains of validity involved in any two of the three cases will have a domain of overlap. Since the three proofs required are substantially the same, we shall give a complete proof only for the first case.

Proof of Case I. If we define ξ and G by

$$\xi = \tfrac{1}{2}n(1 - e^{-2\phi}), \tag{2.9}$$

$$G = \sum_{k=3}^{\infty} \frac{[1 + e^{-2\phi}(1 - 2^{k-1})]i^k n\theta^k}{k!}, \tag{2.10}$$

then the Maclaurin expansion of nF about $\theta = 0$ gives

$$nF = -\xi\theta^2 + G. \tag{2.11}$$

Since $0 \leqslant \arg \xi < \pi/2$ ($\arg \xi$ might converge to $\pi/2$ as $\tau \to \infty$), there is no ambiguity in defining $\sqrt{\xi}$ in the usual way. The substitution, $\theta = u/\sqrt{\xi}$ into (2.11) makes it seem plausible that the first term will be the important term as long as $n/\xi^{3/2} \to 0$. We, therefore, choose our asymptotic variable τ to be

$$\tau = \xi^{3/2}/n = \sqrt{n}(1 - e^{-2\phi})^{3/2}/2^{3/2}. \tag{2.12}$$

From

$$2^{3/2}|\tau| \geqslant \sqrt{n}(1 - e^{-2\alpha})^{3/2} \geqslant K^{3/2}n^{3\lambda/2}, \tag{2.13}$$

we obtain

$$\lim_{n\to\infty} \tau = \infty.$$

Thus the sequence τ^{-k} can be used to give meaning to the Erdélyi definitions of asymptotic equality and asymptotic expansions.

The inequality $|\tau| \leqslant n^{1/2}$ is implied by $|\xi| \leqslant n$. For every real $\lambda > 0$, $n^\lambda \geqslant |\tau|^{2\lambda}$ and terms of the form $n^s \exp(-Kn^\lambda) \approx 0$ for every fixed real number s. An easy calculation, from (2.7), gives

$$\begin{aligned} \operatorname{Re}(nF) &= -2n \sin^2(\tfrac{1}{2}\theta)(1 - e^{-2\alpha}\cos(\theta - 2\beta)) \\ &\leqslant -2n(1 - e^{-2\alpha}) \sin^2(\tfrac{1}{2}\theta), \qquad \text{uniformly in } \beta. \end{aligned} \tag{2.14}$$

For every choice of $k > 0$ and $|\theta| \geqslant \epsilon = 2kn^{-1/3}$, we obtain

$$\operatorname{Re}(nF) \leqslant -\gamma n^\lambda, \qquad \text{for some } \gamma > 0. \tag{2.15}$$

Thus $\exp(nF) \approx 0$, when $|\theta| \geqslant \epsilon$ and

$$h_n \approx \int_{-\epsilon}^{\epsilon} \exp(nF)d\theta. \tag{2.16}$$

The function G of (2.10) can be written

$$\begin{aligned} G &= \sum_{k=1}^{\infty} [1 + e^{-2\phi}(1 - 2^{k+1})]i^{k+2}n\theta^{k+2}/(k+2)! \\ &= \sum_{k=1}^{\infty} [1 + e^{-2\phi}(1 - 2^{k+1})]i^{k+2}[\tfrac{1}{2}(1 - e^{-2\phi})]^{k-1}(\sqrt{\xi}\,\theta)^{k+2}\tau^{-k}/(k+2)!\,. \end{aligned} \tag{2.17}$$

Defining $a_k(\phi)$, u, w by

$$a_k = a_k(\phi) = [1 + e^{-2\phi}(1 - 2^{k+1})]i^{k+2}[\tfrac{1}{2}(1 - e^{-2\phi})]^{k-1}/(k+2)!, \tag{2.18}$$

$$u = \sqrt{\xi}\,\theta, \qquad w = \tau^{-1}, \tag{2.19}$$

we have

$$G = \sum_{k=1}^{\infty} a_k u^{k+2} w^k. \tag{2.20}$$

Let us, for the moment, consider ϕ, u, and w to be independent complex variables. The fact that $|a_k| \leqslant 1$ implies that G is a regular function of w if

$$|w| \leqslant 2\sigma/|u|(1 + |u|^2), \qquad \text{for every } \sigma \text{ in } 0 \leqslant \sigma < 2. \tag{2.21}$$

In this region $\exp G$ has a Maclaurin expansion of the form

$$\exp G = \sum_{k=0}^{\infty} G_k w^k, \qquad G_0 = 1. \tag{2.22}$$

The following properties of the G_k are easily established.

(a) $G_k = O(|u|^k(1 + |u|^2)^k/(2\sigma)^k)$, uniformly in k, ϕ, and u.

(b) $G_k = G_k(e^{-2\phi}, u)$ is a polynomial in each of its arguments.

(c) G_{2k} is an even polynomial in u and G_{2k+1} is odd.

(d) The degree of G_k, as a polynomial in u, does not exceed $3k$.

(e) The coefficients of all powers of u in G_k are polynomials in the a_p's. For each fixed k, these coefficients are uniformly bounded in ϕ.

