Fourth Meeting, February 10th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On the inequality
$$mx^{m-1}(x-1) \gtreqless x^m - 1 \gtreqless m(x-1)$$
and its consequences.

By PROFESSOR CHRYSTAL.

§ 1. The object of this note is to establish the above inequality in as general a form as possible, and to prove by means of it two of the principal propositions in the theory of inequalities, one of which is usually proved by means of infinite series. The logical advantage in making the theory of inequalities independent of that of infinite series is obvious, when it is remarked that the discussion of the convergency of infinite series is strictly speaking a part of the theory of inequalities.

§ 2. *If x, p, q, are all positive, and p and q are integers, then $(x^p - 1)/p > < (x^q - 1)/q$ according as $p > < q$.*

Since p and q are integers
$$(x^p - 1)/p > < (x^q - 1)/q,$$
as
$$q(x^p - 1) > < p(x^q - 1),$$
as
$$(x-1)\{q(x^{p-1} + x^{p-2} + \dots + x + 1) - p(x^{q-1} + x^{q-2} + \dots + x + 1)\} > < 0.$$

Suppose $p > q$; and denote the expression on the left side of the last inequality by X. Then
$$X = (x-1)\{q(x^{p-1} + x^{p-2} + \dots + x^q) - (p-q)(x^{q-1} + x^{q-2} + \dots + x + 1)\}.$$

Now, if $x > 1$, $\quad x^{p-1} + x^{p-2} + \dots + x^q > (p-q)x^q$,

and $\quad x^{q-1} + x^{q-2} + \dots + 1 < qx^{q-1}$;

and therefore
$$\begin{aligned} X &> (x-1)\{q(p-q)x^q - (p-q)qx^{q-1}\} \\ &> q(p-q)x^{q-1}(x-1)^2 \\ &> 0. \end{aligned}$$

Again, if $x < 1$,
$$\begin{aligned} X &> (x-1)\{q(p-q)x^{p-1} - (p-q)q\} \\ &> q(p-q)(x-1)(x^{p-1} - 1) \\ &> 0. \end{aligned}$$

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Hence, in both cases, $(x^p-1)/p > (x^q-1)/q$.

By the same reasoning, if $q>p$,

$$(x^q-1)/q > (x^p-1)/p\,;$$

that is, if $p<q$, $\quad (x^p-1)/p < (x^q-1)/q$.

§ 3. If x is positive, and different from unity, then

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1)$$

unless m lie between 0 and 1, in which case

$$mx^{m-1}(x-1) < x^m - 1 < m(x-1).$$

From last paragraph we have

$$(\xi^p - 1) >< (p/q)(\xi^q - 1) \tag{1}$$

according as $p><q$; where ξ is any positive quantity, different from unity, and p and q are positive integers. In (1) we may put $x^{1/p}$ for ξ, where x is any positive quantity different from unity, the real positive value of the q^{th} root being taken; and we may put m for p/q, where m is any positive commensurable quantity.

The inequality then becomes

$$x^m - 1 >< m(x-1) \tag{2}$$

according as $m><1$; which is part of the theorem.

In (2) put $1/x$ for x, then

$$(1/x)^m - 1 >< m(1/x - 1)$$

or $\quad mx^{m-1}(x-1) >< x^m - 1. \quad (3)$

according as $\quad m><1.$

The theorem is thus established for positive values of m.

Next, let $m=-n$, then

$$x^{-n} - 1 >< (-n)(x-1),$$

according as $\quad 1 - x^n >< -nx^n(x-1),$

according as $\quad x^n - 1 >< nx^n(x-1),$

according as $\quad nx^{n+1} - nx^n >< x^n - 1,$

according as $\quad (n+1)x^n(x-1) >< x^{n+1} - 1.$

Now, since n is positive, $n+1>1$; therefore by (3)

$$(n+1)x^n(x-1) > x^{n+1} - 1\,;$$

and therefore $\quad x^{-n} - 1 > (-n)(x-1). \quad (4)$

In (4) write $1/x$ for x, then

$$(1/x)^{-n} - 1 > (-n)(1/x - 1)$$

therefore $\quad x^{-n} - 1 < (-n)x^{-n-1}(x-1)$

that is, $\quad (-n)x^{-n-1}(x-1) > x^{-n} - 1.$

Hence, if m is negative,

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1)\,;$$

which completes the demonstration.

