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DIGRAPHS WITH EULERIAN CHAINS

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Abstract

An *eulerian chain* in a directed graph is a continuous directed route which traces every arc of the *digraph* exactly once. Such a route may be finite or infinite, and may have 0, 1 or 2 end vertices. For each kind of eulerian chain, there is a characterization of those digraphs possessing such a route. In this survey paper we strealine these characterizations, and then synthesize them into a single description of all digraphs having some eulerian chain. Similar work has been done for eulerian chains in undirected graphs, so we are able to compare corresponding results for graphs and digraphs.

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1. Introduction

An *eulerian chain* in an undirected graph is a continuous route which traces every edge exactly once; such a route in a digraph is a *eulerian chain*, provided it is directed. For both graphs and digraphs, there are four different kinds of eulerian chain, namely *eulerian trail*, *eulerian circuit*, *eulerian one-way chain*, and *eulerian two-way chain*. Definitions of these are given in Eggleton and Skilton [1]. The subject of eulerian chains in graphs is discussed in [1], while this paper is concerned with eulerian chains in digraphs, and is a sequel to [1].

The characterizations of those digraphs which have an eulerian trail, and those digraphs having an eulerian circuit, are both well known. Specfications of the digraphs with an eulerian one-way chain or eulerian two-way chain also exist, and are due to Nash-Williams [5]. In this paper we streamline these four characterizations, and then synthesize them into a single description of all digraphs which have an eulerian chain, without explicit separation into cases dependent on the nature of that chain. We finish with a brief discussion of the parallel between these characterizations and their counterparts for undirected graphs.

Throughout this paper we use the terminology introduced in [1], together with the following notions. If V is a set of vertices in an undirected graph G, the cutset ∂V is the set of edges of G which have precisely one vertex in V. A cutset is *odd* or *even* according as its cardinality is *odd* or *even*. In a digraph D, the *cutset* ∂V is the set of arcs of D with precisely one vertex in V. Moreover, we denote by $d^+(V)$ the cardinality of the set of arcs in ∂V which have their tail in V, while $d^-(V)$ denotes the number of arcs in ∂V which have their head in V. Whenever $|\partial V|$ is finite, we define the *flux* of V by $f(V) := d^+(V) - d^-(V)$. This is also called the flux of the cutset ∂V. We define the *flux-magnitude sum* of D by $\Sigma_V |f(V)|$, where V is singleton. That is, the flux-magnitude sum is the sum of the magnitudes of the fluxes at the finite vertices of D.

2. Characterizing eulerian digraphs

The characterizations of those digraphs with an eulerian trail, or with an eulerian circuit, may be stated as follows.

THEOREM 1. *A connected finite digraph has an eulerian circuit precisely if it has flux-magnitude sum* 0, *and has an eulerian trail precisely when it has flux-magnitude sum* 2.

An eulerian one-way chain in an undirected graph always has an initial vertex. However, an eulerian one-way chain in a digraph may not have an initial vertex, but rather a final vertex. Nash-Williams [5] identified necessary and sufficient conditions for a digraph to have an eulerian one-way chain with prescribed initial vertex. The corresponding conditions for when the final vertex is prescribed follow easily from this characterization, as Nash-Williams pointed out. It is not difficult to obtain from these conditions a single description of all digraphs having an eulerian one-way chain. Before giving such a characterization, we introduce the following definition. A digraph D is *balanced* if for each set V of vertices in D, the terms $d^+(V)$ and $d^-(V)$ are either both finite or both infinite. Clearly, if a digraph has an eulerian one-way chain it must be balanced, as well as connected and countable.

THEOREM 2 (Nash-Williams). *A connected, countable, balanced digraph has an eulerian one-way chain precisely if it has cofinite rank* 1, *has flux-magnitude sum at most* 1, *and has at least one infinite vertex if it has flux-magnitude sum* 0.

If a digraph has an eulerian two-way chain, then it clearly is connected, countable and balanced, and has flux magnitude sum 0. Such a digraph will be called *quasi-eulerian*, by analogy with the definition of quasi-eulerian graphs in [1]. Nash-Williams [5] identified those digraphs having an eulerian two-way chain, and his characterization may be stated as follows.

THEOREM 3 (Nash-Williams). *A quasi-eulerian digraph has an eulerian two-way chain precisely if it has cofinite rank at most* 2, *and has a cutset with flux* 1 *if its cofinite rank equals* 2.

Theorems 1, 2 and 3 can easily be amalgamated to yield the following specification of all eulerian digraphs.

