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Maps Preserving Complementarity of Closed Subspaces of a Hilbert Space

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Abstract. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces and $\operatorname{Lat}\mathcal{H}$ the lattice of all closed subspaces oh $\mathcal{H}$. We describe the general form of pairs of bijective maps $\phi, \psi\colon \operatorname{Lat}\mathcal{H} \to \operatorname{Lat}\mathcal{K}$ having the property that for every pair $U, V \in \operatorname{Lat}\mathcal{H}$ we have $\mathcal{H} = U \oplus V \iff \mathcal{K} = \phi(U) \oplus \psi(V)$. Then we reformulate this theorem as a description of bijective image equality and kernel equality preserving maps acting on bounded linear idempotent operators. Several known structural results for maps on idempotents are easy consequences.

1 Introduction

Throughout the paper $\mathcal{H}$ and $\mathcal{K}$ are separable infinite-dimensional Hilbert spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Denote the lattice of all closed subspaces of $\mathcal{H}$ by $\operatorname{Lat}\mathcal{H}$ and the algebra of all bounded linear operators on $\mathcal{H}$ by $B(\mathcal{H})$. For any positive integer n, we introduce

$$\operatorname{Lat}_n\mathcal{H} = \{U \in \operatorname{Lat}\mathcal{H} : \dim U = n\},$$

that is, the set of all n-dimensional subspaces, and

$$\operatorname{Lat}_{-n}\mathcal{H} = \{U \in \operatorname{Lat}\mathcal{H} : \dim U^{\perp} = n\},$$

that is, the set of all subspaces of codimension n. Here, $U^{\perp}$ stands for the orthogonal complement of U. Further denote the set of all closed subspaces with infinite dimension and infinite codimension by $\operatorname{Lat}_{\infty}\mathcal{H}$ and the set of trivial subspaces by $\operatorname{Lat}_0\mathcal{H} = \{\{0\}, \mathcal{H}\}$. Then $\{\operatorname{Lat}_n\mathcal{H}\}_{n\in\mathbb{Z}\cup\{\infty\}}$ is a partition of the set $\operatorname{Lat}\mathcal{H}$. The set of indices $\mathbb{Z} \cup \{\infty\}$ will be shortly denoted by $\mathbb{Z}_{\infty}$. It will be convenient to use the following arithmetic: $-\infty = \infty$ and $\infty \pm 1 = \infty$. We will sometimes represent finite-dimensional subspaces as $[x_1, \dots, x_n]$, the linear span of vectors $x_1, \dots, x_n$.

The subject of our interest is the set of unordered pairs of closed subspaces of $\mathcal{H}$ that are complemented, that is, $\mathcal{C}_{\mathcal{H}} = \{\{U, V\} : U, V \in \operatorname{Lat}\mathcal{H}, U \oplus V = \mathcal{H}\}$. Here, the direct sum need not be orthogonal. Our aim is to characterize pairs of bijective maps $\phi, \psi\colon \operatorname{Lat}\mathcal{H} \to \operatorname{Lat}\mathcal{K}$ that preserve complementarity of subspaces:

$$\{U, V\} \in \mathcal{C}_{\mathcal{H}} \iff \bigl\{\phi(U), \psi(V)\bigr\} \in \mathcal{C}_{\mathcal{K}}, \quad U, V \in \operatorname{Lat}\mathcal{H}. \tag{1.1}$$

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We mention here that it is natural to consider this problem in the Hilbert space setting. Indeed, [9, Theorem 1] states that if every closed subspace of a Banach space X is complemented, then X is a Hilbert space.

Clearly, a pair (ϕ,ψ) satisfies (1.1) if and only if the pair (ψ,ϕ) satisfies (1.1). Note that $\{U,V\}\in\mathcal{C}_{\mathcal{H}}$ and $U\in\operatorname{Lat}_n\mathcal{H}$ for some $n\in\mathbb{Z}_\infty$ imply $V\in\operatorname{Lat}_{-n}\mathcal{H}$. For every $n\in\mathbb{Z}_\infty$ and any set $\mathcal{S}\subset\operatorname{Lat}_n\mathcal{H}$, we introduce

$$\mathcal{S}'=\{V\in\operatorname{Lat}_{-n}\mathcal{H}:\{U,V\}\notin\mathcal{C}_{\mathcal{H}}\text{ for every }U\in\mathcal{S}\}\quad\text{and}\quad\mathcal{S}''=(\mathcal{S}')'.$$

Let us give some examples of pairs of maps satisfying (1.1). First, define the set $\operatorname{BCI}(\mathcal{H},\mathcal{K})=\{A\colon\mathcal{H}\to\mathcal{K}:A\text{ is bounded, invertible, linear, or conjugate linear}\}$. Note that in the case $\mathbb{F}=\mathbb{R}$, operators from $\operatorname{BCI}(\mathcal{H},\mathcal{K})$ are linear. If $A\in\operatorname{BCI}(\mathcal{H},\mathcal{K})$, then the maps $\phi(U)=\psi(U)=A(U)$, $U\in\operatorname{Lat}\mathcal{H}$, clearly fulfil (1.1). We can get further examples if we notice that the behavior of ϕ and ψ on sets $\operatorname{Lat}_n\mathcal{H}$ and $\operatorname{Lat}_m\mathcal{H}$ are completely unrelated if $m\neq -n$. Thus, we may define ϕ and ψ as follows. Let $(A_n)_{n\in\mathbb{Z}_\infty}$ be a sequence in $\operatorname{BCI}(\mathcal{H},\mathcal{K})$. For every $n\in\mathbb{Z}_\infty$ and $U\in\operatorname{Lat}_n\mathcal{H}$ define $\phi(U)=A_n(U)$ and $\psi(U)=A_{-n}(U)$. Then the maps ϕ and ψ satisfy (1.1). Another class of examples comes from the orthogonal complementation. Let $U,V\in\operatorname{Lat}\mathcal{H}$. Then

$$\{U,V\}\in\mathcal{C}_{\mathcal{H}}\iff\{U^\perp,V^\perp\}\in\mathcal{C}_{\mathcal{H}}.\tag{1.2}$$

Indeed, it is enough to prove implication in one direction. Assume that $\mathcal{H}=U\oplus V$. Let P be the idempotent operator with $\operatorname{im}P=U$ and $\ker P=V$. Then $\mathcal{H}=\ker P^*\oplus\operatorname{im}P^*=U^\perp\oplus V^\perp$. Thus, we may construct a pair of complementarity preserving maps $\phi,\psi\colon\operatorname{Lat}\mathcal{H}\to\operatorname{Lat}\mathcal{K}$ in the following way. We will call a partition of $\mathbb{Z}_\infty$ into a disjoint union of two subsets $\mathbb{Z}_\infty=\mathbb{Z}_\infty(\mathrm{id})\sqcup\mathbb{Z}_\infty(\perp)$ regular if for every $n\in\mathbb{Z}_\infty$ we have $n\in\mathbb{Z}_\infty(\mathrm{id})\iff -n\in\mathbb{Z}_\infty(\mathrm{id})$. Assume that $\mathbb{Z}_\infty=\mathbb{Z}_\infty(\mathrm{id})\sqcup\mathbb{Z}_\infty(\perp)$ is a regular partition. Then the pair of maps $\phi,\psi\colon\operatorname{Lat}\mathcal{H}\to\operatorname{Lat}\mathcal{K}$ defined by

- if $U\in\operatorname{Lat}_n\mathcal{H}$ for some $n\in\mathbb{Z}_\infty(\mathrm{id})$, then $\phi(U)=\psi(U)=U$;
- if $U\in\operatorname{Lat}_n\mathcal{H}$ for some $n\in\mathbb{Z}_\infty(\perp)$, then $\phi(U)=\psi(U)=U^\perp$,

obviously satisfies (1.1). Our main theorem states that any pair of bijective maps ϕ,ψ satisfying (1.1) is a composition of pairs of maps described in the last two examples.