Property (a) follows from Cauchy's inequalities. The remaining properties either follow from (a) or can be obtained by mathematical induction.

In the domain $|w| \leqslant \sigma/|u|(1 + |u|^2)$, we can, for every fixed non-negative integer N, write

$$\exp G = \sum_{k=0}^{2N+1} G_k w^k + O(|u|^{2N+2}(1 + |u|^2)^{2N+2}|w|^{2N+2}), \tag{2.23}$$

uniformly in all arguments. When $u = \sqrt{\xi}\theta$, $w = \tau^{-1} = n/\xi^{3/2}$, we have

$$\begin{aligned} |u|(1 + |u|^2)|w| &= \frac{n|\theta|}{|\xi|} + n|\theta|^3 \leqslant \frac{2|\theta|}{(1 - e^{-2\alpha})} + n|\theta|^3 \\ &\leqslant \frac{2|\epsilon|}{(1 - e^{-2\alpha})} + n|\epsilon|^3, \qquad \text{if } |\theta| \leqslant |\epsilon|, \\ &\leqslant 8k^3 + o(1), \qquad \text{for } \epsilon = 2kn^{-1/3}. \end{aligned} \tag{2.24}$$

Clearly the choice $k = 1/4$ will ensure the validity of (2.23) for $u = \sqrt{\xi}\,\theta$, $w = \tau^{-1}$. Thus

$$\exp G = \sum_{k=0}^{2N+1} G_k(e^{-2\phi}, \sqrt{\xi}\,\theta)\tau^{-k} + O(|\sqrt{\xi}\,\theta|^{2N+2}(1 + |\xi\theta^2|)^{2N+2}|\tau|^{-(2N+2)}). \tag{2.25}$$

It is important to note that (2.25) holds for complex θ, providing $|\theta| \leqslant \epsilon$.

From (2.11), we obtain

$$nF = -\xi\theta^2 + O(1), \qquad |\theta| \leqslant \epsilon. \tag{2.26}$$

Since $0 \leqslant \arg \xi < \pi/2$, (2.26) implies that

$$\operatorname{Re}(nF) \leqslant -Kn^{\lambda}, \qquad \text{for some } K > 0, \tag{2.27}$$

on the sectors of the circle $|\theta| = \epsilon$ specified by $-\delta \leqslant \arg \theta \leqslant 0$ and $\pi - \delta \leqslant \arg \theta \leqslant \pi$. Hence on these sectors $\exp(nF) \approx 0$. Thus

$$h_n \approx \int_{-\epsilon}^{\epsilon} \exp(nF)d\theta \approx \int_L \exp(nF)d\theta, \tag{2.28}$$

where L is the path of integration shown in Figure 1.

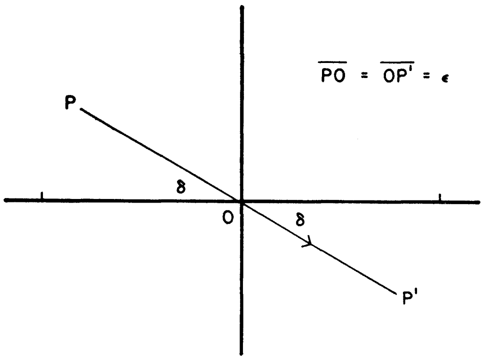

FIGURE 1

The result

$$\int_{-\epsilon}^{\epsilon} \exp(nF)d\theta \approx \sum_{k=0}^{2N+1} \int_L G_k e^{-\xi\theta^2} d\theta \tau^{-k} + O\left(\frac{\tau^{-(2N+2)}}{\sqrt{\xi}}\right) \tag{2.29}$$
$$\approx \sum_{k=0}^{N} \int_L G_{2k} e^{-\xi\theta^2} d\theta \tau^{-2k} + O\left(\frac{\tau^{-(2N+2)}}{\sqrt{\xi}}\right)$$

follows from the fact that $\int_L |e^{-u^2}|u|^{2N+2}(1 + |u|^2)^{2N+2}du|/|\sqrt{\xi}|$ exists even if we extend the path of integration from $-\infty$ to ∞ along the ray POP'.

Finally we consider, for a fixed non-negative integer m,

$$\int_{P'}^{\infty e^{-i\delta}} (\sqrt{\xi}\,\theta)^m e^{-\xi\theta^2} d\theta = \frac{1}{\sqrt{\xi}} \int_{P'\sqrt{\xi}}^{\infty e^{i\psi}} u^m e^{-u^2} du, \qquad \psi = \tfrac{1}{2}\arg \xi - \delta. \tag{2.30}$$

Since $|P'\sqrt{\xi}| = |\epsilon\sqrt{\xi}| \geqslant \sqrt{2}kn^{1/6}(1 - e^{-2\alpha})^{1/2} \geqslant Kn^{\lambda/2}$ for some $K > 0$, we must have $|P'\sqrt{\xi}| \to \infty$. From the known asymptotic behaviour of integrals of this type,

$$(2.31)\qquad \int_{P'}^{\infty e^{-i\delta}} (\sqrt{\xi}\,\theta)^m e^{-\xi\theta^2} d\theta = O(e^{-Kn^\lambda}), \qquad \text{for some } K > 0,$$
$$\approx 0.$$

