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ISOMETRIES OF SPACES OF UNBOUNDED CONTINUOUS FUNCTIONS

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By the classical Banach-Stone Theorem any surjective isometry between Banach spaces of bounded continuous functions defined on compact sets is given by a homeomorphism of the domains. We prove that the same description applies to isometries of metric spaces of unbounded continuous functions defined on non compact topological spaces.

Let X, X' be topological spaces and let $A(X), A(X')$ be metric vector spaces of continuous functions on X and X', respectively. If $A(X)$, $A(X')$ consist of only bounded functions, for example if X, X' are compact, then, in most cases, we have a complete description of all surjective linear isometries $T : A(X) \to A(X')$. One may check [**2, 3, 4, 5, 6, 7, 10, 11, 12, 13**] for descriptions of isometries of spaces of bounded continuous scalar and vector valued functions, bounded analytic functions, absolutely continuous functions, bounded Lipschitz functions, differentiable functions, and many other classes of bounded functions. Usually (but not always, see for example [**1**] or [**9**]) any such an isometry is given by a homeomorphism $\varphi : X' \to X$ followed by a multiplication by a continuous scalar valued function κ; that is,

$$Tf = \kappa f \circ \varphi \text{ for all } f \in A(X).$$

Any map of the above form will be called canonical. The most classical result in this area is the Banach-Stone Theorem which states that any surjective isometry of the space of all *bounded* continuous functions on a compact set X equipped with the usual sup norm, onto the space of bounded continuous functions on a compact set X', is canonical. In this paper we discuss isometries of spaces of unbounded continuous functions.

Assume that X is a σ-compact topological space and let $X_1, X_2, \ldots$ be a sequence of compact subsets of X such that

$$X_1 \subsetneq X_2 \subsetneq \ldots \subsetneq X_n \subsetneq \ldots, \text{ and } \bigcup_{n=1}^{\infty} X_n = X.$$

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The two metrics most often considered on the space of unbounded continuous functions on X are:

$$\|f\| = \sum_{n=1}^{\infty} a_n \|f\|_n, \text{ and} \tag{1}$$

$$d(f,g) = \sum_{n=1}^{\infty} a_n \frac{\|f-g\|_n}{1+\|f-g\|_n}, \tag{2}$$

where $\|\cdot\|_n$ is the usual sup norm on the set X_n and $(a_n)_{n=1}^{\infty}$ is a fixed sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$; for example, $a_n = 1/2^n$. The formula (1) defines a Banach space norm; for fixed sequences (X_n) and (a_n) we shall denote this space by $\widetilde{C}(X)$. The space $\widetilde{C}(X)$ contains unbounded continuous functions of restricted rate of growth; it does not contain all the continuous functions. The second formula defines a complete metric on the space $C(X)$ of *all* the continuous functions on X. In this paper we prove that all surjective isometries of the spaces $\left(\widetilde{C}(X), \|\cdot\|\right)$ and $\left(C(X), d(\cdot,\cdot)\right)$ are canonical. The results are valid both in the real and in the complex case; however at some crucial points the arguments will be different in the two cases. We denote by $\mathbb{K}$ the set of scalars ($\mathbb{R}$ or $\mathbb{C}$), and by $\mathbb{K}_1$ the set of scalars of absolute value one.

Theorem 1. *Let $\left(\widetilde{C}(X), \|\cdot\|\right)$ be the Banach space of continuous functions on a σ-compact set X with the norm given by (1). Let $\left(\widetilde{C}(X'), \|\cdot\|\right)$ be the analogous space on X'. Assume that T is a linear isometry from $\widetilde{C}(X)$ onto $\widetilde{C}(X')$. Then there is a homeomorphism φ of X' onto X with $\varphi(X_n') = X_n$, for $n \in \mathbb{N}$, and a unimodular, continuous, scalar valued function κ on X' such that*

$$Tf = c \cdot \kappa \cdot f \circ \varphi \text{ for all } f \in \widetilde{C}(X),$$

where $c = \left(\sum_{n=1}^{\infty} a_n\right) / \left(\sum_{n=1}^{\infty} a_n'\right)$. Moreover, $a_n = c a_n'$, for $n \in \mathbb{N}$.

