REFERENCE FRAMES AND GRAVITATIONAL EFFECTS IN THE GENERAL THEORY OF RELATIVITY

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ABSTRACT. In connection with the precision growth in modern astrometry, General Relativity (GR) has become a necessary basis of celestial mechanics and of computations of ephemerides of planets and other space objects, from the point of view of both the laws of motion and the expression of measurable (observable) quantities in terms of Reference Frames (RFs) notions. In this report we summarize the RF theory in GR (in the monad as well as in the tetrad representation) and give examples of specific gravitational effects. In any theory of RFs an important element consists on separation of the RF and system-of-coordinates (SC) notions, with a constructive description of projectors onto the physical temporal and spatial directions, of the observables as invariants (scalars) under SC transformations, as well as of the transformation laws of these observables under transitions between different RFs. For a more detailed though still incomplete synopsis of the monad and tetrad methods of RF representation in GR, see our preprint [1].

1. INTRODUCTION

In the principal monographs and textbooks on GR [2 - 5] (which are used also by astronomers in computation of ephemerides for determination of the relativistic corrections: see, e.g., [6]), there is no complete enough formulation of the RF theory, while its notions are entering

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only occasionally and often inconsistently. The monad approach allows to choose a special class of SCs in which a given RF may be described in terms of the more familiar formalism of Chronometric Invariants (CI, see [7] and also [8 - 11]) usually considered as being more closely relevant to celestial mechanics. Many authors (including ourselves) developed the RF theory in GR in its general form as well as its special representations. It was stressed by Lichnerowicz [10] that a physical interpretation of effects of GR is possible only with the help of RFs based on the 4-vector of the observer's velocity being a part of the tetrad depending on the world point; such an approach eliminates also a great many of difficulties of the former theory. In the modern theory of RFs, ambiguities in the definition of every specific observable are removed, and limits of applicability of these methods are determined together with the interconnections between them. We give below some examples of transition to observables from the objects referred to SCs, in the tetrad and monad approaches (Table I).

2. FUNDAMENTALS OF THE REFERENCE FRAMES THEORY

The "minimal" representation of a RF consists of assigning a time-like congruence in the given region of the space-time manifold (with a certain metric tensor of the Lorentzian signature, + - - -, so that a pseudo-Riemannian manifold is meant). The field of unit 4-vectors (4-velocities), the monad $\tau^{\mu} = e_{(0)}{}^{\mu}$ (cf. the statement by Lichnerowicz above), corresponds to the lines of this congruence representing the motion of test particles of the reference body (observers and/or measuring devices). We use 4-dimensional Greek indices running from 0 to 3, and 3-dimensional Latin ones, while indices in parentheses are local (Lorentz) indices referred to the RF. Since the 4-dimensional metric, $g_{\mu\nu}$, is considered as preassigned, the monad determines also the projector onto the local physical 3-space of the RF orthogonal to the congruence, this projector ($b_{\mu\nu}$) being at the same time the metric tensor on this submanifold.

The manifold of GR is at any regular point tangent to a flat vector space of the same Lorentzian signature; that means that one can always introduce locally the Galilean metric of Special Relativity (SR) which corresponds to the local basis (tetrad) with the time-like vector coinciding with the monad (see Eq. (1) in Table I).

One can start the construction of RF theory (as in the case of non-relativistic mechanics) with introduction of a triad $e_{(i)}{}^{\mu}$ prior to the congruence (see Eq. (8)ff).

TABLE I (description of reference frames)

Basis vectors:
in Classical Mechanics i, j, k ($\underline{e}_{(k)} = \underline{e}_x, \underline{e}_y, \underline{e}_z$)
in Special Relativity $\underline{e}_{(0)}, \underline{e}_{(k)}$
in General Relativity, the same local basis with properties

$$e_{(\lambda)}{}^{\mu} e_{(\sigma)}{}^{\nu} g_{\mu\nu} = \eta_{(\lambda)(\sigma)} := \mathrm{diag}\,(1,-1,-1,-1) \qquad (1)$$

$$g_{\mu\nu} = e^{(0)}{}_{\mu} e_{(0)\nu} + e^{(k)}{}_{\mu} e_{(k)\nu} \qquad (2)$$