We remark that (1.1) and (1.2) yield the equivalence

$$\{U,V\}\in\mathcal{C}_{\mathcal{H}}\iff\{\phi(U^\perp)^\perp,\psi(V^\perp)^\perp\}\in\mathcal{C}_{\mathcal{K}},\quad U,V\in\operatorname{Lat}\mathcal{H},\tag{1.3}$$

which will prove useful.

The motivation for this research is twofold. Denote by $G(m,m+n)$, $m,n>1$, the Grassmannian consisting of all m-dimensional subspaces of a vector space V with $\dim V=m+n$. Two elements $U,W\in G(m,m+n)$ are said to be adjacent if $\dim(U\cap W)=m-1$ (or equivalently, $\dim(U+W)=m+1$). The classical Chow's theorem [3] states that if ϕ is an adjacency preserving bijection of $G(m,m+n)$ onto itself, then it is induced by a semilinear automorphism of V, or $m=n$ and ϕ is induced by a semilinear isomorphism from a dual space V' onto V. For a pair of different subspaces of the same dimension, one extremal

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position is when they are adjacent, and the other extreme is that they are complementary subspaces. This motivated Blunck and Havlicek [1] to investigate bijective maps ϕ on $G(m,2m)$ with the property that for every pair $U,W \in G(m,2m)$ we have $V = U \oplus W \iff V = \phi(U) \oplus \phi(W)$. They proved that such maps preserve adjacency, and therefore one can apply Chow's result to describe the general form of such maps. They have also included the infinite-dimensional case in their considerations as much as possible. In particular, they pointed out that their proof techniques cannot be extended to the infinite-dimensional case. These observations certainly made the infinite-dimensional case interesting. When dealing with the infinite-dimensional case we have two possibilities, either to work in a purely algebraic setting or to work, say, with Hilbert spaces, where some analysis is involved. We believe that the second possibility is more interesting, and, as we shall soon see, it is also more important because of applications. If we followed Blunck and Havlicek, we would restrict ourselves to bijective mappings acting on the set of all closed subspaces of infinite dimension and infinite codimension. But there is no reason to restrict to such subspaces. So we consider the whole lattice of closed subspaces. Of course, this makes the problem more difficult as we first need to show the somewhat surprising fact that the subspaces of a finite dimension m are mapped into subspaces that either have dimension m or codimension m, while the subspaces of infinite dimension and infinite codimension are mapped into subspaces of the same type. Next, with each pair of complementary closed subspaces we can associate a pair of bounded idempotent operators whose images and kernels are these two subspaces. This observation brings us to the second motivation for our research. Namely, an easy consequence of our main result is a structural theorem for bijective maps on the set of bounded idempotent operators preserving the equality of images and the equality of kernels. We will also prove the dual form of this result. In fact, because of these two applications it is more natural to work with a complementarity preserving pair of maps on $\operatorname{Lat}\mathcal{H}$ instead of with a single map as in the result of Blunck and Havlicek. Once we have the description of image equality and kernel equality preserving bijections on the set of idempotent operators we can reprove a whole set of characterizations of various maps on idempotent operators. We will illustrate this with two examples.

For a given Hilbert space $\mathcal{H}$ we denote the set of all bounded idempotent operators on $\mathcal{H}$ by $\mathcal{I}(\mathcal{H})$,

$$\mathcal{I}(\mathcal{H}) = \{P \in B(\mathcal{H}) : P^2 = P\}.$$

It is well known that $\mathcal{I}(\mathcal{H})$ is a poset with the partial order defined by

$$P \le Q \iff PQ = QP = P.$$

It is easy to verify that for $P, Q \in \mathcal{H}$ we have $P \le Q$ if and only if there is a direct sum decomposition of $\mathcal{H}$ into a direct sum of closed subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ (here, some of subspaces $\mathcal{H}_j$, $j = 1,2,3$, may be trivial) such that P and Q have the following matrix representations with respect to this decomposition:

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A bijective map $\phi\colon \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{K})$ is a poset isomorphism if for every pair of idempotents $P, Q \in \mathcal{I}(\mathcal{H})$ we have $P \leq Q$ if and only if $\phi(P) \leq \phi(Q)$. Motivated by some problems in mathematical physics, Ovchinnikov [11] described the general form of such maps. We will show that this beautiful result can be quickly deduced from our main theorem.

Recently, quite a lot of attention has been paid to the study of 2-local automorphisms (see [8] and the references therein). Let S be a ring. A map $\phi\colon S \to S$ is called a 2-local automorphism if for every pair of elements $a, b \in S$ there exists an automorphism $\phi_{a,b}\colon S \to S$ (depending on a and b) such that $\phi(a) = \phi_{a,b}(a)$ and $\phi(b) = \phi_{a,b}(b)$. Of course, the natural question is whether for a given ring S each 2-local automorphism is an automorphism. Clearly, if ϕ is a 2-local automorphism of S, then ϕ maps every idempotent element of S into some idempotent element. And clearly, for every pair of idempotents $p, q \in S$, the product pq is a nonzero idempotent if and only if $\phi(p)\phi(q)$ is a nonzero idempotent. The study of maps having this property is also motivated by a well developed theory of additive, linear, and multiplicative preservers (see [2, 7, 12] and the references therein). We will conclude our paper by reproving a structural result for bijective maps on $\mathcal{I}(\mathcal{H})$ preserving the nonzero idempotency of the products of idempotent operators.

2 Maps on $\operatorname{Lat}\mathcal{H}$

This section is devoted to our main result. We start with some definitions.

For $U, V \in \operatorname{Lat}\mathcal{H}$ denote $U \ominus V = U \cap V^{\perp}$. If $U, V \in \operatorname{Lat}_n\mathcal{H}$ for some $n \in \mathbb{Z}_\infty$, then we say that U and V are *adjacent* if $\dim U \ominus (U \cap V) = \dim V \ominus (U \cap V) = 1$. Note that this is equivalent to $\dim(U+V)\ominus U = \dim(U+V)\ominus V = 1$. When $n \neq \infty$, the definition simplifies to: U and V are adjacent if and only if $U \cap V \in \operatorname{Lat}_{n-1}\mathcal{H}$ if and only if $U + V \in \operatorname{Lat}_{n+1}\mathcal{H}$.