Proof of the Theorem

Definition 1: We call a nonempty subset Ω of $\widetilde{C}(X)$ a *peaking set* if

$$\|f_1 + \dots + f_p\| = \|f_1\| + \dots + \|f_p\| \text{ for all } f_1, \dots, f_p \in \Omega.$$

A subset of $\widetilde{C}(X)$ is called a *maximal peaking set* if it is a peaking set and it is not contained properly in any other peaking set.

The definition of a peaking set involves only the norm and the addition operation of the vector space. Since surjective isometries preserve both the metric and the linear structure they map peaking sets onto peaking sets and maximal peaking sets onto maximal peaking sets.

DEFINITION 2: For a given peaking set Ω we define the complement $\Omega^{\perp}$ of Ω by

$$\Omega^{\perp} = \{g \in \widetilde{C}(X) : \exists r > 0 \quad \exists f \in \Omega \quad \forall \alpha \in \mathbb{K}_1 \ \ \|f + r\alpha g\| = \|f\|\}.$$

Again, linear isometries map complements of peaking sets onto complements of peaking sets.

DEFINITION 3: For $x \in X$ we define the *index* of x by

$$\operatorname{ind}(x) = \min\{n : x \in X_n\}.$$

We shall say that $(x_n)_{n=1}^{\infty}$ is *an increasing sequence in* X if $\operatorname{ind}(x_1) = 1$, and for any $n \in \mathbb{N}$, we have

$$\operatorname{ind}(x_{n+1}) > \operatorname{ind}(x_n) \qquad \text{or} \qquad x_{n+1} = x_n,$$

and the set $\{x_n : n \in \mathbb{N}\}$ has no limit points. □

LEMMA 2. *A subset Ω of $\widetilde{C}(X)$ is a maximal peaking set if and only if there is an increasing sequence $(x_n)_{n=1}^{\infty}$ in X and a sequence $(\theta_n)_{n=1}^{\infty}$ of scalars of absolute value one such that*

$$\Omega = \Omega\big((x_n), (\theta_n)\big) \overset{df}{=} \{f \in \widetilde{C}(X) : \forall n \in \mathbb{N}, \ \ \|f\|_n = \theta_n f(x_n)\}.$$

PROOF: Assume $\Omega = \Omega\big((x_n), (\theta_n)\big)$. For any $f_1, \ldots, f_p \in \Omega$, we have

$$\begin{aligned} \|f_1 + \ldots + f_p\| &= \sum_{n=1}^{\infty} a_n \|f_1 + \ldots + f_p\|_n \\ &= \sum_{n=1}^{\infty} a_n \theta_n \big(f_1(x_n) + \ldots + f_p(x_n)\big) = \|f_1\| + \ldots + \|f_p\|, \end{aligned}$$

so Ω is a peaking set. Assume that Ω_0 is a strictly larger peaking set and let $f_0 \in \Omega_0 - \Omega$. Since $f_0 \notin \Omega$ then we have two cases:

1. There is a positive integer k such that $\|f_0\|_k > |f_0(x_k)|$, or
2. there is a positive integer k such that $\|f_0\|_k = |f_0(x_k)|$, but $\|f_0\|_k \neq \theta_k f_0(x_k)$.

Let (y_n) be a sequence of nonzero scalars such that for any n, $|y_n| = \theta_n y_n$, the series $\sum_{n=1}^{\infty} a_n |y_n|$ is convergent, and $y_n = y_{n+1}$ if $x_n = x_{n+1}$ and $|y_n| < |y_{n+1}|$, if $x_n \neq x_{n+1}$. In the first case we introduce a continuous function f on X such that

$$f(x) = 0 \text{ for any } x \in X_k \text{ with } |f_0(x)| = \|f_0\|_k, \text{ and}$$
$$\|f\|_n = \theta_n f(x_n) = \theta_n y_n, \text{ for any } n \in \mathbb{N}.$$