A similar result holds for the extension of the ray OP to ∞. Thus

$$(2.32)\qquad \int_P^{P'} (\sqrt{\xi}\,\theta)^m e^{-\xi\theta^2} d\theta \approx \int_{-\infty^{-i\delta}}^{\infty e^{-i\delta}} (\sqrt{\xi}\,\theta)^m e^{-\xi\theta^2} d\theta$$
$$= \frac{1}{\sqrt{\xi}} \int_{-\infty e^{i\psi}}^{\infty e^{i\psi}} u^m e^{-u^2} du, \qquad \psi = \tfrac{1}{2}\arg\xi - \delta.$$

Since G_{2k} is a polynomial in $\sqrt{\xi}\,\theta$ with bounded coefficients we can write

$$(2.33)\qquad \int_P^{P'} G_{2k} e^{-\xi\theta^2} d\theta \approx \int_{-\infty e^{-i\delta}}^{\infty e^{i\delta}} G_{2k} e^{-\xi\theta^2} d\theta = \frac{1}{\sqrt{\xi}} \int_{-e^{i\psi}}^{\infty e^{i\psi}} G_{2k}(e^{-2\phi}, u) e^{-u^2} du.$$

If we define $g_k(\phi)$ by

$$(2.34)\qquad g_k = \sqrt{\xi/\pi} \int_{-\infty e^{-i\delta}}^{\infty e^{-i\delta}} G_{2k} e^{-\xi\theta^2} d\theta,$$

then $g_0 = 1$ and

$$(2.35)\qquad h_n \approx \frac{\sqrt{\pi}}{\sqrt{\xi}} \left[\sum_{k=0}^{N} g_k \tau^{-2k} + O(\tau^{-2(N+1)}) \right]$$

and

$$(2.36)\qquad h_n \sim \sqrt{\pi/\xi} \sum_{k=0}^{\infty} g_k \tau^{-2k}.$$

We therefore have obtained for $h_n(\phi)$ a uniform expansion of Poincaré type.

Proof for Case II. The domain of validity for this case is as follows: If $\beta_0 = n^{-\frac{1}{3}+\lambda}$ for a fixed, arbitrarily small λ, then $\alpha = o(\beta_0)$ and $\beta_0 = O(\beta^2)$. Hence $\alpha = o(\beta^2)$.

From (2.14),

$$(2.37)\qquad \operatorname{Re}(nF) = -2n\sin^2 \tfrac{1}{2}\theta(1 - e^{-2\alpha}\cos(\theta - 2\beta)).$$

Uniformly in α,

$$(2.38)\qquad \operatorname{Re}(nF) \leqslant -4n\sin^2(\tfrac{1}{2}\theta)\sin^2(\tfrac{1}{2}\theta - \beta).$$

With the same choice $\epsilon = 2kn^{-1/3}$,

$$(2.39)\qquad \operatorname{Re}(nF) \leqslant -4n\sin^2 \tfrac{1}{2}\epsilon \sin^2 \beta, \quad |\theta| \geqslant \epsilon \quad \text{and} \quad |\theta - 2\beta| \geqslant \epsilon$$

and

$$(2.40)\qquad \operatorname{Re}(nF) \leqslant -\gamma n^\lambda, \qquad \text{for some } \gamma > 0.$$

Thus $\exp(nF) \approx 0$ for $|\theta| \geqslant \epsilon$ and $|\theta - 2\beta| \geqslant \epsilon$.

The only difference is that we now have two critical points, one at $\theta = 0$ and the other at $\theta = -2i\phi$, which we must consider. Since our integrand is periodic, with period 2π, we may use the range of integration $(-\beta, 2\pi - \beta)$ and modify the path of integration as shown in Figure 2. This modification in the range of integration is only necessary to accommodate the possibility

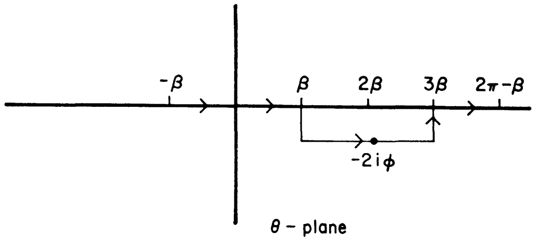

FIGURE 2

that $\beta \to \pi/2$. No modification of our previous proof is necessary except to show that the contribution along the vertical lines is ≈ 0. Since $\alpha \to 0$, this is not difficult to prove. If we show the explicit dependence of ξ, τ, and g_k on ϕ by $\xi = \xi(\phi)$, $\tau = \tau(\phi)$, $g_k = g_k(\phi)$, we define

$$(2.41) \qquad \xi_0 = \xi(-\phi), \qquad \tau_0 = \tau(-\phi), \qquad g_k{}^0 = g_k(-\phi).$$