Separation of RF and SC; specialization of RFs (projectors)

for monad approach	for tetrad (triad) approach
$g_{\mu\nu}\tau^{\mu}\tau^{\nu} = 1$ (3)	$e_{(0)\alpha} = (g_{0\alpha} - e^{(k)}{}_{\alpha} e_{(k)0})(g_{00} - e^{(k)}{}_{0} e_{(k)0})^{-1/2}$ (8)
Projector (3-metric) $b_{\mu\nu} = \tau_{\mu}\tau_{\nu} - g_{\mu\nu}$ (4)	$b_{\mu\nu} = - e^{(k)}{}_{\mu} e_{(k)\nu} = e^{(0)}{}_{\mu} e_{(0)\nu} - g_{\mu\nu}$ (9)

Quantities depending on RF in monad and tetrad approaches

1) Monad-dependent and triad-(tetrad)-dependent SC-scalars

$d\tau = \tau_{\alpha}\, dx^{\alpha}$ (5)	$dx^{(0)} = e^{(0)}{}_{\alpha} dx^{\alpha} = (g_{0\alpha} dx^{\alpha} - e_{(k)0} dx^{(k)})(g_{00} - e^{(k)}{}_{0} e_{(k)0})^{-1/2}$ (10)
Energy density: $w = \tau_{\mu}\tau_{\nu} T^{\mu\nu}$	$w = e_{(0)}{}^{\alpha} e_{(0)}{}^{\beta} T_{\alpha\beta}$

2) Observables dependent on SC-scalar quadratic forms

$dl^2 = b_{\mu\nu}\, dx^{\mu} dx^{\nu}$ (5a)	see also (9): $dl^2 = \eta_{(k)(N} dx^{(k)} dx^{(n)}$ (10a)
$\cos\Phi = b_{\mu\nu} p^{\mu} q^{\nu} (b_{\alpha\beta} \cdot p^{\alpha} p^{\beta} b_{\sigma\tau} q^{\sigma} q^{\tau})^{-1/2}$ (6)	$\cos\Phi = p_{(k)} q^{(k)} \cdot (p_{(m)} p^{(m)} q_{(n)} q^{(n)})^{-1/2}$ (11)

3) Monad- and triad-(tetrad)-dependent SC-non-scalar quantities and SC-scalars depending on triad rotation

$d\tilde{x}^{\mu} = b^{\mu}_{\nu}\, dx^{\nu}$,	$dx^{(k)} = e^{(k)}{}_{\nu}\, dx^{\nu}$,
Poynting vector: $S^{\mu} = \tau_{\nu} T^{\lambda\nu} b^{\mu}_{\lambda}$, ...(7)	$S^{(k)} = e_{(0)}{}^{\mu} e^{(k)\nu} T_{\mu\nu}$, ... (12) $T_{(k)(l)} = e_{(k)}{}^{\mu} e_{(l)}{}^{\nu} T_{\mu\nu}$

INTERPRETATION:
Measurable Quantities (Observables) Must Be SC-Scalars!

(continued on the next page)

TABLE I (continued)

CI Formalism: SCs are co-moving with the RF	
$\tau^{\alpha} = \delta^{\alpha}_{0}/(g_{00})^{1/2} = \begin{cases} 0, & \alpha = k \\ (g_{00})^{-1/2}, & \alpha = 0 \end{cases}$ (13)	$e^{(k)}{}_{0} = 0,\ e^{(0)}{}_{\alpha} = g_{0\alpha}/(g_{00})^{1/2}$ (16)
$\tau_{\alpha} = g_{0\alpha}/(g_{00})^{1/2}$	$dx^{(k)} = e^{(k)}{}_{n} dx^{n}$
$x^{0'} = x^{0'}(x^{0}, x^{1}, x^{2}, x^{3}),$ $x^{k'} = x^{k'}(x^{1}, x^{2}, x^{3}),$ $\partial x^{k'}/\partial x^{0} = 0$ (14)	$(b_{\alpha\beta})_{e^{(k)}{}_{0}=0} = \begin{cases} b_{0\beta} = 0 \\ b_{kn}, \end{cases}$ (17) $b_{kn} = -e^{(m)}{}_{k} e_{(m)n} = g_{0k}g_{0n}/g_{00} - g_{kn}$
$d\tau = g_{0\alpha} dx^{\alpha}/(g_{00})^{1/2}$ (15)	As to other CI quantities and their representation, it is easy to build them by combining preceding formulae, e.g. for $\cos\Phi$ see (6), (11) & (17).