Let U and V be distinct elements of $\operatorname{Lat}_n\mathcal{H}$ for some $n \in \mathbb{Z}_\infty \setminus \{0\}$ and let k be a positive integer. We call a sequence $U_0, U_1, \dots, U_k \in \operatorname{Lat}_n\mathcal{H}$ an adjacent chain between U and V if $U_0 = U$, $U_k = V$, and U_j, U_{j+1} are adjacent for all $j = 0, \dots, k-1$. Furthermore, we say that k is the length of a chain.

We continue with a few lemmas about adjacent subspaces that we will need in the sequel.

Lemma 2.1 *Let k be a positive integer, $n \in \mathbb{Z}_\infty \setminus \{0\}$, and let $U_0, U_1, \dots, U_k \in \operatorname{Lat}_n\mathcal{H}$ be an adjacent chain. Then for $j = 0, \dots, k-1$,*

$$\dim U_j \ominus (U_0 \cap U_1 \cap \dots \cap U_k) = \dim U_{j+1} \ominus (U_0 \cap U_1 \cap \dots \cap U_k) \leq k.$$

Proof For every $j \in \{0, 1, \dots, k\}$ define

$$V_j = U_j \ominus (U_0 \cap U_1 \cap \dots \cap U_k).$$

Then $V_j \cap V_{j+1} = (U_j \cap U_{j+1}) \ominus (U_0 \cap \dots \cap U_k)$ implies

$$\begin{aligned} V_j \ominus (V_j \cap V_{j+1}) &= U_j \ominus \big((U_0 \cap \dots \cap U_k) + \big((U_j \cap U_{j+1}) \ominus (U_0 \cap \dots \cap U_k) \big) \big) \\ &= U_j \ominus (U_j \cap U_{j+1}), \end{aligned}$$

where the last equality holds because $U_0 \cap \cdots \cap U_k \subset U_j \cap U_{j+1}$. Analogously we get $V_{j+1} \ominus (V_j \cap V_{j+1}) = U_{j+1} \ominus (U_j \cap U_{j+1})$. Because U_j and U_{j+1} are adjacent, the same is true for V_j and V_{j+1}. In particular, $\dim V_j = \dim V_{j+1}$, and it remains to show that $\dim V_0 \leq k$. In order to do this, note that for every $j \in \{0, \ldots, k-1\}$, there exists $u_j \in U_j \setminus \{0\}$ such that $U_j \cap U_{j+1} = U_j \ominus [u_j]$. Moreover, let $P \in \mathcal{I}(\mathcal{H})$ be the projection onto U_0. Then

$$\begin{aligned} V_0 &= U_0 \ominus \big((U_0 \cap \cdots \cap U_{k-1}) \ominus [u_{k-1}]\big) = \cdots = U_0 \ominus \big(U_0 \ominus [u_0, \ldots, u_{k-1}]\big) \\ &= [Pu_0, \ldots, Pu_{k-1}] \end{aligned}$$

is at most k-dimensional. ∎

Lemma 2.2 *Let $U, V \in \operatorname{Lat}\mathcal{H}$ be such that*

$$\dim U \ominus (U \cap V) = \dim V \ominus (U \cap V) = d$$

for some positive integer d. Then

- *there exists an adjacent chain between U and V of length d;*
- *any adjacent chain between U and V has length at least d.*

Proof We start by proving the first statement in the lemma. Set $U_0 = U$ and $U_d = V$. If $d = 1$, then U and V are adjacent and we are done. If $d > 1$, choose a basis $u_1, \ldots, u_d$ of $U \ominus (U \cap V)$ and a basis $v_1, \ldots, v_d$ of $V \ominus (U \cap V)$. For every $j \in \{1, \ldots, d-1\}$ define $U_j = [v_1, \ldots, v_j, u_{j+1}, \ldots, u_d] \oplus (U \cap V)$. Then clearly $U_0, U_1, \ldots, U_d$ is an adjacent chain between U and V.

Turn now to the proof of the second statement. Let r be a positive integer and $V_0, \ldots, V_r$ an adjacent chain between U and V. By Lemma 2.1 we have

$$r \geq \dim U \ominus (V_0 \cap V_1 \cap \cdots \cap V_r) \geq \dim U \ominus (U \cap V) = d,$$

as desired. ∎

Lemma 2.3 *Let $U_1, U_2, U_3 \in \operatorname{Lat}\mathcal{H}$ be pairwise adjacent. If $U_1 \cap U_2 \not\subset U_3$, then $U_3 \subset U_1 + U_2$ and $\dim(U_1 + U_2) \ominus U_3 = 1$.*

Proof We will first show that $U_1 \cap U_2 \not\subset U_3$ implies

$$U_1 \cap U_3 \not\subset U_2 \quad \text{and} \quad U_2 \cap U_3 \not\subset U_1. \tag{2.1}$$

Indeed, denote $U = U_1 \cap U_2 \cap U_3$. Suppose that $U_1 \cap U_3 \subset U_2$, or, equivalently, $U = U_1 \cap U_3$. Since U_1 and U_3 are adjacent, this yields $\dim U_1 \ominus U = 1$. This and the adjacency of U_1 and U_2 now imply $U = U_1 \cap U_2$. It follows that $U_1 \cap U_2 \subset U_3$, a contradiction. By exchanging the roles of U_1 and U_2 in the previous argument we also get $U_2 \cap U_3 \not\subset U_1$, as desired.

We next claim that

$$\dim U_j \ominus U = 2, \quad j = 1, 2, 3.$$

The inclusions (2.1) guarantee that the assumptions on U_1, U_2, and U_3 are completely symmetric, so it is enough to show that $\dim U_3 \ominus U = 2$. Since U_1, U_2, U_3 is an adjacent chain, Lemma 2.1 implies that $\dim U_3 \ominus U \leq 2$. It is clear that $\dim U_3 \ominus U$ cannot equal 0. Moreover, neither can it equal 1, because this would imply $U = U_1 \cap U_3$, a contradiction with (2.1).