It is trivial to show that $|\xi/\xi_0| \leqslant 1$ and $|\tau/\tau_0| \leqslant 1$. We can then write the corresponding result for Case II as

$$(2.42) \quad h_n \approx \sqrt{\pi/\xi}\left[\sum_{k=0}^{\infty} \{g_k + \sqrt{\xi/\xi_0}\, g_k{}^0(\tau/\tau_0)^{2k} \exp[n(\sinh 2\phi - 2\phi)]\}\tau^{-2k}\right].$$

In this domain the zeros of the Hermite polynomials show their influence on the asymptotic expansion. It is interesting to note how (2.42) handles such zeros. When $\phi = \pi i/2$, $z = 0$. Further $g_k = g_k{}^0$, $\xi = \xi_0$, $\tau = \tau_0$, and $\sinh 2\phi = 0$. Thus

$$(2.43) \qquad h_n(0) \approx \sqrt{\pi/\xi}\,(1 + (-1)^n) \sum_{k=0}^{\infty} g_k \tau^{-2k}$$

and

$$(2.44) \qquad h_{2n+1}(0) \approx 0.$$

Although the behaviour at the other zeros is not quite so spectacular, we can get asymptotic information about these zeros from (2.42). Since $g_0 = g_0{}^0 = 1$, a first approximation must be given by solutions of

$$(2.45) \qquad \exp n(\sinh 2\phi - 2\phi) = -\sqrt{(\xi_0/\xi)} = i \exp \phi.$$

Approximate solutions for ϕ are not difficult to obtain. Olver has discussed in detail the procedure by means of which we can obtain asymptotic information about the zeros of h_n from formulae of the type (2.42).

3. Asymptotic formulae in terms of Airy functions. In order to complete our set of formulae, we must investigate the third case for which $\phi \to 0$. If we define

$$\rho^3 = \tfrac{1}{2}n(3e^{-2\phi} - 1) = \tfrac{1}{2}n\{(3e^{-2\alpha}\cos 2\beta - 1) - 3e^{-2\alpha}\, i \sin 2\beta\}, \tag{3.1}$$

$$\begin{aligned} M &= \sum_{k=4}^{\infty} [(1 + e^{-2\phi}(1 - 2^{k-1}))i^k n\theta^k]/k! \\ &= \sum_{k=1}^{\infty} [(1 + e^{-2\phi}(1 - 2^{k+2}))i^{k+3} n\theta^{k+3}]/(k+3)!, \end{aligned} \tag{3.2}$$

then

$$nF = -\xi\theta^2 + i\rho^3\theta^3/3 + M. \tag{3.3}$$

As $\phi \to 0$, $\arg \rho^3 \to 0$. Therefore we have three choices for ρ. In one choice $\arg \rho \to 0$, in another $\arg \rho \to 2\pi/3$, and in the third $\arg \rho \to -2\pi/3$.

We make the first choice and take ρ^{-k} as our asymptotic sequence. As $\phi \to 0$, $\rho \sim n^{1/3}$ and

$$\overline{\lim_{n \to \infty}}\, \rho = \infty.$$

Thus ρ^{-k} is a suitable sequence.

We have already shown that

$$\operatorname{Re}(nF) \leqslant -4n \sin^2(\tfrac{1}{2}\theta) \sin^2(\tfrac{1}{2}\theta - \beta), \qquad \text{uniformly in } \alpha. \tag{3.4}$$

Instead of a variable ϵ, we shall now require ϵ to be fixed. Since $\beta \to 0$,

$$\operatorname{Re}(nF) \leqslant -\gamma n, \qquad \text{for some } \gamma > 0, \text{ and } |\theta| \geqslant \epsilon. \tag{3.5}$$

In the range $|\theta| \geqslant \epsilon$, $\exp(nF) \approx 0$.

When $|\theta| \leqslant \epsilon$,

$$nF = -\xi\theta^2 + i\rho^3\frac{\theta^3}{3} - \frac{(7e^{-2\phi} - 1)n\theta^4}{4!} + O(n\theta^5). \tag{3.6}$$

If $0 \leqslant \arg \xi < \tfrac{1}{2}\pi - \Delta$, $\Delta > 0$, it is possible to find a fixed $\delta > 0$ such that

$$\arg(nF) \leqslant -\gamma n, \qquad \text{for some } \gamma > 0, \tag{3.7}$$

on the sections of the circle $|\theta| = \epsilon$, for which $0 \leqslant \arg \theta \leqslant \delta$, and $\pi - \delta \leqslant \arg \theta \leqslant \pi$. Since $\arg \delta$ is bounded away from $\pi/2$, a sufficiently small δ can be chosen so that $-\operatorname{Re}(\xi\theta^2) \leqslant 0$. Further we are in a region for which $\phi \to 0$ and $\xi = 0$ is not excluded. Thus this inequality is best possible for the first term of (3.6). Since $\operatorname{Re}(i\rho^3\theta^3/3) \geqslant 0$, our proof consists of showing that the desired result is obtained from the relative behaviour of the cubic and quartic terms. This behaviour is easily established.

This implies that

$$h_n \approx \int_{-\epsilon}^{\epsilon} \exp(nF)d\theta \approx \int_L \exp(nF)d\theta, \tag{3.8}$$

where L is the path of integration shown in Figure 3.