The tetrad method introduces locally in GR the orthonormal basis of SR, together with its interpretation which involves a certain measurement method: the expansion of any vector, $\underline{A} = A_{(\alpha)} e^{(\alpha)} = \text{inv.}$, bears similarity to the "fundamental equation of measurement" of the theoretical metrology. In refs. [1, 8, 11 - 13], one can find discussion of the methods of representation of RFs, their classification (of a single observer, co-moving, Killing, geodesic, etc), and construction of CI quantities. In the theory of RFs, there are formulated both the algebraic formalism and the differential one, consisting of differential operations and quantities which describe evolution of RF in terms analogous to those of hydrodynamics (acceleration, rotation, rate of strain), since a RF may be considered as an idealization of a continuous medium.

3. PHYSICAL OBSERVABLES

One of the basic assumptions of the RF theory, is that of the invariance of physical observables (<u>all measurable quantities have to be scalars</u>). Under transitions between different RFs, the observables undergo transformations (if any). It is easy to see that the RF transformations reduce to the local Lorentz transformations [12, 14]. In general, the RF transformations have nothing in common with the SC transformations, but if some special convention is imposed (e.g. in the CI Formalism or when $e^{(k)}{}_{0} = 0$), the corresponding connection between coordinate and local Lorentz transformations emerges [1, 8, 14 - 16](cf. Table II). Those ob-

TABLE II (RF transformations while SC remains fixed)

for monad approach	for tetrad (triad) approach
$\tau'_{\mu} = (\tau_{\mu} - b_{\mu\nu}v^{\nu})(1 - v^2/c^2)^{-1/2}$ (18) $v = dl/d\tau$	$e_{(k)'}{}^{\mu} = L_{(k)'}{}^{(\nu)} e_{(\nu)}{}^{\mu}$ (21) $e^{(0)'}{}_{\alpha} = (g_{0\alpha} - e^{(k)'}{}_{\alpha} e_{(k)'0})(g_{00} - e^{(n)'}{}_{0} e_{(n)'0})^{-1/2} = L^{(0)'}{}_{(\nu)} e^{(\nu)}{}_{\alpha}$ (22)
$b'_{\mu\nu} = \tau'_{\mu}\tau'_{\nu} - g_{\mu\nu}$ (19)	$b'_{\mu\nu} = - e^{(k)'}{}_{\mu} e_{(k)'\nu}$
$d\tau' = \tau'_{\alpha} dx^{\alpha}$ (20)	$dx^{(0)'} = L^{(0)'}{}_{(\alpha)} e^{(\alpha)}{}_{\beta} dx^{\beta}$ (23)

For RF transformation in the CI Formalism combine these formulae with (13) - (17) of Table I

servables which are related to a world point and not to the system as a whole, do depend on coordinates of that point, which are determined by the adopted arithmetization of the space-time. This does not lead to any contradiction, since the property of the observables to be invariants (scalars) under SC transformations, should not be confused with the constancy (which is not the case) of these quantities as functions of the world point. When the SC is changed, the form of these expressions undergoes a change, but the corresponding numerical values at a fixed world point, must coincide when the coordinates of that point are properly substituted. Another feature of observables is their dependence on configuration of the physical system under consideration (e.g. on locations of planets in the Solar system at the given epoch). This reflects in fact the many-particle nature of the problem, and though its solutions depend on the coordinates of individual particles, the numerical predictions as well as observables in general, do depend not on the choice of SC, but on the corresponding world points which are invariant geometrical objects. Thus, if one calculates observables using scalar expressions found from the GR equations as functions of coordinates, one has to know the latters. If the observables are related to celestial objects, one has to know their positions. In the relativistic astrometry, the construction of a RF requires in practice taking into account effects of GR both for motion of the objects and for propagation of electromagnetic waves (light) used in observations. As a result, such calculations of observables relative to a given RF together with a choice of the most suitable SC, are sophisticated self-consistent problems, though the numerical values of all observables (constructed in accordance with the RF theory of GR) cannot change whichever SCs were admitted in these calculations.