Because U_1, U_2, and U_3 are pairwise adjacent, the previous paragraph yields the existence of linearly independent u_1, u_2, and u_3 such that

$$U_1 = U \oplus [u_1, u_2], \quad U_2 = U \oplus [u_2, u_3], \quad U_3 = U \oplus [u_1, u_3].$$

Hence, $U_1 + U_2 = U \oplus [u_1, u_2, u_3]$ contains U_3 as a subspace with codimension 1. ■

For any $U \in \operatorname{Lat} \mathcal{H} \setminus \{\mathcal{H}\}$ denote

$$\mathcal{A}_U = \{V \in \operatorname{Lat} \mathcal{H} : V \supset U, \dim V \ominus U = 1\} = \{U \oplus [x] : x \in U^{\perp} \setminus \{0\}\},$$

and for any $U \in \operatorname{Lat} \mathcal{H} \setminus \{\{0\}\}$ denote

$$\mathcal{B}_U = \{V \in \operatorname{Lat} \mathcal{H} : V \subset U, \dim U \ominus V = 1\} = \{U \ominus [x] : x \in U \setminus \{0\}\}.$$

We remark that

$$\mathcal{B}_U = \{V \in \operatorname{Lat} \mathcal{H} : V^{\perp} \in \mathcal{A}_{U^{\perp}}\}. \tag{2.2}$$

A subset $\mathcal{S} \subset \operatorname{Lat}_n \mathcal{H}$ is called a *maximal adjacent set* if any two distinct elements of $\mathcal{S}$ are adjacent and $\mathcal{S}$ is maximal among all such sets. Clearly, $\operatorname{Lat}_1 \mathcal{H} = \mathcal{A}_{\{0\}}$ and $\operatorname{Lat}_{-1} \mathcal{H} = \mathcal{B}_{\mathcal{H}}$ are maximal adjacent sets.

Lemma 2.4 *Let $\mathcal{S} \subset \operatorname{Lat}_n \mathcal{H}$, where $n \in \mathbb{Z}_\infty \setminus \{-1, 0, 1\}$. Then the following two statements are equivalent:*

- *$\mathcal{S}$ is a maximal adjacent set;*
- *either there exists $U \in \operatorname{Lat}_{n-1} \mathcal{H}$ such that $\mathcal{S} = \mathcal{A}_U$, or there exists $U \in \operatorname{Lat}_{n+1} \mathcal{H}$ such that $\mathcal{S} = \mathcal{B}_U$.*

Proof It is clear that $\mathcal{A}_U$, $U \neq \mathcal{H}$, and $\mathcal{B}_U$, $U \neq \{0\}$, are adjacent sets. If, moreover, $U \notin \operatorname{Lat}_{-2} \mathcal{H} \cup \operatorname{Lat}_{-1} \mathcal{H}$, then $\mathcal{A}_U$ is a maximal adjacent set. Indeed, for every $W \in \operatorname{Lat} \mathcal{H}$ such that $W \notin \mathcal{A}_U$ we will find $V \in \mathcal{A}_U$ so that V and W are not adjacent. So, let $W \in \operatorname{Lat} \mathcal{H} \setminus \mathcal{A}_U$. We distinguish three cases.

Case 1 $W \supset U$ and $\dim W \ominus U \geq 2$. Choose any nonzero $w \in W \ominus U$ and set $V = U \oplus [w] \in \mathcal{A}_U$. Then V is contained in W and is therefore not adjacent to W.

Case 2 $W \not\supset U$ and $\overline{U + W} \neq \mathcal{H}$. Choose nonzero $v \in (U + W)^{\perp}$ and set $V = U \oplus [v] \in \mathcal{A}_U$. It follows from $W \not\supset U$ that $\overline{U + W}$ strictly contains W, hence W has codimension at least 2 in $\overline{V + W} = [v] \oplus \overline{U + W}$. Thus, W and V are not adjacent.

Case 3 $\overline{U + W} = \mathcal{H}$. Because U has codimension at least 3 in $\mathcal{H}$, any $V \in \mathcal{A}_U$ has codimension at least 2 in $\overline{V + W} = \mathcal{H}$. Hence, V and W are not adjacent.

Suppose now that $U \notin \mathrm{Lat}_1\mathcal{H} \cup \mathrm{Lat}_2\mathcal{H}$. We already know that $\mathcal{A}_{U^\perp}$ is a maximal adjacent set. Since orthogonal complementation preserves adjacency, (2.2) implies that $\mathcal{B}_U$ is also a maximal adjacent set.

Assume on the other hand that $\mathcal{S} \subset \mathrm{Lat}_n\mathcal{H}$, $n \notin \{-1, 0, 1\}$, is a maximal adjacent set such that $\mathcal{S} \neq \mathcal{A}_U$ for every $U \in \mathrm{Lat}_{n-1}\mathcal{H}$. We need to show that there exists $Z \in \mathrm{Lat}_{n+1}\mathcal{H}$ such that $\mathcal{S} = \mathcal{B}_Z$. Indeed, there exist $U_1, U_2, U_3 \in \mathcal{S}$ such that $U_3 \notin \mathcal{A}_{U_1 \cap U_2}$. We assert that $U_1 \cap U_2 \not\subset U_3$. Suppose on the contrary that $U_1 \cap U_2 \subset U_3$. Then $U_1 \cap U_2$ and $U_1 \cap U_3$ are comparable and have both codimension 1 in U_1. Hence, they are the same and consequently, $U_3 \in \mathcal{A}_{U_1 \cap U_2}$, a contradiction. Lemma 2.3 now yields that $U_3 \in \mathcal{B}_{U_1+U_2}$. Denote $Z = U_1 + U_2 \in \mathrm{Lat}_{n+1}\mathcal{H}$. By what we have already proved we have

$$U_j = Z \ominus [v_j], \quad j = 1, 2, 3,$$

for some linearly independent $v_1, v_2, v_3 \in Z$.

Choose any $V \in \mathcal{S}$ different from U_1, U_2, U_3. We will show that $V \in \mathcal{B}_Z$. We claim that V does not contain at least one of $U_1 \cap U_2$ and $U_1 \cap U_3$. Indeed, if V contained both, $U_1 \cap U_2 = Z \ominus [v_1, v_2]$ and $U_1 \cap U_3 = Z \ominus [v_1, v_3]$, then it would also contain $(U_1 \cap U_2) + (U_1 \cap U_3) = Z \ominus [v_1] = U_1$, which would contradict adjacency of U_1 and V. Suppose that $U_1 \cap U_3 \not\subset V$, as the other case is symmetric. Then Lemma 2.3 implies that $V \in \mathcal{B}_{U_1+U_3} = \mathcal{B}_Z$, as desired. Thus, $\mathcal{S} \subset \mathcal{B}_Z$, and by the maximality of $\mathcal{S}$, $\mathcal{S} = \mathcal{B}_Z$. ∎

Let $n \in \mathbb{Z}_\infty \setminus \{0\}$ and r be a positive integer, $r \geq 2$. We will investigate the set $\{U_1, \dots, U_r\}''$, where $U_1, \dots, U_r \in \mathrm{Lat}_n\mathcal{H}$. Recall that for $U \in \mathrm{Lat}_n\mathcal{H}$ we have $U \in \{U_1, \dots, U_r\}''$ if and only if

$$\{U, V\} \notin \mathcal{C}_\mathcal{H} \tag{2.3}$$

for every $V \in \mathrm{Lat}_{-n}\mathcal{H}$ satisfying

$$\{U_j, V\} \notin \mathcal{C}_\mathcal{H}, \quad j = 1, \dots, r. \tag{2.4}$$

We continue with several rather simple statements that will help the reader to easily absorb the proof of the main theorem. In these statements we fix n, r and pairwise distinct $U_1, \dots, U_r$ as above.