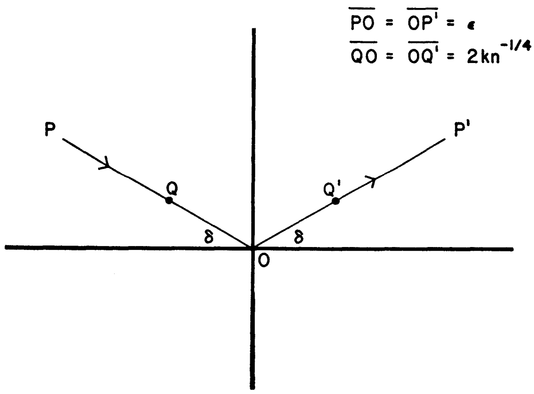

FIGURE 3

Along OP', $\theta = re^{i\delta}$ and $\mathrm{Re}(i\rho^3\theta^3/3) \to -\frac{1}{3}n \sin \delta r^3$. This is sufficient to prove that

$$\int_{Q'}^{P'} \exp(nF)d\theta \approx 0$$

and a similar result holds for

$$\int_{P}^{Q} \exp(nF)d\theta.$$

This enables us to shorten our path of integration to QOQ'.

The function M of (3.2) can be written as

$$M = \sum_{k=1}^{\infty} a_k u^{k+3} w^k, \tag{3.9}$$

where

$$u = \rho\theta, \quad w = \rho^{-1}, \quad a_k = 2[1 + e^{-2\phi}(1 - 2^{k+2})]i^{k+3}/(k+3)!(3e^{-2\phi} - 1). \tag{3.10}$$

Expanding $\exp M$ in a Maclaurin series in w, we have

$$\exp M = \sum_{k=1}^{\infty} M_k w^k, \tag{3.11}$$

where $M_k = M_k(\phi, u)$ is a polynomial in u. Following the same pattern used in our first case, we find that

$$\begin{aligned} h_n &= \frac{2\pi}{\rho}\left[\sum_{k=0}^{N} m_k \rho^{-k} + O(1/\rho^N)\right], \qquad \text{uniformly in } \phi \ (\phi \to 0), \\ &\sim \frac{2\pi}{\rho}\left[\sum_{k=0}^{\infty} m_k \rho^{-k}\right], \end{aligned} \tag{3.12}$$

where

$$m_k = \frac{\rho}{2\pi}\int_{-\infty e^{-i\delta}}^{\infty e^{i\delta}} M_k(\phi, \rho\theta) \exp\left(-\xi\theta^2 + i\frac{\rho^3\theta^3}{3}\right) d\theta. \tag{3.13}$$

In particular

$$(3.14) \qquad m_0 = [\exp(2\xi^3/3\rho^6)]A_i(\xi^2/\rho^4),$$

where $A_i(z)$ is the Airy function. It is possible to show that m_k is a linear combination of m_0 and the derivative of m_0 with respect to the variable ξ/ρ^2.

A discussion of the possibility $\frac{1}{2}\pi - \Delta \leqslant \arg \xi \leqslant \frac{1}{2}\pi$, $\phi \to 0$, will then yield our definitive set of formulae. In this range, the zeros of $H_n(z)$ again exert their influence and we must take into account the second critical point. Beyond the indication of a suitable path of integration, no other modifications need be made. This path is indicated in Figure 4.

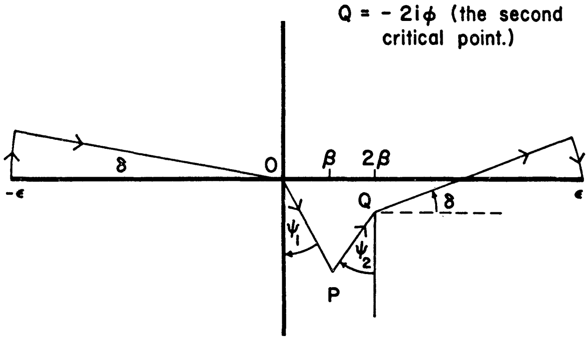

FIGURE 4

It is necessary to ensure that OP and QP are greater than or equal to $2kn^{-1/4}$. Since $\phi \to 0$, we must allow ψ_1 and ψ_2 to be variable angles that approach 0. In fact for $\beta = 0$, $\psi_1 = \psi_2 = 0$. We then obtain

$$(3.15) \qquad h_n \sim \frac{2\pi}{\rho} \sum_{k=0}^{\infty} [\mu_k + \exp n(\sinh 2\phi - 2\phi)\mu_k^0(\rho/\rho_0)^k]\rho^{-k},$$

where

$$(3.16) \qquad \mu_k = \frac{\rho}{2\pi} \int_{-\infty e^{-i\delta}}^{-i\infty e^{i\psi}} M_k(\phi, \rho\theta) \exp\left(-\xi\theta^2 + \frac{i\rho^3\theta^3}{3}\right) d\theta,$$

$$(3.17) \qquad \rho_0 = \rho(-\phi), \qquad \xi_0 = \xi(-\phi),$$

$$(3.18) \qquad u_k^0 = \frac{\rho}{2\pi} \int_{-i\infty e^{-\psi}}^{\infty e^{i\delta}} M_k(-\phi, \rho_0\theta) \exp\left(-\xi_0\theta^2 + \frac{i\rho_0^3\theta^3}{3}\right) d\theta.$$