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To construct such a function f we first construct inductively a continuous function $\widetilde{f}$ on X: after it is defined on $\bigcup_{j=1}^{n} X_j$ we extend it to $\bigcup_{j=1}^{n} X_j \cup \{x_{n+1}\}$ by putting $\widetilde{f}(x_{n+1}) = y_{n+1}$, and then extend to $\bigcup_{j=1}^{n+1} X_j$ with the same sup norm [8]. Put

$$K = \left\{ x \in X_k : \left| f_0(x) \right| = \|f_0\|_k \right\}.$$

K is a closed set and does not contain x_k, nor any x_n for $n > k$; since (x_n) is an increasing sequence neither does it contain any x_n with $n < k$. Let g be a continuous function on X with norm one and such that $g = 0$ on K and $g = 1$ on $\{x_n : n \in \mathbb{N}\}$. Put $f = g\widetilde{f}$. The function f belongs to Ω; however $\|f + f_0\| < \|f\| + \|f_0\|$ since $\|f + f_0\|_k < \|f\|_k + \|f_0\|_k$, so Ω_0 is not a peaking set.

In the second case using the same method, and the same function $\widetilde{f}$, we construct a continuous function f on X such that

$$f(x) = 0 \quad \text{for any } x \in X_k \text{ with } \left| f_0(x) + \widetilde{f}(x) \right| = \|f_0\|_k + \left\| \widetilde{f} \right\|_k, \text{ and}$$
$$\|f\|_n = \theta_n f(x_n) = \theta_n y_n, \quad \text{for any } n \in \mathbb{N}.$$

Again $\|f + f_0\| < \|f\| + \|f_0\|$ since $\|f + f_0\|_k < \|f\|_k + \|f_0\|_k$, so Ω_0 is not a peaking set.

Assume now that Ω is a maximal peaking set. For any $f \in \widetilde{C}(X)$ put

$$M_f = \left\{ (x_n, \theta_n) \in \prod_{n=1}^{\infty} X_n \times \mathbb{K}_1 : \|f\|_n = \theta_n f(x_n) \right\}.$$

The sets M_f are compact and, since Ω is a peaking set the family $\{M_f : f \in \Omega\}$ has the finite intersection property, consequently

$$M = \left\{ (x_n, \theta_n) \in \prod_{n=1}^{\infty} X_n \times \mathbb{K}_1 : \|f\|_n = \theta_n f(x_n) \text{ for all } f \in \Omega \right\}$$

is not empty. Assume M contains two distinct points (x_n^1, θ_n^1) and (x_n^2, θ_n^2). The sets $\{x_n^i : n \in \mathbb{N}\}$, $i = 1, 2$ can not have a limit point so we may select $f \in \widetilde{C}(X)$ such that $(x_n^1, \theta_n^1) \in M_f$ and $(x_n^2, \theta_n^2) \notin M_f$. Then $\Omega \cup \{f\}$ is a peaking set contrary to our assumption that Ω is maximal. Hence M is a singleton, $M = \{(x_n, \theta_n)\}$, consequently (x_n) is increasing and Ω is contained in

$$\Omega((x_n), (\theta_n)) = \{ f \in \widetilde{C}(X) : \forall n \in \mathbb{N}, \ \|f\|_n = \theta_n f(x_n) \}.$$

Again from the maximality of Ω, since by the first part of the proof $\Omega((x_n), (\theta_n))$ is a peaking set, we have $\Omega = \Omega((x_n), (\theta_n))$. ∎

Lemma 3. *For any maximal peaking set $\Omega((x_n), (\theta_n))$ we have*

$$\bigcap_{n=1}^{\infty} \ker \delta_{x_n} \subset \Omega^{\perp}((x_n), (\theta_n)),$$

where the functional δ_x on $\widetilde{C}(X)$ is defined by $\delta_x(h) := h(x)$ for $h \in \widetilde{C}(X)$.