§ 4. *The arithmetic mean of n positive quantities is not less than the geometric mean.*

Let us suppose the theorem to hold for n quantities a, b, c, $\ldots\ldots k$; and let l be one more.

By hypothesis, $(a+b+c+\ \ldots\ldots\ +k)/n \nless (abc\ \ldots\ldots\ k)^{1/n}$

that is, $\quad a+b+c+\ \ldots\ldots\ +k \nless n(abc\ \ldots\ldots\ k)^{1/n}$

Therefore $\quad a+b+c+\ \ldots\ldots\ +k+l \nless n(abc\ \ldots\ldots\ k)^{1/n}+l.$

Now, $\quad n(abc\ \ldots\ldots\ k)^{1/n}+l \nless (n+1)(abc\ \ldots\ldots\ kl)^{1/(n+1)},$

provided $\quad n(abc\ \ldots\ldots\ k/l^n)^{1/n}+1 \nless (n+1)(abc\ \ldots\ldots\ kl/l^{n+1})^{1/(n+1)},$

$$\nless (n+1)(abc\ \ldots\ldots\ k/l^n)^{1/(n+1)};$$

that is, provided $\quad n\xi^{n+1}+1 \nless (n+1)\xi^n,$

where $\quad \xi^{n(n+1)} = abc\ \ldots\ldots\ k/l^n,$

that is, provided $\quad (n+1)\xi^n(\xi-1) \nless \xi^{n+1}-1,$

which is true, by the theorem of last paragraph.

Hence, if the theorem hold for n quantities, it will hold for $n+1$; and it is obviously true for two quantities, and hence it is true generally.

Corollary.—If a, b, $\ldots\ldots k$ are n positive quantities, and p, q, $\ldots\ldots t$, n positive commensurable quantities, then

$$\frac{pa+qb+\ \ldots\ldots\ +tk}{p+q+\ \ldots\ldots\ +t} > (a^pb^q\ \ldots\ldots\ k^t)^{1/(p+q+\ \ldots\ldots\ +t)}$$

It is obvious that we are only concerned with the ratios of the quantities p, q, $\ldots\ldots t$; and we may, therefore, suppose these quantities to be integral. The theorem is thus seen to be a particular case of that just proved—namely, that the arithmetic mean of $p+q+\ \ldots\ldots\ t$ positive quantities, of which p are equal to a, q to b, and so on, is greater than their geometric mean.

§ 5. *If a, b, $\ldots\ldots k$, are n positive quantities, and p, q, $\ldots\ldots t$, are n commensurable quantities, then*

$$\frac{pa^m+qb^m+\ \ldots\ldots\ +tk^m}{p+q+\ \ldots\ldots\ +t} \nless \ngtr \left(\frac{pa+qb+\ \ldots\ldots\ +tk}{p+q+\ \ldots\ldots\ +t}\right)^m \qquad (1)$$

according as m does not or does lie between 0 and 1.

If we denote $p(p+q+\ \ldots\ldots\ +t)$, $q(p+q+\ \ldots\ldots\ +t)$, etc., by λ, μ, $\ldots\ldots \tau$; and $a/(\lambda a+\mu b+\ \ldots\ldots\ +\tau k)$, $b/(\lambda a+\mu b+\ \ldots\ldots\ +\tau k)$, etc., by x, y, $\ldots\ldots w$, then

$$\lambda+\mu+\ \ldots\ldots\ +\tau=1$$

$$\lambda x+\mu y+\ \ldots\ldots\ +\tau w=1.$$

Dividing both sides of (1) by $\{(pa+qb+\ \ldots\ tk)/(p+q+\ \ldots\ +t)\}^m$ we have to prove that

$$\lambda x^m + \mu y^m + \ \dots\ \dots\ + \tau w^m \nless \ngtr 1$$

according as m does not or does lie between 0 and + 1.