In particular

$$(3.19) \qquad \mu_0 = e^{-i\pi/3} \exp(2\xi^3/3\rho^6)A_i(e^{2\pi i/3}\xi^2/\rho^4),$$

$$(3.20) \qquad \mu_0^0 = e^{i\pi/3} \exp(2\xi_0^3/3\rho_0^6)A_i(e^{-2\pi i/3}\xi_0^2/\rho_0^4).$$

Since $H_n(z) = H_n(\sqrt{2n}\cosh\phi) = A_n h_n$, we have a definitive set, consisting of four formulae, that give the asymptotic behaviour of $H_n(z)$ as $n \to \infty$, when z is restricted to lie in the first quadrant and may or may not

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tend to infinity. The behaviour in the other four quadrants follows from $H_n(z) = (-1)^n H_n(-z) = H_n(\bar{z})$.

If we return to our first two cases, it is clear that we could have used the technique of the third case by retaining the cubic term and expanding only the function M. This would have led to formulae for these cases expressed in terms of Airy functions. It is true, but far from obvious, that we can reduce the number of formulae to two. This would have been an accomplishment if this procedure had reduced the work involved in obtaining our four formulae and at the same time allowed us to derive our formulae as special cases. Unfortunately we have not found this to be the case.

4. Discussion of the formulae. For purposes of discussion, we list the members of our definitive set of formulae, their domains of validity, and the type of expansion. The notation is that used in the previous sections.

$$(4.1)\qquad H_n(z) = H_n(\sqrt{2n}\cosh\phi) \sim \frac{n!(2/n)^{\frac{1}{2}n}\exp n(\phi + 1 + \frac{1}{2}e^{-2\phi})}{2\sqrt{(\pi\xi)}}\left[1 + \sum_{k=1}^{\infty} g_k\tau^{-2k}\right].$$

Type of expansion. (4.1) is a uniform expansion of Poincaré type.

Domain of validity. For some λ, $0 < \lambda \leqslant 1/3$, $n^{\lambda} = O(n^{1/3}(1 - e^{-2\alpha}))$.

The result contained in (4.1) is sufficient to determine the complete asymptotic behaviour of $H_n(z)$ except in a domain that in the ϕ plane is rectangular in shape and infinitesimally close to the β axis. The width of the exceptional region approaches 0 as $n \to \infty$.

$$(4.2)\qquad H_n(z) \sim \frac{n!(2/n)^{\frac{1}{2}n}\exp[n(\phi + 1 + \frac{1}{2}e^{-2\phi})]}{2\sqrt{(\pi\xi)}} \times\left[\sum_{k=0}^{\infty}\left\{g_k + \sqrt{\frac{\xi}{\xi_0}}\, g_k{}^0\left(\frac{\tau}{\tau_0}\right)^{2k}\exp[n(\sinh 2\phi - 2\phi)]\right\}\tau^{-2k}\right].$$

Type of expansion. (4.2) is a uniform expansion of Erdélyi type.

Domain of validity. Let λ be an arbitrarily small, fixed, positive number and suppose $\beta_0 = n^{-\frac{1}{3}+\lambda}$. The domain of validity of (4.2) is $\alpha = o(\beta_0)$, and $\beta_0 = O(\beta^2)$. This domain will now include all but an infinitesimal part of the β axis, and the dimensions of the exceptional region approach 0 as $n \to \infty$.

$$(4.3)\qquad H_n(z) \sim \frac{n!(2/n)^{\frac{1}{2}n}\exp[n(\phi + 1 + \frac{1}{2}e^{-2\phi})]}{\rho}\left[\sum_{k=0}^{\infty} m_k\rho^{-k}\right].$$

Type of expansion. (4.3) is a uniform expansion of Poincaré type.

Domain of validity. $\phi \to 0$, $0 \leqslant \arg\phi \leqslant \frac{1}{2}\pi - \Delta$.

$$(4.4)\quad H_n(z) \sim \frac{n!(2/n)^{\frac{1}{2}n}\exp[n(\phi + 1 + \frac{1}{2}e^{-2\phi})]}{\rho} \times\left[\sum_{k=0}^{\infty}\left\{\mu_k + \exp[n(\sinh 2\phi - 2\phi)]\mu_k{}^0\left(\frac{\rho}{\rho_0}\right)^{k+1}\right\}\rho^{-k}\right].$$

Type of expansion. (4.4) is a uniform expansion of Erdélyi type.

Domain of validity. Let Δ be an arbitrarily small, fixed, positive number. The domain of validity of (4.4) is $\phi \to 0$, $\frac{1}{2}\pi - \Delta \leqslant \arg \phi \leqslant \frac{1}{2}\pi$.