Lemma 2.5 *Let $U \in \{U_1, \dots, U_r\}''$. If $V \in \mathrm{Lat}\,\mathcal{H}$ is finite-dimensional such that $V \cap U_j \neq \{0\}$ for every $j = 1, \dots, r$, then $V \cap U \neq \{0\}$.*

Proof Assume on the contrary that $V \cap U = \{0\}$. Obviously, $U \oplus V$ is a closed subspace of $\mathcal{H}$, $(U \oplus V)^\perp \cap V = \{0\}$, and $V \oplus (U \oplus V)^\perp \in \mathrm{Lat}_{-n}\mathcal{H}$. The subspace $V \oplus (U \oplus V)^\perp$ satisfies (2.4) due to the assumption $V \cap U_j \neq \{0\}$, $j = 1, \dots, r$. On the other hand, $\mathcal{H} = (U \oplus V) \oplus (U \oplus V)^\perp = U \oplus (V \oplus (U \oplus V)^\perp)$, and therefore $V \oplus (U \oplus V)^\perp$ satisfies (2.4), but not (2.3), a contradiction. ∎

Corollary 2.6 *Let $U \in \{U_1, \dots, U_r\}''$. Then*

$$U_1 \cap \cdots \cap U_r \subset U \subset \overline{U_1 + \cdots + U_r}.$$

Proof To prove the first inclusion we take a nonzero $x \in U_1 \cap \cdots \cap U_r$ and apply the previous lemma with $V = [x]$ to get $U_1 \cap \cdots \cap U_r \subset U$. It follows from (1.2), (2.3), and (2.4) that $U^\perp \in \{U_1^\perp, \dots, U_r^\perp\}''$, and therefore, by what we have just proved, we have $(U_1 + \cdots + U_r)^\perp = U_1^\perp \cap \cdots \cap U_r^\perp \subset U^\perp$, or equivalently,

$$U \subset \overline{U_1 + \cdots + U_r}. \qquad \blacksquare$$

Lemma 2.7 *If $U_1, \dots, U_r \in \mathcal{A}_W$ for some $W \in \operatorname{Lat}_{n-1}\mathcal{H}$, then*

$$\{U_1, \dots, U_r\}'' = \{U \in \mathcal{A}_W : U \subset U_1 + \cdots + U_r\}.$$

Proof Assume that $U \in \{U_1, \dots, U_r\}''$. Since $U_1 + \cdots + U_r$ is closed, we get from the previous corollary that $W \subset U \subset U_1 + \cdots + U_r$. We need to show that $\dim(U \ominus W) = 1$.

Choose $\widetilde{U} \in \operatorname{Lat}\mathcal{H}$ such that $U_1 + \cdots + U_r = U \oplus \widetilde{U}$. Of course, $\widetilde{U}$ is finite-dimensional. Because $\mathcal{H} = (U \oplus \widetilde{U}) \oplus (U_1 + \cdots + U_r)^\perp$, the subspace $V = \widetilde{U} \oplus (U_1 + \cdots + U_r)^\perp$ does not satisfy (2.3), and consequently we have $\mathcal{H} = U_j \oplus V = (U_j \oplus \widetilde{U}) \oplus (U_1 + \cdots + U_r)^\perp$ for some $j \in \{1, \dots, r\}$. Since $U_j, \widetilde{U} \subset U_1 + \cdots + U_r$, we conclude that

$$U_j \oplus \widetilde{U} = U_1 + \cdots + U_r = U \oplus \widetilde{U},$$

yielding that U_j and U have the same finite codimension in $U_1 + \cdots + U_r$. It follows that $\dim(U \ominus W) = \dim(U_j \ominus W) = 1$, as desired.

To prove the opposite inclusion we first write

$$U_j = W \oplus [u_j], \qquad j = 1, \dots, r,$$

for some nonzero vectors $u_j \in W^\perp$, and then choose $U \in \mathcal{A}_W$ with $U \subset U_1 + \cdots + U_r$. We must show that for each $V \in \operatorname{Lat}_{-n}\mathcal{H}$ such that $\mathcal{H} = U \oplus V$ (in other words, V does not satisfy (2.3)) we have $\mathcal{H} = U_j \oplus V$ for some $j \in \{1, \dots, r\}$.

We claim that we have $U_j \cap V = \{0\}$ for some j. Suppose on the contrary that for every $j \in \{1, \dots, r\}$ there exists a nonzero $v_j \in U_j \cap V$. We have $W \cap [v_1, \dots, v_r] = \{0\}$. We also know that $W \cap [u_1, \dots, u_r] = \{0\}$, because $[u_1, \dots, u_r] \subset W^\perp$. Now, $v_j \in W \oplus [u_j]$, $j = 1, \dots, r$, and therefore,

$$U_1 + \cdots + U_r = W \oplus [u_1, \dots, u_r] \subset W \oplus [v_1, \dots, v_r] \subset U_1 + \cdots + U_r.$$

Since $W \subset U \subset U_1 + \cdots + U_r$ and W has codimension one in U, we have $U \cap [v_1, \dots, v_r] \neq \{0\}$, contradicting $\mathcal{H} = U \oplus V$.

Hence, there is $j \in \{1, \dots, r\}$ such that $U_j \cap V = \{0\}$. Because $\mathcal{H} = U \oplus V$, we have $u_j = u + v$ for some $u \in U$ and some $v \in V$. We distinguish two cases.

If $u \in U_j$, then because $U_j \cap V = \{0\}$, we have $u = u_j$, and consequently, $U = U_j$. We are done.

If, on the other hand $u \notin U_j$, then clearly $u \notin W$, and therefore, $W \oplus [u] = U$. It follows that

$$U_j + V = W + [u_j] + V = W + [u] + V = U + V = \mathcal{H},$$

as desired. $\blacksquare$

Corollary 2.8 *If $U_1,\dots,U_r \in \mathcal{B}_Z$ for some $Z \in \mathrm{Lat}_{n+1}\mathcal{H}$, then*

$$\{U_1,\dots,U_r\}'' = \{U \in \mathcal{B}_Z : U_1 \cap \cdots \cap U_r \subset U\}.$$

Proof This statement follows directly from the previous lemma, (1.2), and (2.2). ■

Lemma 2.9 *If $0 < n \le \infty$ and $W \in \mathrm{Lat}_{n-1}\mathcal{H}$, then for every collection $U_1,\dots,U_r \in \mathcal{A}_W$ we have*

$$\{U_1,\dots,U_r\}'' \neq \mathcal{A}_W.$$

Proof From $n > 0$ we conclude that $U_1 + \cdots + U_r \neq \mathcal{H}$, and then the statement is a straightforward consequence of Lemma 2.7. ■

Lemma 2.10 *If $0 < n < r$ and $Z \in \mathrm{Lat}_{n+1}\mathcal{H}$, then there exist $U_1,\dots,U_r \in \mathcal{B}_Z$ such that*

$$\{U_1,\dots,U_r\}'' = \mathcal{B}_Z.$$

Proof Since $n < r$, one can choose $U_1,\dots,U_r \in \mathcal{B}_Z$ such that $U_1 \cap \cdots \cap U_r = \{0\}$. Applying Corollary 2.8 we complete the proof. ■

A purely algebraic analogue of the next lemma can be found in [1, Theorem 3.2].