PROOF: Let $g \in \bigcap_{n=1}^{\infty} \ker \delta_{x_n}$. For any $n \in \mathbb{N}$ let U_n be an open neighbourhood of x_n and k_n a continuous function on X such that for all $n, m \in \mathbb{N}$

$$U_n \cap U_m = \emptyset \text{ if } x_n \neq x_m, \text{ and } 0 \leqslant k_n(x) \leqslant 1, \; k_n(x_n) = 1, \text{ and } k_n \equiv 0 \text{ on } X \setminus U_n.$$

Put

$$f(x) := \sum_{n=1}^{\infty} k_n(x)\overline{\theta_n}\Big(\|g\|_n - |g(x)|\Big), \text{ for } x \in X.$$

It is clear that $f \in \Omega\big((x_n), (\theta_n)\big)$. For each $n \in \mathbb{N}$ and $x \in X_n$ we have

$$k_n(x)\Big(\|g\|_n - |g(x)|\Big) \leqslant \|g\|_n - |g(x)|,$$

so $k_n(x)\Big(\|g\|_n - |g(x)|\Big) + |g(x)| \leqslant \|g\|_n$, hence

$$|f(x)| + |g(x)| \leqslant |f(x_n)| = \|f\|_n,$$

which proves that $g \in \Omega^{\perp}\big((x_n), (\theta_n)\big)$. ∎

LEMMA 4. *Assume $\mathbb{K} = \mathbb{C}$. Let f, g be elements of $\widetilde{C}(X)$ such that*

$$\exists r > 0 \quad \forall \alpha \in \mathbb{K}_1 \quad \|f + r\alpha g\| = \|f\|$$

and let (x_n) be an increasing sequence in X such that $\|f\|_n = |f(x_n)|$ for $n \in \mathbb{N}$. Then

$$g(x_n) = 0 \text{ for } n \in \mathbb{N}.$$

PROOF: We have

$$\|f + e^{2\pi i\theta}g\| = \sum_{n=1}^{\infty} a_n \|f + e^{2\pi i\theta}g\|_n \geqslant \sum_{n=1}^{\infty} a_n |f(x_n) + e^{2\pi i\theta}g(x_n)| \overset{df}{=} \xi(\theta). \tag{3}$$

By a simple computation one can check that for any nonzero scalar y we have

$$\int_0^1 |1 + e^{2\pi i\theta}y|\, d\theta > 1.$$

It follows that if the $g(x_n)$ are not all equal to zero then

$$\int_0^1 \xi(\alpha)\, d\alpha = \sum_{n=1}^{\infty} a_n \int_0^1 |f(x_n) + e^{2\pi i\alpha}g(x_n)| d\alpha > \sum_{n=1}^{\infty} a_n |f(x_n)| = \|f\|.$$

Hence $\xi(\theta_0) > \|f\|$ for some real number θ_0. From (3) $\|f + e^{2\pi i\theta_0}g\| > \|f\|$ which contradicts our assumption. ∎

From the last two lemmas it follows that in the complex case

$$\Omega\big((x_n), (\theta_n)\big) = \bigcap_{n=1}^{\infty} \ker \delta_{x_n}.$$

The above formula is not true in the real case in general. However, the next lemma shows that it remains true if (x_n) is constant.

Lemma 5. *Assume $\Omega = \Omega\big((x_n),(\theta_n)\big)$ is a maximal peaking set, and assume that the sequence (x_n) contains only one value, that is $x_n = x_1$ for every $n \in \mathbb{N}$. Then*

$$\Omega^{\perp}\big((x_n),(\theta_n)\big) = \ker \delta_{x_1}.$$

Proof: Suppose that $g \in \widetilde{C}(X)$ satisfies $g(x_1) \neq 0$. Then given any $f \in \Omega\big((x_n),(\theta_n)\big)$ we can find an arbitrarily small $\alpha \in \mathbb{K}$ such that

$$\big|f(x_1)\big| < \big|\alpha g(x_1) + f(x_1)\big|.$$

Hence

$$\|f\| = \sum_{n=1}^{\infty} a_n \|f\|_n < \sum_{n=1}^{\infty} a_n \|\alpha g + f\|_n = \|\alpha g + f\|,$$

so $g \notin \Omega^{\perp}\big((x_n),(\theta_n)\big)$. This implies that $\Omega^{\perp}\big((x_n),(\theta_n)\big) \subset \ker \delta_{x_1}$. The other inclusion follows from Lemma 3. ∎

Lemma 6. *Assume that $\mathbb{K} = \mathbb{R}$ and that $\Omega = \Omega\big((x_n),(\theta_n)\big)$ is a maximal peaking set. Then*

$$\Omega^{\perp}\big((x_n),(\theta_n)\big) = \ker\left(\sum_{n=1}^{\infty} a_n \theta_n \delta_{x_n}\right).$$