We would like to point out that the domains of validity of these formulae are arbitrary in the sense that each of the formulae retains its validity in domains that contain the given domains as proper subsets. To illustrate, let us consider

$$(4.5) \qquad \begin{aligned} \operatorname{Re}(\sinh 2\phi - 2\phi) &= \sinh 2\alpha \cos 2\beta - 2\alpha \\ &= -4\alpha \sin^2 \beta + O(\alpha^3), \qquad \text{as } \alpha \to 0 \\ &= -4\alpha \sin^2 \beta(1 + o(1)) \qquad \text{(in the stated domain of validity of (4.2)).} \end{aligned}$$

If γ is any fixed number in $0 < \gamma < 1$, then $\exp[2n(\sinh 2\phi - 2\phi)] \approx 0$ providing $\alpha \sin^2 \beta \geqslant K/n^\gamma$, for some $K > 0$. Thus (4.2) will reduce to (4.1) and we obtain an extension to the stated domain of validity of (4.1). Although we attempt no proof, we shall give a reason why we believe that the union of the domains of validity of (4.1) and (4.2) is characterized by the simple condition that $\tau \to \infty$.

Let us consider (4.3). We have already noted that $\rho \sim n^{1/3}$ as $n \to \infty$. In order to compare (4.1) and (4.3), we first compare the two asymptotic sequences involved. If $1/\rho$ is not asymptotically equal to zero with respect to the asymptotic sequence $\{\tau^{-2k}\}$, then for some fixed integer m, we must have $|\tau|^{2m}/\rho$ bounded away from zero. In the stated domain of validity of (4.3) ($\phi \to 0$, $0 \leqslant \arg \phi \leqslant \frac{1}{2}\pi - \Delta$), the condition that ρ^{-1} is not ≈ 0; $\{\tau^{-2k}\}$, implies that $|\phi| \geqslant Kn^{(-3+m)/9}$, for some $K > 0$. This latter condition, coupled with $0 \leqslant \arg \phi \leqslant \frac{1}{2}\pi - \Delta$, implies that $|\alpha| \geqslant K_1 n^{(-3+m)/9}$, for some $K_1 > 0$. This means that we are in the domain of validity of (4.1). If we are satisfied with the asymptotic information that can be obtained from the asymptotic sequence $\{\tau^{-2k}\}$, then we need only consider (4.3) when $1/\rho \approx 0$. This would of course imply a simplification in that this condition would allow us to write, instead of (4.3),

$$(4.6) \qquad h_n \approx \frac{2\pi m_0}{\rho} = \frac{2\pi}{\rho} [\exp(2\xi^3/3\rho^6) A_i(\xi^2/\rho^4)]$$

and, in this sense, we can write

$$(4.7) \qquad H_n(z) \approx \frac{n!(2/n)^{\frac{1}{2}n} \exp[n(\phi + 1 + \frac{1}{2}e^{-2\phi})]}{\rho} [\exp(2\xi^3/3\rho^6) A_i(\xi^2/\rho^4)].$$

Since $\xi^2/\rho^4 \to \infty$ as $\tau \to \infty$ and $0 \leqslant \arg(\xi^2/\rho^4) \leqslant \pi - \Delta$, we can replace $A_i(\xi^2/\rho^4)$ by its known asymptotic expansion. In particular

$$(4.8) \qquad H_n(z) = \frac{n!(2/n)^{\frac{1}{2}n} \exp[n(\phi + 1 + \frac{1}{2}e^{-2\phi})]}{2\sqrt{(\pi\xi)}} [1 + O(\tau^{-2})],$$

which is identical with the result we obtain from (4.1). Similar results can be obtained by using (4.2) and (4.4). This is the reason for our belief that the union of the domains of validity of (4.1) and (4.2) is completely characterized by the one condition, $\tau \to \infty$. We shall return to these results after a comparison is made between our formulae and those obtained by Olver.

In using his technique, Olver finds it convenient to introduce and retain the single asymptotic variable $\mu = \sqrt{(2n+1)}$. His expansions are of Poincaré type or linear combinations of expansions of Poincaré type in which the asymptotic sequence consists of powers of μ^{-2}. One of the formulae he obtains is

$$(4.9) \qquad H_n(\mu t) \sim \frac{2^{\frac{1}{2}n}[\exp \mu^2(\frac{1}{2}t^2 - \xi)]g(\mu)}{(t^2-1)^{1/4}} \sum_{s=0}^{\infty} \frac{A_s(\xi)}{\mu^{2s}},$$

where $g(\mu)$ and $A_s(\xi)$ are well-defined functions of their arguments and

$$(4.10) \qquad \xi = \tfrac{1}{2}t(t^2-1)^{\frac{1}{2}} - \tfrac{1}{2}\ln\{t + (t^2-1)^{\frac{1}{2}}\}.$$

One of the conditions used to determine the validity of (4.9) is $|t \pm 1| \geqslant \delta > 0$. Formula (4.9) compares with our formula (4.1). They are, however, completely different. In order to obtain (4.1) from (4.9) we would have to make the substitutions

$$(4.11) \quad t = \left(1 + \frac{1}{2n}\right)^{-\frac{1}{2}} \cosh \phi, \qquad \mu = \sqrt{2n+1} = \sqrt{2n}\left(1 + \frac{1}{2n}\right)^{\frac{1}{2}}.$$

In addition, re-expansion of the individual expressions and a regrouping of the terms that arise would be involved. A similar analysis would be involved in the converse problem of obtaining (4.9) from our formula (4.1).