Lemma 2.11 *Let $n \in \mathbb{Z}_\infty \setminus \{0\}$ and $U_1, U_2 \in \mathrm{Lat}_n\mathcal{H}$ be distinct. Then U_1 and U_2 are adjacent if and only if $\{U_1,U_2\}'' \neq \{U_1,U_2\}$.*

Proof Applying Lemma 2.7 with $r = 2$ and $W = U_1 \cap U_2$, we get $\{U_1,U_2\}'' \neq \{U_1,U_2\}$ if $U_1,U_2 \in \mathrm{Lat}_n\mathcal{H}$ are adjacent. Suppose now that for distinct $U_1,U_2 \in \mathrm{Lat}_n\mathcal{H}$ there exists $U \in \{U_1,U_2\}''$ such that $U \notin \{U_1,U_2\}$. We will prove that U_1 and U_2 are adjacent in four steps.

Step 1 $U_1 \not\subset U$ and $U_2 \not\subset U$. By symmetry, it suffices to prove the first equation. It holds automatically if $n \neq \infty$, so assume the opposite. Suppose that $U_1 \subset U$. We will distinguish two cases. First, assume that we also have $U_2 \subset U$. If $V \in \mathrm{Lat}_\infty \mathcal{H}$ is such that $\{U,V\} \in \mathcal{C}_\mathcal{H}$, then $U_1 + V$ and $U_2 + V$ are strictly contained in $U + V = \mathcal{H}$. Thus, $\{U_1,V\} \notin \mathcal{C}_\mathcal{H}$ and $\{U_2,V\} \notin \mathcal{C}_\mathcal{H}$, a contradiction with (2.3) and (2.4). Turn now to the second case, when $U_2 \not\subset U$. Choose $u_2 \in U_2 \setminus U$ and set $V = [u_2] \oplus ([u_2] \oplus U)^\perp \in \mathrm{Lat}_\infty \mathcal{H}$. Then $\{U,V\} \in \mathcal{C}_\mathcal{H}$, $U_1 + V \neq \mathcal{H}$, and $U_2 \cap V \neq \{0\}$ again contradict (2.3) and (2.4).

Step 2 $U_1 \cap U \in \mathcal{B}_{U_1}$ and $U_2 \cap U \in \mathcal{B}_{U_2}$. We will again prove only the first equation and then refer to symmetry. The equation clearly holds if $n = 1$, so assume that $n \neq 1$. Because $U_1 \not\subset U$, $U_1 \cap U$ is strictly contained in U_1. Hence, it is enough to show that for any pair of linearly independent $u_1, v_1 \in U_1$ we have $[u_1,v_1] \cap U \neq \{0\}$. So choose linearly independent $u_1, v_1 \in U_1$. Since $U_2 \not\subset U$, there exists $u_2 \in U_2 \setminus U$. If u_1 and u_2 were linearly dependent, they would be contained in $U_1 \cap U_2$, which is contained in U by Corollary 2.6, a contradiction. Thus, u_1 and u_2 are linearly independent and the same argument applies to v_1 and u_2. It follows that u_1, v_1, u_2 are linearly independent. Otherwise u_2 would belong to $[u_1,v_1]$ and therefore we would have $u_2 \in U_1 \cap U_2 \subset U$, a contradiction. Lemma 2.5 yields that there exist $0 \neq u \in [u_1,u_2] \cap U$ and $0 \neq v \in [v_1,u_2] \cap U$. We claim that u and v are linearly independent.

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Indeed, $u = \lambda_1 u_1 + \lambda_2 u_2$ and $v = \mu_1 v_1 + \mu_2 u_2$ for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}$. It follows from $u_2 \notin U$ that $\lambda_1, \mu_1 \neq 0$. The linear independence of u_1 and v_1 now implies the linear independence of u and v. So, $[u, v]$ and $[u_1, v_1]$ are two-dimensional subspaces of $[u_1, u_2, v_1]$ and have therefore non-trivial intersection. This completes Step 2.

Step 3 $\overline{U_1 + U_2} = U \oplus [u_2]$ for some $u_2 \in U_2 \setminus U$. We know from Step 2 that $U_1 = (U_1 \cap U) \oplus [u_1]$ and $U_2 = (U_2 \cap U) \oplus [u_2]$ for some $u_1, u_2 \notin U$. It follows from Lemma 2.5 that the intersection $[u_1, u_2] \cap U$ is non-trivial. Because $u_1, u_2 \notin U$, this intersection must be equal to some $[u] \notin \{[u_1], [u_2]\}$. Hence,

$$U_1 + U_2 = (U_1 \cap U) + (U_2 \cap U) + [u_1, u_2] \subset U + [u_1, u_2] = U + [u, u_2] = U \oplus [u_2].$$

But the latter term is contained in $\overline{U_1 + U_2}$ by Corollary 2.6, and therefore $\overline{U_1 + U_2} = U \oplus [u_2]$.

Step 4 U_1 and U_2 are adjacent. Let $u_2 \in U_2 \setminus U$ be as in the previous step. Denote $V = [u_2] \oplus (U_1 + U_2)^\perp \in \mathrm{Lat}_{-n}\mathcal{H}$. Then $U \oplus V = (U \oplus [u_2]) \oplus (U_1 + U_2)^\perp = \mathcal{H}$ and $0 \neq u_2 \in U_2 \cap V$. Hence, $\{U_2, V\} \notin \mathcal{C}_{\mathcal{H}}$, so (2.3) and (2.4) yield $\{U_1, V\} \in \mathcal{C}_{\mathcal{H}}$. It is now clear that $U_1 \oplus [u_2] \subset \overline{U_1 + U_2}$, and these two subspaces have a common complement, namely $(U_1 + U_2)^\perp$. Hence, they are the same and consequently, $U_1 + U_2$ is closed. Moreover, U_1 has codimension 1 in $U_1 + U_2$ and, by symmetry, the same is true for U_2. ■