4. EXAMPLES OF GRAVITATIONAL EFFECTS

As illustrative examples of GR observables and some relations of the RF theory, we consider now some gravitational effects predicted in GR. In 4.1 the RF is represented by the time coordinate congruence (two of them are in fact considered); in 4.2 the Killing vector orbits are used as the monad congruence, and a convenient choice of the invariant time parameter is made; in 4.3 solutions of the geodesic equation are taken, the integrals of motion, E and H, playing the role of parameters of the congruence. The obtained results depend on coordinates, and the simplicity of corresponding expressions is due to the choice of SC, but this does not play any crucial role since observables are scalars under arbitrary changes of SCs, the RF being kept fixed.

4.1. Effects of oscillation of test particles on orbiting platforms

The motion of test particles in RFs orbiting about the Schwarzschild or Kerr centres, gives rise to an important class of problems requiring the use of RF theory. In particular, these are the problems of relative oscillations of free test particles (Shirokov's effect), oscillations of a torsion pendulum (Karpov's effect), of a linear oscillator (the Braginsky-Polnaryov effect) etc. To this class are related also the phenomena considered by Mashhoon and Theiss [17] (however, we express our doubts in their results).

These effects were revisited on the basis of the monad representation of the RF theory in the CI gauge [18] with the use of the Kerr metric in the Boyer-Lindquist coordinates. The CI RF taken in these coordinates is realized by a system of observers being at rest relative to the fixed stars. Now we perform the following three coordinate transformations: 1) passage to the RF in which the plane of the circular orbit does not rotate; 2) rotation of the spatial SC by the angle $\pi/2 - \theta_0$ corresponding to inclination of the orbital plane; 3) passage to the RF orbiting together with the platform. The CI RF related to the resulting SC, is the orbital RF. Its monad congruence is formed by the geodesic world line of the platform complemented by the world lines rigidly connected with it. In such a RF the angular velocity tensor represents the Coriolis acceleration. Calculations of the mentioned effects were done with the help of the geodesic equations in their monad form, the evolution parameter being τ (the observable time in this RF, see Table I). The oscillation periods (and frequencies) in different projections are directly measurable quantities.

We have got, e.g., for Shirokov's effect in the Kerr metric, oscillation frequencies in the plane perpendicular to the orbit and in the radial direction [11],

$$\omega_n = \pm c\Omega(1 - (-1)^n 3r_g(4r_0)^{-1} \mp (-1)^n 3a\Omega), \quad n = 1, 2; \tag{24}$$

$$\Delta t = 2\pi(c\Omega)^{-1}(3r_g(2r_0)^{-1} \mp 6a\Omega) \tag{25}$$

where $\Omega = r_g/(2r_0^3)^{1/2}$, r_0 being the radius of the platform orbit, +/- corresponding to the orbital motions in the direct or opposite sense to the rotation of the Kerr source. For the Braginsky-Polnaryov effect the frequency of a "free" oscillator was found in such a RF to be equal to [18]

$$\omega = k + \Omega^2 \cos^2\alpha - 3\Omega^2 \cos^2\beta \tag{26}$$

where k is the proper dynamical frequency of the oscillator, the cosines being the direction cosines of the spring orientations. For the torsion pendulum (Karpov's effect) the forced oscillations period is equal to the orbiting period, and the tidal accelerations are related to the Newtonian ones as $h = (r_g/r_0)(a/b)\cos\theta_0$ (in this RF), θ_0 being inclination of the orbit, a and b the angular momenta (of the field source and the orbital ones) per unit mass of the corresponding object.