Proof: If the sequence (x_n) is constant the lemma follows from the previous one so we may assume that (x_n) is not constant. For each $n \in \mathbb{N}$, let U_n be an open neighbourhood of x_n such that $U_n \cap U_m = \emptyset$ if $x_n \neq x_m$. Let $g \in \widetilde{C}(X)$ be such that

$$\|g\|_n = \big|g(x_n)\big| \text{ for } n \in \mathbb{N},\ \sum_{n=1}^{\infty} a_n \theta_n g(x_n) = 0, \text{ and } g \equiv 0 \text{ on } X \setminus \left(\bigcup_{n=1}^{\infty} U_n\right).$$

If (x_n) contains at least two values then we can always find a g that is not equal to zero at these two points. Moreover, if $h \in \widetilde{C}(X)$ is such that $\sum_{n=1}^{\infty} a_n \theta_n h(x_n) = 0$, then we can always find a function g that satisfies all the above conditions and a function $g' \in \bigcap_{n=1}^{\infty} \ker \delta_{x_n}$ such that $g + g' = h$. From Lemma 3 we know that $\Omega^{\perp}\big((x_n),(\theta_n)\big)$ contains $\bigcap_{n=1}^{\infty} \ker \delta_{x_n}$ so to prove the lemma it is sufficient to show that the function g is contained in $\Omega^{\perp}\big((x_n),(\theta_n)\big)$.

Let $f \in \Omega\big((x_n),(\theta_n)\big)$ be such that for some sequence of numbers (c_n) with absolute values equal to one, we have

$$f(x) = c_n g(x) \quad \text{if } x \in U_n \text{ and } g(x_n) \neq 0,$$
$$f(x) = 0 \quad \text{if } x \in X \setminus \left(\bigcup_{n=1}^{\infty} U_n\right).$$

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Notice that c_n is positive if and only if $g(x_n)\,\theta_n$ is positive. For any $0 \leqslant r \leqslant 1$ we have

$$\|f \pm rg\|_n = \left(1 \pm \frac{r}{c_n}\right)|f(x_n)| \text{ if } g(x_n) \neq 0, \text{ and}$$
$$\|f \pm rg\|_n = \|f\|_n \text{ otherwise,}$$

hence

$$\|f \pm rg\| = \sum_{n=1}^{\infty} a_n \|f \pm rg\|_n = \sum_{n=1}^{\infty} a_n \|f\|_n \pm r \sum_{n=1}^{\infty} a_n sgn(c_n)\,|g(x_n)|$$
$$= \|f\| \pm r \sum_{n=1}^{\infty} a_n \theta_n g(x_n) = \|f\|,$$

so $g \in \Omega^{\perp}((x_n),(\theta_n))$. ∎

Proof of Theorem 1: Notice that the isometry T maps maximal peaking sets onto maximal peaking sets. We shall show that T maps the maximal peaking sets $\Omega((x_n),(\theta_n))$ that correspond to the constant sequence $(x_n) = (x_1)$, onto the same type of maximal peaking sets: $T(\Omega((x_1),(\theta_1))) = \Omega((x_1'),(\theta_1'))$. We shall also show that x_1' does not depend on the value of θ_1, but only on x_1.

In the complex case, by the previous lemmas, $\Omega^{\perp}((x_n),(\theta_n))$ is of codimension one if and only if (x_n) is constant. Since T preserves the codimension and the maximal peaking sets and their complements, it maps maximal peaking sets that correspond to constant sequences in X, onto the same type maximal peaking sets. For a constant sequence (x_n), $\Omega^{\perp}((x_n),(\theta_n))$ is equal to $\ker \delta_{x_1}$ and hence it does not depend on θ_1.