Corollary 2.12 *Let $n, m \in \mathbb{Z}_\infty \setminus \{0\}$, and $\phi : \mathrm{Lat}_n\mathcal{H} \to \mathrm{Lat}_m\mathcal{K}$, $\psi\colon \mathrm{Lat}_{-n}\mathcal{H} \to \mathrm{Lat}_{-m}\mathcal{K}$ be bijective maps such that*

$$\{U, V\} \in \mathcal{C}_{\mathcal{H}} \iff \{\phi(U), \psi(V)\} \in \mathcal{C}_{\mathcal{K}}, \quad U \in \mathrm{Lat}_n\mathcal{H}, V \in \mathrm{Lat}_{-n}\mathcal{H}.$$

Then for all $U_1, U_2 \in \mathrm{Lat}_n\mathcal{H}$ we have

$$U_1 \textit{ and } U_2 \textit{ are adjacent} \iff \phi(U_1) \textit{ and } \phi(U_2) \textit{ are adjacent.}$$

Moreover, for all $V_1, V_2 \in \mathrm{Lat}_{-n}\mathcal{H}$ we have

$$V_1 \textit{ and } V_2 \textit{ are adjacent} \iff \psi(V_1) \textit{ and } \psi(V_2) \textit{ are adjacent.}$$

Proof The first statement is a direct consequence of Lemma 2.11 and the fact that $\phi(\{U_1, U_2\}'') = \psi(\{U_1, U_2\}')' = \{\phi(U_1), \phi(U_2)\}''$. We get the second statement if we exchange the roles of ϕ and ψ in the last argument. ■

Proposition 2.13 *Let n be a positive integer. Suppose that $\phi\colon \mathrm{Lat}_n\mathcal{H} \to \mathrm{Lat}_n\mathcal{K}$ and $\psi\colon \mathrm{Lat}_{-n}\mathcal{H} \to \mathrm{Lat}_{-n}\mathcal{K}$ are bijective maps such that*

$$\{U, V\} \in \mathcal{C}_{\mathcal{H}} \iff \{\phi(U), \psi(V)\} \in \mathcal{C}_{\mathcal{K}}, \quad U \in \mathrm{Lat}_n\mathcal{H}, V \in \mathrm{Lat}_{-n}\mathcal{H}.$$

Then there exists $A_n \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ such that

$$\phi(U) = A_n(U),\ U \in \mathrm{Lat}_n\mathcal{H}, \quad \textit{and} \quad \psi(V) = A_n(V),\ V \in \mathrm{Lat}_{-n}\mathcal{H}.$$

Proof We apply induction on n. Start with $n = 1$. Lemma 2.7, applied to $r = 2$ and $W = \{0\}$, tells us that for $[x] \in \mathrm{Lat}_1\mathcal{H}$ and distinct $[y], [z] \in \mathrm{Lat}_1\mathcal{H}$ we have

$$[x] \subset [y] + [z] \iff [x] \in \{[y], [z]\}''.$$

Hence,

$$[x] \subset [y] + [z] \iff \phi([x]) \subset \phi([y]) + \phi([z]).$$

By the fundamental theorem of projective geometry [4] there exists a bijective semilinear map $A_1 : \mathcal{H} \to \mathcal{K}$ such that $\phi([x]) = [A_1 x]$, $x \in \mathcal{H} \setminus \{0\}$. In other words, $\phi(U) = A_1(U)$, $U \in \mathrm{Lat}_1\mathcal{H}$. Now (1.3) yields that the same is true for the map $U \mapsto \psi(U^\perp)^\perp$, $U \in \mathrm{Lat}_1\mathcal{H}$, so $\psi(V) = B_1(V^\perp)^\perp$, $V \in \mathrm{Lat}_{-1}\mathcal{H}$, for some bijective semilinear map $B_1 : \mathcal{H} \to \mathcal{K}$.

We next show that A_1 is linear or conjugate linear and is bounded. Note that for any pair $x, y \in \mathcal{H} \setminus \{0\}$, we have $\{[x], [y]^\perp\} \notin \mathcal{C}_\mathcal{H} \iff [x] \cap [y]^\perp \neq \{0\} \iff x \perp y$, which implies

$$x \perp y \iff A_1 x \perp B_1 y, \quad x, y \in \mathcal{H}. \tag{2.5}$$

Hence, A_1 carries closed hyperplanes in $\mathcal{H}$ into closed hyperplanes in $\mathcal{K}$, and the analogous statement is true for A_1^{-1}. If $\mathbb{F} = \mathbb{R}$, then A_1 is automatically linear and is bounded by [10, Lemma B]. On the other hand, if $\mathbb{F} = \mathbb{C}$, the (conjugate) linearity and the boundedness follow from [5, Lemmas 2 and 3]. We already know that for every $V \in \mathrm{Lat}_{-1}\mathcal{H}$ we have $\psi(V) = B_1(V^\perp)^\perp$, which equals $A_1(V)$ by (2.5).

Continue with the induction step. Let $n \geq 2$ and suppose that any pair of bijective maps defined on $\mathrm{Lat}_{n-1}\mathcal{H}$ and $\mathrm{Lat}_{-n+1}\mathcal{H}$ that preserves complementarity is induced by an operator from $\mathrm{BCI}(\mathcal{H}, \mathcal{K})$. We want to see that the same is true for the maps defined on $\mathrm{Lat}_n\mathcal{H}$ and $\mathrm{Lat}_{-n}\mathcal{H}$. Choose and fix an arbitrary $W \in \mathrm{Lat}_{n-1}\mathcal{H}$. Corollary 2.12 and Lemma 2.4 yield that we have either $\phi(\mathcal{A}_W) = \mathcal{A}_{W'}$ for some $W' \in \mathrm{Lat}_{n-1}\mathcal{K}$ or $\phi(\mathcal{A}_W) = \mathcal{B}_Z$ for some $Z \in \mathrm{Lat}_{n+1}\mathcal{K}$. Suppose that the second condition is fulfilled. Let r be any integer greater that n. By Lemma 2.10 there exist $U_1, \dots, U_r \in \mathcal{B}_Z$ such that $\{U_1, \dots, U_r\}'' = \mathcal{B}_Z$. After applying ϕ^{-1} on the last equation, we get $\{\phi^{-1}(U_1), \dots, \phi^{-1}(U_r)\}'' = \mathcal{A}_W$, a contradiction to Lemma 2.9. Thus, for every $W \in \mathrm{Lat}_{n-1}\mathcal{H}$ there exists (unique) $W' \in \mathrm{Lat}_{n-1}\mathcal{K}$ such that $\phi(\mathcal{A}_W) = \mathcal{A}_{W'}$. Consequently, ϕ induces a map $\tau : \mathrm{Lat}_{n-1}\mathcal{H} \to \mathrm{Lat}_{n-1}\mathcal{K}$, determined by

$$\phi(\mathcal{A}_W) = \mathcal{A}_{\tau(W)}, \quad W \in \mathrm{Lat}_{n-1}\mathcal{H},$$

or equivalently

$$W \subset U \iff \tau(W) \subset \phi(U), \quad W \in \mathrm{Lat}_{n-1}\mathcal{H},\ U \in \mathrm{Lat}_n\mathcal{H}. \tag{2.6}$$