Now, if m does not lie between 0 and 1,

$$x^m - 1 \nless m(x-1),\ y^m - 1 \nless m(y-1), \text{ etc.}$$

Therefore
$$\begin{aligned}\Sigma\lambda(x^m - 1) &\nless \Sigma\lambda m(x-1)\\ &\nless m\{\Sigma\lambda x - \lambda\}\\ &\nless m(1-1)\\ &\nless 0;\end{aligned}$$

that is,
$$\Sigma\lambda x^m \nless \Sigma\lambda \nless 1.$$

In like we may show that if m does lie between 0 and 1, then

$$\Sigma\lambda x^m \ngtr 1.$$

Corollary.—If we make $p = q = \ \dots\ \dots\ = t$, we have

$$(a^m + b^m + \ \dots\ \dots\ + k^m)/n \nless \ngtr \{(a + b + \ \dots\ \dots\ + k)/n\}^m,$$

that is to say, *the arithmetical mean of the m^{th} powers of n positive quantities is not less or not greater than the m^{th} power of their arithmetical mean, according as m does not or does lie between* 0 *and* 1.

§ 6. The inequality just discussed is by no means new, nor has its importance been overlooked, as may be seen from the elegant use of it by Schlömilch in the second chapter of his *Algebraische Analysis.* (See also *Zeitschrift für Mathematik*, Bd. III., p. 387 (1858), and Bd. VII., p. 46 (1862); also G. F. Walker, *Messenger of Mathematics*, vol. XII, p. 37). The inequality has not, however, usually been stated in quite so general a form as the one I have given; and, possibly in consequence, its application to the demonstration of the theorem of § 5 seems to have hitherto escaped notice. This theorem is usually proved by a somewhat awkward combination of induction and the use of infinite series.

The history of the theorem is a little obscure. At first I suspected that it was due to Canchy, but it does not appear in his *Analyse Algébrique* (Paris, 1821). The earliest reference to it which I have discovered was given me by Mr A. Y. Fraser, and occurs in *Problèmes et Dévelopmens sur diverses Parties des Mathématiques*, par M. Reynaud et M. Duhamel. It is there deduced from the maximum and minimum values of $x^m + y^m$ subject to the condition $ax + by = c$. I can scarcely believe that this is the earliest occurrence, and I should be glad if any of our members could furnish me with reference to an earlier.

§ 7. The inequality $mx^{m-1}(x-1) \gtreqless x^m - 1 \gtreqless m(x-1)$ has the merit of binding together a great variety of algebraical theorems which are usually put before the student without any organic connection whatever; and for this reason I have brought it specially under the notice of the younger members of the Mathematical Society. Its power is not surprising when we reflect on its close connection with the theorem $\underset{x=1}{\mathrm{L}}(x^m - 1)/(x - 1) = m$, which is the fundamental proposition in the differentiation of algebraic functions.

Mr W. PEDDIE exhibited and described a model of the thermodynamic surface which represents the state of water-substance in terms of pressure, volume, and temperature. Various lines, the equations of which are $\frac{dp}{dt} = \text{const.}$, $\frac{dp}{dv} = \text{const.}$, &c., were drawn upon the surface.

Fifth Meeting, March 9th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

Sur un système de cercles tangents à une circonférence et orthogonaux à une autre circonférence.

Par M. PAUL AUBERT.

On donne deux cercles S et Σ ayant pour centres les points O et ω, pour rayons r et ρ. Le cercle S est supposé intérieur au cercle Σ, et le point ω intérieur au cercle S.

I. *Tous les cercles T tangents extérieurement au cercle S et orthogonaux au cercle Σ sont tangents à un troisième cercle fixe.*

Figures 18, 19.

Soit T un cercle tangent au cercle S et orthogonal à Σ. Prenons la figure inverse par rapport au point I comme pôle, la puissance d'inversion étant la puissance k^2 du point I par rapport au cercle S. Ce cercle reste invariable, et le cercle Σ se transforme en une droite perpendiculaire au diamètre Iω en un point P′ tel que

$$\mathrm{IP}\cdot\mathrm{IP}' = k^2.$$

Le cercle T se transforme en un cercle T′ tangent au cercle S et coupant à angle droit la droite P′D; son centre est donc sur cette droite, et par suite le cercle T′ est aussi tangent à la circonférence S_1