JOINT SPECTRA OF OPERATORS ON BANACH SPACE

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(Received 23 January, 1985)

Let X be a complex Banach space. We denote by $B(X)$ the algebra of all bounded linear operators on X. Let $\hat{T}=(T_1,\ldots,T_n)$ be a commuting n-tuple of operators on X. And let $\sigma_T(\hat{T})$ and $\sigma''(\hat{T})$ by *Taylor's joint spectrum* and the *doubly commutant spectrum* of $\hat{T}$, respectively. We refer the reader to Taylor [**8**] for the definition of $\sigma_T(\hat{T})$ and $\sigma''(\hat{T})$. A point $z=(z_1,\ldots,z_n)$ of $\mathbb{C}^n$ is in the *joint approximate point spectrum* $\sigma_\pi(\hat{T})$ of $\hat{T}$ if there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\|(T_i-z_i)x_k\|\to 0 \quad \text{as} \quad k\to\infty \quad \text{for} \quad i=1,2,\ldots,n.$$

A point $z=(z_1,\ldots,z_n)$ of $\mathbb{C}^n$ is in the *joint approximate defect spectrum* $\sigma_\delta(\hat{T})$ of $\hat{T}$ if there exists a sequence $\{f_k\}$ of norm one functionals in X^* (dual space of X) such that

$$\|(T_i-z_i)^*f_k\|\to 0 \quad \text{as} \quad k\to\infty \quad \text{for} \quad i=1,2,\ldots,n.$$

A point $z=(z_1,\ldots,z_n)$ of $\mathbb{C}^n$ is said to be a *joint eigenvalue* of $\hat{T}$ if there exists a non-zero vector x such that

$$T_ix=z_ix \quad \text{for} \quad i=1,2,\ldots,n.$$

It is well known that $\sigma_\pi(\hat{T})\cup\sigma_\delta(\hat{T})\subset\sigma_T(\hat{T})\subset\sigma''(\hat{T})$.

We denote by Π the subset of the Cartesian product $X\times X^*$ defined by

$$\Pi=\{(x,f):\|f\|=f(x)=\|x\|=1\}.$$

The *joint numerical range* $V(\hat{T})$ of $\hat{T}=(T_1,\ldots,T_n)$ is defined by

$$V(\hat{T})=\{(f(T_1x),\ldots,f(T_nx)):(x,f)\in\Pi\}.$$

Let $S\in B(X)$ and A be a commutative Banach subalgebra containing S. The usual spectrum of S, the spectrum of S in A and (spatial) numerical range of S are denoted by $\sigma(S)$, $\sigma_A(S)$ and $V(S)$, respectively. We refer the reader to Bonsall and Duncan [**1**].

The *joint operator norm*, *joint spectral radius* and *joint numerical radius* of $\hat{T}=(T_1,\ldots,T_n)$, denoted by $\|\hat{T}\|$, $r(\hat{T})$ and $v(\hat{T})$ respectively, are defined by

$$\|\hat{T}\|=\sup\left\{\left(\sum_{i=1}^{n}\|T_ix\|^2\right)^{1/2}:\|x\|=1\right\},$$

$$r(\hat{T})=\sup\left\{\left(\sum_{i=1}^{n}|z_i|^2\right)^{1/2}:(z_1,\ldots,z_n)\in\sigma_T(\hat{T})\right\}$$

and

$$v(\hat{T})=\sup\left\{\left(\sum_{i=1}^{n}|f(T_ix)|^2\right)^{1/2}:(x,f)\in\Pi\right\},$$

respectively.

Given E, let co E and $\bar{E}$ denote the convex hull and the closure of E, respectively. For $(x,f)\in\Pi$, a functional f_x in $B(X)^*$ is defined by

$$f_x(S)=f(Sx) \quad \text{for} \quad S\in B(X).$$

THEOREM 1. *Let A be a commutative Banach subalgebra of $B(X)$. Then $\Phi_A \subset$ w^*-cl co$\{f_x:(x,f)\in\Pi\}$, where Φ_A is the carrier space of A.*

We shall need the following two facts.

THEOREM A (Crabb [5]). *Let $S\in B(X)$. Then* co $\sigma(S)\subset\overline{V(S)}$.

THEOREM B (Dekker [6]). *Let $S\in B(X)$ and A be a commutative Banach subalgebra containing S. Then* co $\sigma_A(S)=$ co $\sigma(S)$.

Proof of Theorem 1. Let $\phi\in\Phi_A$. We assume that

$$\phi\notin w^*\text{-cl co}\{f_x:(x,f)\in\Pi\}.$$

By the separation theorem for convex set, this implies the existence of $S\in A$ such that

$$\sup_{(x,f)\in\Pi}\operatorname{Re} f(Sx)<\operatorname{Re}\phi(S).$$

Hence $\phi(S)\notin\overline{V(S)}$. On the other hand $\phi(S)\in\sigma_A(S)\subset\text{co }\sigma_A(S)=\text{co }\sigma(S)\subset\overline{V(S)}$, by Theorem B and Theorem A.

This yields a contradiction. So the proof is complete.

This fact yields the following result.

THEOREM 2. *Let $\hat{T}=(T_1,\dots,T_n)$ be a commuting n-tuple of operators on X. Then* $\sigma_T(\hat{T})\subset\sigma''(\hat{T})\subset\overline{\text{co}}\,V(\hat{T})$.

COROLLARY 3. *Let $\hat{T}=(T_1,\dots,T_n)$ be a commuting n-tuple of operators on X. Then* $r(\hat{T})\leqq v(\hat{T})\leqq\|\hat{T}\|$.