4.2. Effect of the meeting point drift for two test particles in the Kerr field

The Kerr field admits two Killing vectors since it is both stationary and axisymmetric. In the Boyer-Lindquist coordinates they are $\xi = \partial_t$ and $\zeta = \partial_\varphi$. Hence there exists a unique privileged RF (the Killing RF), the Killing time t (cf. [13]) being holonom and differing from the standard monad time (which is nonholonom: the congruence rotates) by a scalar factor (the norm of ξ) along the RF congruence. Hence t may be considered as an observable (this is the case also for the angle φ). In terms of this Killing time we describe now the motion of two spinless test particles in the opposite directions along one and the same circular orbit in the equatorial plane in the Kerr field. We find that the siderial periods of revolution of these particles along their common orbit are equal to [19]

$$T_\pm = 2\pi\left[(r^3/(\gamma m))^{1/2} \pm L/(mc^2)\right] = T_N \pm \Delta T \tag{27}$$

where +/- means the same as in 4.1, m being the mass of the Kerr centre, L its proper angular momentum, γ the Newtonian gravitational constant, c the velocity of light. The remark-

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able fact is here that the component ΔT (additional to the "Newtonian" period T_N) does not depend on the orbital radius and gravitational constant (this result is an exact and not approximate one). The corresponding difference in cyclic frequencies of the orbital motion of test particles,

$$\omega_\pm = \dot{\varphi}_\pm = \pm \left[(r^3/(\gamma m))^{1/2} \pm L/(mc^2) \right]^{-1}, \qquad (28)$$

leads to a drift of the meeting point of these two test particles, being equal in the angular measure to

$$\Delta\varphi = 2\pi \Delta T/T_N \qquad (29)$$

per revolution. In Table III, we give estimates of the predicted effect [19] for different realistic rotating central masses where the orbital radius is assumed to be equal to $3^{1/3} = 1.44$ of that of the central body, R. We give, for a comparison, the estimates of the perihelion advance for a slightly elliptical orbit with the same large semi-axis about the given central mass. Perihelion advance and meeting-point drift are mutually independent and qualitatively different effects. In fact, our effect coincides with that proposed by Ciufolini at Austin, Texas (cf. the paper by B. Bertotti in this volume).

TABLE TABLE III

central body	effects, in arc sec/century drift	perihel. advance
Sun	600	10^6
Earth	4.4	670
Jupiter	280	10^4
Class B star, fast rotation	$7.6\ 10^4$	10^6
Neutron star rotating at the stability limit	93 rad/sec	750 rad/sec

4.3. Test particles delay effect in the Schwarzschild field as a result of desynchronization

The separation of RF and SC results in separation of synchronizations with respect to the coordinate and physical times. Synchronization of the coordinate time leads to a desynchronization of the RF time:

$$(dx^{(0)})_{dx^0 = 0} := dx^{(0)}_{desyn} = \tau_k dx^k = e^{(0)}{}_k dx^k. \qquad (30)$$

The measurable time element is here on the left-hand side;

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the synchronization condition $dx^0 = 0$ is convenient due to its unambiguity (dx^0 is a total differential). If the RF time is synchronized, then, analogously,

$$(dx^0)_{d\tau=0} := dx^0_{desyn} = -\tau_k dx^k/\tau_0. \tag{31}$$

The separation of synchronizations implies certain effects on test particles (and signals) delay [12]. If a test particle performs plane motion along a quasi-elliptic orbit in the Schwarzschild field in curvature coordinates ("standard SC"), dx^0_{desyn} takes the form (see [12], pp. 55 and 201)

$$dx^0_{desyn} = (a^2E/H - (EH)^{-1})r^2 d\varphi, \quad e^{(0)}{}_0 = E, \quad e^{(0)}{}_3 = -H \tag{32}$$

where E and H are integrals of motion, $a^2 = (1 - 2m/r)^{-1}$. In the limiting case when $E, H \to \infty$, $H/E = b$ (the impact parameter), dx^0_{desyn} represents the time lag interval for electromagnetic signals. For a test particle on a circular orbit, we get per one revolution $\Delta x^0_{desyn} = 2\pi(mr)^{1/2}$. If converted into the angular measure (with the help of $e^{(3)}{}_0$ or $d\varphi/dx^0$) this yields

$$\Delta\varphi_{desyn} = d\varphi/dx^0 \Delta x^0_{desyn} = e^{(3)}{}_0 \Delta x^0_{desyn} = m/r, \tag{33}$$

$d\varphi/dx^0 = H/(a^2Er)$, thus determining the "angular delay" $\Delta\varphi_{desyn}$ which is an observable.

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