For our asymptotic variable τ, we have

$$(4.12) \qquad \tau^2 = \xi^3/n^2 = \tfrac{1}{2}n(1 - e^{-2\phi})^3.$$

Hence

$$(4.13) \qquad \tau^2/\mu^2 = n(1 - e^{-2\phi})^3/2(2n+1).$$

Since $\alpha \geqslant 0$, τ^2/μ^2 would be bounded away from both zero and infinity, when ϕ is bounded away from zero. This restriction corresponds to the one required of (4.9), $|t \pm 1| \geqslant \delta$.

Even though formulae (4.1) and (4.9) are completely different, the important asymptotic features of both are essentially the same in their common domain of validity. Since both are members of definitive sets of formulae, it would seem somewhat pointless to attempt to derive one formula from the other.

Although our set of formulae is equivalent to the set obtained by Olver, the individual members of our set are derived by a different procedure and are based on a different point of view. We illustrate in the following way. The starting point of the problem was the discussion of the asymptotics of $H_n(z)$ as $n \to \infty$ with z unrestricted. The technique developed by Olver finds it convenient to introduce two new functions, μ and t, by

$$(4.14) \qquad \mu = \sqrt{(2n+1)}, \qquad t = z/\mu.$$

This implies that

$$\lim_{n\to\infty} \mu = \infty$$

and that $\{\mu^{-k}\}$ is an asymptotic sequence. Olver retains the same asymptotic sequence throughout his set of formulae and at every step his error terms are of the form $O(\mu^{-k})$ uniformly in t. The a priori demand that we obtain asymptotic information containing this amount of detail imposes certain restrictions on the domains of validity of the individual formulae of the set.

On the other hand, we have found it convenient to introduce two functions, τ and ϕ, by

$$\tau = \sqrt{n}\,[\tfrac{1}{2}(1 - e^{-2\phi})]^{3/2}, \qquad \cosh\phi = z/\sqrt{2n}. \tag{4.15}$$

The function τ does not have the property

$$\lim_{n\to\infty} \tau = \infty$$

if we allow ϕ to be restricted. In order for τ to possess this property, we need not restrict ϕ to be bounded away from zero. It is sufficient to require

$$\lim_{n\to\infty} n^{1/3}\,|\phi| = \infty.$$

With this condition, $\{\tau^{-k}\}$ is an asymptotic sequence, and formulae (4.1) and (4.2) are derived by means of this particular choice of asymptotic sequence. Our error terms at each stage are of the form $O(\tau^{-s})$ and are uniform in ϕ. When ϕ is bounded away from zero, the asymptotic variables μ and τ become equivalent and no distinction in the asymptotic information given by (4.1) and (4.9) is apparent. However, (4.1) seems to have a domain of validity greater than that of (4.9). The only reason for this extension is the fact that we have designed our formulae to be capable of giving asymptotic information containing a smaller amount of detail. Thus, for example, if $\phi = n^{-33/100}$, $\tau \sim n^{1/100}$ and the estimate of error in (4.8) is $O(n^{-1/50})$.

Since Olver's set of formulae is definitive, we should point out that there is no important asymptotic result that we can obtain from any one of our formulae that he would not obtain by, at worst, a combination of two of his formulae. The converse is also true. For this reason, we do not believe that the type of extension in domains of validity that are possible is of any significant importance. We do believe, however, that the point of view that we have illustrated is important. Many of the general theorems of asymptotics, such as Watson's lemma and Olver's work on the asymptotics of the solutions of differential equations, have been designed to apply to classes of functions, which we call the class of admissible functions, for which a detail of asymptotic information exists that is exemplified by the choice of $\mu = \sqrt{(2n+1)}$ as the asymptotic variable. The a priori demand of asymptotic information of this type places a severe restriction on the class of admissible functions that it is difficult to justify. In some of the applications of asymptotics, especially those of a theoretical nature, formulae that have uniform error terms of the type $o(1)$ are sufficient. It is for this reason that we believe that many of the classical theorems of asymptotics should be re-examined. The basic assumptions of these theorems should be weakened to the extent that they yield asymptotic information that is in keeping with some of the more modern definitions of an asymptotic expansion.

5. Conclusion. Our purpose in writing the present paper has been twofold. It is our desire to introduce an integral representation procedure that yields asymptotic formulae that are similar in nature to recent results obtained by a study of differential equation techniques. The procedure, which we have outlined in some detail, is capable of generalization and a paper is in progress showing that the procedure will apply to a very extensive class of admissible functions. An examination of known work makes it seem unlikely that a single formula exists that will satisfactorily describe the complete asymptotic behaviour of functions of several complex variables. This behaviour will usually be obtained by a set of formulae whose individual members need not be unique. The loss of uniqueness makes it seem desirable to abandon the rigid form required by the Poincaré type of expansion. Among the infinity of possible generalized type expansions, there may well exist, for certain functions, some formulae that are better than others for certain specific purposes.

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