A Banach space X will be said to be uniformly convex if to each ϵ, $0\leqq\epsilon\leqq 2$, there corresponds a $\delta>0$ such that the conditions

$$\|x\|=\|y\|=1, \qquad \|x-y\|\geqq\epsilon$$

imply

$$\left\|\frac{x+y}{2}\right\|\leqq 1-\delta.$$

THEOREM C (Theorem 1 in Clarkson [4]). *The Cartesian product of finitely many uniformly convex Banach spaces can be given a uniformly convex norm.*

THEOREM 4. *Let X be uniformly convex and $\hat{T}=(T_1,\dots,T_n)$ be a commuting n-tuple of operators on X. Then*

$$\{z\in\overline{V(\hat{T})}:|z|=\|\hat{T}\|\}\subset\sigma_\pi(\hat{T}).$$

https://doi.org/10.1017/S0017089500006352 Published online by Cambridge University Press

Proof. Let $z \in \overline{V(\hat{T})}$ and $|z| = \|\hat{T}\|$. We may assume that $|z| = \|\hat{T}\| = 1$. Then there exist $(x_k, f_k) \in \Pi$ such that

$$(f_k(T_1x_k), \ldots, f_k(T_nx_k)) \to z \quad \text{as} \quad k \to \infty.$$

Since

$$\sum_{i=1}^{n} |\tfrac{1}{2}f_k(z_ix_k + T_ix_k)|^2 \to 1 \quad \text{as} \quad k \to \infty$$

and

$$1 \geqq \left(\sum_{i=1}^{n} \|\tfrac{1}{2}(z_ix_k + T_ix_k)\|^2\right)^{1/2} \geqq \left(\sum_{i=1}^{n} |f(\tfrac{1}{2}(z_ix_k + T_ix_k))|^2\right)^{1/2},$$

it follows that $\left(\sum_{i=1}^{n} \|z_ix_k + T_ix_k\|^2\right)^{1/2} \to 2$ as $k \to \infty$. So by Theorem C, we have

$$\left(\sum_{i=1}^{n} \|(z_i - T_i)x_k\|^2\right)^{1/2} \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, $z \in \sigma_\pi(T)$. So the proof is complete.

COROLLARY 5. *If X is uniformly convex and $v(\hat{T}) = \|\hat{T}\|$, then $r(\hat{T}) = \|\hat{T}\|$.*

A Banach space X is said to be *strictly convex* if and only if x and y are linearly dependent whenever

$$\|x + y\| = \|x\| + \|y\|.$$

LEMMA 6. *Let X be a strictly convex Banach space. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be vectors in $X \times \ldots \times X$. Then the relation*

$$\left(\sum_{i=1}^{n} \|x_i + y_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} + \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{1/2}$$

implies that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are linearly dependent.

Proof. The relation

$$\left(\sum_{i=1}^{n} \|x_i + y_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} + \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{1/2}$$

implies that

$$\left(\sum_{i=1}^{n-1} \|x_i + y_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{n-1} \|x_i\|^2\right)^{1/2} + \left(\sum_{i=1}^{n-1} \|y_i\|^2\right)^{1/2}$$

by Hölder's inequality. So it is easy to verify by induction.

THEOREM 7. *Let X be a strictly convex Banach space, and let $\hat{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on X. Let $z = (z_1, \ldots, z_n) \in V(\hat{T})$ and let $|z| = \|\hat{T}\|$. Then z is a joint eigenvalue of $\hat{T}$.*

Proof. We may assume that $\|\hat{T}\| = |z| = 1$. Then there exists $(x, f) \in \Pi$ such that $(f(T_1x), \ldots, f(T_nx)) = z$. Therefore,

$$2 \geqq \left(\sum_{i=1}^{n} \|z_ix\|^2\right)^{1/2} + \left(\sum_{i=1}^{n} \|T_ix\|^2\right)^{1/2} \geqq \left(\sum_{i=1}^{n} \|T_ix + z_ix\|^2\right)^{1/2}$$
$$\geqq \left(\sum_{i=1}^{n} |f(T_ix) + z_if(x)|^2\right)^{1/2} = 2.$$

This implies that

$$\left(\sum_{i=1}^{n} \|T_ix + z_ix\|^2\right)^{1/2} = \left(\sum_{i=1}^{n} \|T_ix\|^2\right)^{1/2} + \left(\sum_{i=1}^{n} \|z_ix\|^2\right)^{1/2}$$

and so $(T_1x, \ldots, T_nx)$ and $(z_1x, \ldots, z_nx)$ are linearly dependent. It is easy to show that

$$T_ix = z_ix \quad \text{for} \quad i = 1, 2, \ldots, n.$$

So the proof is complete.

Problem. Let $\hat{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on X. Is it then true that $\sigma_T(\hat{T}) \subset \overline{V(\hat{T})}$?

It is easy to verify that $\sigma_\pi(\hat{T}) \cup \sigma_\delta(\hat{T}) \subset \overline{V(\hat{T})}$.

Acknowledgement. We would like to express our thanks to the referee for his kind advice.

REFERENCES

1. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras* (Cambridge, 1971).

2. F. F. Bonsall and J. Duncan, *Numerical ranges II* (Cambridge, 1973).

3. M. Chō and M. Takaguchi, Some classes of commuting n-tuple of operators, *Studia Math.*, to appear.

4. J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 396–414.

5. M. J. Crabb, *The numerical range of an operator*, Ph.D. thesis, University of Edinburgh, 1969.

6. N. P. Dekker, *Joint numerical range and joint spectrum of Hilbert space operators*, Ph.D. thesis, University of Amsterdam, 1969.

7. G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.

8. J. L. Taylor, A joint spectrum for several commuting operators, *J. Functional Analysis* **6** (1970), 172–191.

9. J. P. Williams, Spectra of product and numerical ranges, *J. Math. Anal. Appl.* **17** (1967), 214–220.

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