In the real case the situation is more complicated — by the previous lemma

$$\Omega^{\perp}((x_n),(\theta_n)) = \ker\left(\sum_{n=1}^{\infty} a_n \theta_n \delta_{x_n}\right), \tag{4}$$

for any increasing sequence (x_n), so the codimension of $\Omega^{\perp}((x_n),(\theta_n))$ is always one. Since T preserves the maximal peaking sets and their complements as well as the norm, from (4) we get that for any $((x_n),(\theta_n))$ there is a $((x_n'),(\theta_n'))$ in $X' \times \mathbf{K}_1$ such that

$$T^*\left(\sum_{n=1}^{\infty} a_n \theta_n \delta_{x_n}\right) = c \sum_{n=1}^{\infty} a_n' \theta_n' \delta_{x_n'},$$

where $c = \left(\sum_{n=1}^{\infty} a_n\right) \Big/ \left(\sum_{n=1}^{\infty} a_n'\right)$, and T^* is the conjugate of T. Denote by Ξ the set of all functionals of the form $\sum_{n=1}^{\infty} a_n \theta_n \delta_{x_n}$. We shall say that functionals $F_1, F_2 \in \Xi$ are orthogonal if

$$\|F_1\| + \|F_2\| = \|F_1 + F_2\| = \|F_1 - F_2\|.$$

For $F \in \Xi$ we put

$$F^{\perp} = \{F_1 \in \Xi : F_1 \text{ and } F \text{ are orthogonal}\}.$$

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Since the definition of orthogonality involves only the norm and the linear structure it is preserved by T^*. On the other hand $F_1 = \sum_{n=1}^{\infty} a_n\theta_n\delta_{x_n}$ and $F_2 = \sum_{n=1}^{\infty} a_n\widetilde{\theta}_n\delta_{\widetilde{x}_n}$ are orthogonal if and only if the corresponding sequences (x_n) and $(\widetilde{x}_n)$ do not have any common point. Put

$$\Xi_0 = \left\{F^{\perp} : F \in \Xi\right\}.$$

The set Ξ_0 is partially ordered by inclusion, $F^{\perp}$ is a maximal element of Ξ_0 if and only if F is defined by a constant sequence $(x_n) = (x_1)$. All this is again preserved by T, so T maps $\ker\delta_x$, $x \in X_1$, onto $\ker\delta_{x'}$, $x' \in X_1'$.

We proved that in both the complex and the real case there is a bijection φ_1 from X_1' onto X_1 such that $T\left(\ker\delta_{\varphi_1(x)}\right) = \ker\delta_x$, for $x \in X_1'$, equivalently $T^*\left(\delta_x\right) = \kappa_1\left(x\right)\delta_{\varphi_1(x)}$, for $x \in X_1'$, or

$$T\left(f\right)\left(x\right) = \kappa_1(x)f\left(\varphi_1(x)\right), \text{ for } x \in X_1', \tag{5}$$

where κ_1 has the constant absolute value equal to $c = \left(\sum_{n=1}^{\infty} a_n\right) \Big/ \left(\sum_{n=1}^{\infty} a_n'\right)$. Since T^* is continuous in the weak* topology the functions φ_1 and κ_1 must also be continuous.

We proved that T has the canonical form on $X_1 \subset X$, and

$$\|Tf\|_1 = c\,\|f\|_1 \text{ for } f \in \widetilde{C}\left(X\right). \tag{6}$$

Now we need to extend φ_1 and κ_1 to the entire set X. Because of the symmetry we may assume that

$$\frac{a_1}{a_1'} \leqslant \frac{\sum_{n=1}^{\infty} a_n}{\sum_{n=1}^{\infty} a_n'}$$

and define new norms on $\widetilde{C}\left(X\right)$ and $\widetilde{C}\left(X'\right)$, by

$$p\left(f\right) = \|f\| - a_1\|f\|_1 = \sum_{n=2}^{\infty} a_n\|f\|_n\,, \text{ for } f \in \widetilde{C}\left(X\right), \text{ and}$$

$$p'\left(g\right) = \|g\| - \frac{a_1}{c}\|g\|_1 = \left(a_1' - \frac{a_1}{c}\right)\|g\|_1 + \sum_{n=2}^{\infty} a_n'\|g\|_n\,, \text{ for } g \in \widetilde{C}\left(X'\right).$$

By (6) T is an isometry of $\left(\widetilde{C}(X), p(\cdot)\right)$ onto $\left(\widetilde{C}\left(X'\right), p'\left(\cdot\right)\right)$. If $a_1' - (a_1/c) \neq 0$, then, based on what we already proved applied to this new isometry, there is a homeomorphism φ_2 from X_1' onto X_2, and a scalar valued function κ_2 such that

$$T(f)(x) = \kappa_2(x)f\left(\varphi_2(x)\right), \text{ for } x \in X_1',$$

but this contradicts (5). Hence $a_1' - (a_1/c) = 0$ and φ_2 and κ_2, are extensions to X_2' of φ_1 and κ_1, respectively. By continuing this process we extend φ_1 to a homeomorphism of X' onto X.