THEOREM 4. *Let D be a connected, countable, balanced digraph with flux-magnitude sum n and cofinite rank r. Then D has an eulerian chain if and only if $n + r \leqslant 2$ and, whenever $r = 2$, there is a cutset with flux* 1 *in D.*

3. Comparison of eulerian graphs and digraphs

The given characterizations of those digraphs having a finite eulerian chain, or an eulerian one-way chain, are close parallels of their counterparts for undirected graphs, as can be seen by comparing Theorems 1 and 2 of this paper with Theorems 5 and 6 of [1]. Of particular note is the analogy between flux-magnitude sume of a digraph and the number of odd vertices of a graph. On the other hand, there is only a weaker similarity between the original eulerian two-way chain characterization for graphs, due to Erdös, Grünwald and Vázsonyi [2], and Nash-Williams' eulerian two-way chain characterization for digraphs. (These results are stated in Theorem 3 of this paper and Theorem 7 of [1].) Nash-Williams [5] recognized this poor correspondence, and accordingly derived a modified version of the eulerian two-way chain characterization for graphs which more closely resembled its digraph counterpart. He obtained this modified version using the following lemma.

LEMMA (Nash-Williams). *Let G be a quasi-eulerian graph ω with cofinite rank* 2. *Then G has even cofinite rank* 1 *precisely if it has an odd cutset.*

Nash-Williams did not explicitly state this result, and moreover, he left its proof as an exercise for his readers. We now give a suitable argument.

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PROOF. First note that if V is a set of finite vertices in a graph G, then the cutset ∂V is finite and has the same property as $\sum_{v \in V} d(v)$, where $d(v)$ is the degree in G of the vertex v. This is an easy result, and to our knowledge first appears in Nash-Williams [4]. In the particular case when G is quasi-eulerian, this result implies that if ∂V is odd then $\sum_{v \in V} d(v)$ must be infinite, as must be the sum of the degrees of the vertices not in V.

Let G be a quasi-eulerian graph with cofinite rank 2. If G has an odd cutset ∂V then its even cofinite rank is 1. For suppose there is an even subgraphy H of G whose edge deletion $G \setminus EH$ has at least two infinite components. Since H is even and finite it has no odd cutset, and so contains an even number of the edges in ∂V. Hence, some subgraph F of $G \setminus EH$ has an odd number of these edges. Let S be the set of edges of ∂V lying in F. Then S is an odd cutset of F. Moreover, F is quasi-eulerian since H is even, and so the deletion from F of the edges in S yields at least two infinite components. Since F is just one of the infinite components in $G \setminus EH$, it follows that G has cofinite rank greater than 2, which is a contradiction. Now suppose that G has no odd cutset. Nash-Williams [3] has shown that the edges of such a graph can be partitioned into cycles, and it follows from this that G has even cofinite rank at least 2. For if H is a finite subgraph of G, then H is contained in a finite union of edge disjoint cycles, so is contained in an even subgraph H' of G. Moreover, if $G \setminus EH$ has two infinite components then so does $G \setminus EH'$, and so the even cofinite rank of G is at least 2.

This completes the proof.

Using the lemma, Nash-Williams [5] modified the original eulerian two-way chain characterization for graphs to obtain the following version.

THEOREM 5. *A quasi-eulerian graph has an eulerian two-way chain precisely if it has cofinite rank* 2, *and has an odd cutset if its cofinite rank equals* 2.

There is a clear parallel between the above theorem and Theorem 3, its counterpart for digraphs. It is of interest to determine whether Theorem 3 can be modified to yield a close parallel to the original eulerian two-way chain characterization for graphs. The digraph in Figure 1 shows that this is essentially not possible.

Using Theorem 5 we can modify Theorem 1 of [1], the theorem which describes all eulerian graphs. This new version appears below. It clearly bears a strong resemblance to Theorem 4 of this paper, which specifies all eulerian digraphs.

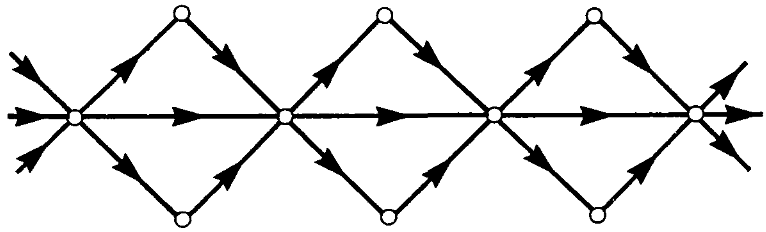

Figure 1. A quasi-eulerian digraph which has cofinite rank 2 and even cofinite rank 1, but has no eulerian two-way chain.

THEOREM 6. *Let G be a connected countable graph with n odd vertices and cofinite rank r. Then G has an eulerian chain if and only if* $n + r \leqslant 2$ *and, whenever* $r = 2$, *there is an odd cutset in G.*

We note that Nash-Williams [5] derived yet another version of the eulerian two-way chain characterization for both graphs and digraphs. These new characterizations were closely parallel, but involved fairly technical properties of directed and undirected graphs.

REMARK. The even cofinite rank of a digraph D is the maximum number of infinite components obtainable by edge deleting finitely many edge disjoint cycles from the underlying graph. Define the *directed* even cofinite rank of D to be the maximum number of infinite components resulting from the deletion of finitely many arc disjoint directed cycles in D. Arguing as in the proof of the lemma, it is possible to show that *if D is quasi-eulerian and has cofinite rank* 2, *then D has directed even cofinite rank* 1 *precisely if it has a cutset with non-zero flux.*

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