We also get

$$\frac{a_1}{a'_1} = \frac{\sum_{n=1}^{\infty} a_n}{\sum_{n=1}^{\infty} a'_n}, \quad \frac{a_2}{a'_2} = \frac{\sum_{n=2}^{\infty} a_n}{\sum_{n=2}^{\infty} a'_n}, \dots,$$

hence for $c = \left(\sum_{n=1}^{\infty} a_n\right) / \left(\sum_{n=1}^{\infty} a'_n\right)$ we have

$$a_n = c a'_n, \text{ for } n \in \mathbb{N}. \qquad \square$$

Theorem 7. *Let $(C(X), d(\cdot,\cdot))$ be the metric vector space of all continuous functions on a σ-compact set X, where the metric is given by (2). Let $(C(X'), d(\cdot,\cdot))$, be the analogous space on X'. Assume that T is a linear isometry from $C(X)$ onto $C(X')$. Then there is a homeomorphism φ of X' onto X with $\varphi(X'_n) = X_n$, for $n \in \mathbb{N}$, and a unimodular, continuous, scalar valued function κ on X' such that*

$$Tf = c\kappa f \circ \varphi \text{ for all } f \in C(X),$$

where $c = \left(\sum_{n=1}^{\infty} a_n\right) / \left(\sum_{n=1}^{\infty} a'_n\right)$. Moreover $a_n = c a'_n$, for $n \in \mathbb{N}$.

Proof: Let $T : C(X) \to C(X')$ be a surjective isometry. For a fixed $f \in C(X)$ we define a function $\alpha_f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\alpha_f(t) \overset{\text{df}}{=} d(tf, 0) = \sum_{n=1}^{\infty} a_n \frac{\|tf\|_n}{1 + \|tf\|_n} = \sum_{n=1}^{\infty} a_n \frac{t\|f\|_n}{1 + t\|f\|_n}.$$

It is easy to check that

$$\frac{d\alpha_f}{dt} = \sum_{n=1}^{\infty} a_n \frac{\|f\|_n}{(1 + \|tf\|_n)^2}, \text{ for } t \geqslant 0.$$

For $t = 0$ the value of $\frac{d\alpha_f}{dt}$ is $\sum_{n=1}^{\infty} a_n \|f\|_n$; it may by finite or may be equal to $+\infty$. Since the function α_f is defined only in terms of the metric and linear structure it is preserved by the isometry. Hence $\alpha_f = \alpha_{Tf}$ and T preserves the quantity

$$\frac{d\alpha_f}{dt}(0) = \|f\| = \sum_{n=1}^{\infty} a_n \|f\|_n.$$

Consequently T restricted to the Banach space

$$\widetilde{C}(X) = \left\{ f \in C(X) : \|f\| = \sum_{n=1}^{\infty} a_n \|f\|_n < \infty \right\}$$

is an isometry, with respect to that Banach space norm, onto

$$\widetilde{C}(X') = \left\{ g \in C(X') : \|g\| = \sum_{n=1}^{\infty} a'_n \|g\|_n < \infty \right\}.$$

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By the previous Theorem there is a homeomorphism φ of X' onto X with $\varphi(X'_n) = X_n$, for $n \in \mathbf{N}$, and a unimodular, continuous, scalar valued function κ on X' such that

$$Tf = \kappa f \circ \varphi \text{ for all } f \in \widetilde{C}(X).$$

Since $\widetilde{C}(X)$ is dense in $\big(C(X), d(\cdot,\cdot)\big)$ the above formula holds for all functions from $C(X)$. ∎

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