We assert that τ is bijective. The injectivity follows directly from the injectivity of ϕ. In order to check the surjectivity choose an arbitrary $W' \in \mathrm{Lat}_{n-1}\mathcal{K}$. Since ϕ^{-1} has the same properties as ϕ, we get $\phi^{-1}(\mathcal{A}_{W'}) = \mathcal{A}_W$ for some $W \in \mathrm{Lat}_{n-1}\mathcal{H}$ and finally $\tau(W) = W'$. It now follows from (1.3) that the map $U \mapsto \psi(U^\perp)^\perp$, $U \in \mathrm{Lat}_n\mathcal{H}$, also maps the sets of the form $\mathcal{A}_W$, $W \in \mathrm{Lat}_{n-1}\mathcal{H}$, into such sets. This, together with

(2.2), implies that ψ induces a bijective map $\rho\colon \mathrm{Lat}_{-n+1}\mathcal{H} \to \mathrm{Lat}_{-n+1}\mathcal{K}$, determined by

$$\psi(\mathcal{B}_Z) = \mathcal{B}_{\rho(Z)}, \quad Z \in \mathrm{Lat}_{-n+1}\mathcal{H}.$$

Of course, the last condition has an equivalent formulation:

$$V \subset Z \iff \psi(V) \subset \rho(Z), \quad V \in \mathrm{Lat}_{-n}\mathcal{H},\ Z \in \mathrm{Lat}_{-n+1}\mathcal{H}. \tag{2.7}$$

We claim that for any pair $W \in \mathrm{Lat}_{n-1}\mathcal{H}$, $Z \in \mathrm{Lat}_{-n+1}\mathcal{H}$ we have

$$\{W, Z\} \in \mathcal{C}_{\mathcal{H}} \iff \forall U \in \mathcal{A}_W \exists V \in \mathcal{B}_Z : \{U, V\} \in \mathcal{C}_{\mathcal{H}}. \tag{2.8}$$

Assume first that $\{W, Z\} \in \mathcal{C}_{\mathcal{H}}$ and let $U \in \mathcal{A}_W$ be arbitrary. Then $U = W \oplus [z]$ for some $z \in Z \setminus \{0\}$. Hence, for $V = Z \ominus [z] \in \mathcal{B}_Z$ we have $U \oplus V = W \oplus ([z] \oplus V) = W \oplus Z = \mathcal{H}$. Suppose now that the second condition in (2.8) holds. Because W is finite-dimensional and Z has finite codimension in $\mathcal{H}$, one can choose $z \in Z \setminus W$. Let $U = W \oplus [z] \in \mathcal{A}_W$. Then there exists $V \in \mathcal{B}_Z$ such that $W \oplus ([z] \oplus V) = U \oplus V = \mathcal{H}$. Since $z \notin V$ and $V \in \mathcal{B}_Z$, we have $[z] \oplus V = Z$, so $W \oplus Z = \mathcal{H}$, as desired.

Equation (2.8) now yields that for any pair $W \in \mathrm{Lat}_{n-1}\mathcal{H}$, $Z \in \mathrm{Lat}_{-n+1}\mathcal{H}$ we have

$$\{W, Z\} \in \mathcal{C}_{\mathcal{H}} \iff \forall U \in \phi(\mathcal{A}_W) \exists V \in \psi(\mathcal{B}_Z) : \{U, V\} \in \mathcal{C}_{\mathcal{K}},$$

and furthermore,

$$\{W, Z\} \in \mathcal{C}_{\mathcal{H}} \iff \{\tau(W), \rho(Z)\} \in \mathcal{C}_{\mathcal{K}}.$$

By the induction hypothesis, there exists $A_n \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ such that $\tau(W) = A_n(W)$, $W \in \mathrm{Lat}_{n-1}\mathcal{H}$, and $\rho(Z) = A_n(Z)$, $Z \in \mathrm{Lat}_{-n+1}\mathcal{H}$. Now (2.6) and (2.7) imply that $\phi(U) = A_n(U)$, $U \in \mathrm{Lat}_n\mathcal{H}$, and $\psi(V) = A_n(V)$, $V \in \mathrm{Lat}_{-n}\mathcal{H}$, respectively. ∎

We are now ready to prove our main theorem.

Theorem 2.14 *Let $\phi, \psi\colon \mathrm{Lat}\,\mathcal{H} \to \mathrm{Lat}\,\mathcal{K}$ be bijective maps such that*

$$\{U, V\} \in \mathcal{C}_{\mathcal{H}} \iff \{\phi(U), \psi(V)\} \in \mathcal{C}_{\mathcal{K}}, \quad U, V \in \mathrm{Lat}\,\mathcal{H}.$$

Then there exist a sequence $(A_n)_{n\in\mathbb{Z}_\infty} \subset \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ and a regular partition $\mathbb{Z}_\infty = \mathbb{Z}_\infty(\mathrm{id}) \sqcup \mathbb{Z}_\infty(\perp)$ such that

- *if $U \in \mathrm{Lat}_n\mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\mathrm{id})$, then $\phi(U) = A_n(U)$ and $\psi(U) = A_{-n}(U)$;*
- *if $U \in \mathrm{Lat}_n\mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\perp)$, then $\phi(U) = A_n(U^\perp)$ and $\psi(U) = A_{-n}(U^\perp)$.*

Proof Our first goal is to show that ϕ and ψ map sets of the form $\mathrm{Lat}_n\mathcal{H}$, $n \in \mathbb{Z}_\infty$, into sets of the same form. Of course, this can be done in the easiest way for $n = 0$.

Indeed, observe that $U \in \mathrm{Lat}_0\mathcal{H}$ if and only if there is exactly one $V \in \mathrm{Lat}\,\mathcal{H}$ such that $\{U, V\} \in \mathcal{C}_{\mathcal{H}}$. Hence, $\phi(\mathrm{Lat}_0\mathcal{H}) = \mathrm{Lat}_0\mathcal{K}$ and $\psi(\mathrm{Lat}_0\mathcal{H}) = \mathrm{Lat}_0\mathcal{K}$. It is clear that either both ϕ and ψ act like the identity on the set $\mathrm{Lat}_0\mathcal{H}$ or they both act like orthogonal complementation. Thus, any $A_0 \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ satisfies the conclusion of the theorem.

We next claim that for any pair of distinct $U, V \in \mathrm{Lat}\,\mathcal{H}$, the following are equivalent:

- $\exists n \in \mathbb{Z}_\infty : U \in \mathrm{Lat}_n\mathcal{H},\ V \in \mathrm{Lat}_{-n}\mathcal{H}$;
- $\exists W, Z \in \mathrm{Lat}\,\mathcal{H} : \{U, W\} \in \mathcal{C}_\mathcal{H}, \{W, Z\} \in \mathcal{C}_\mathcal{H}, \{Z, V\} \in \mathcal{C}_\mathcal{H}$.

A trivial verification shows that the second condition implies the first one. Assume now that the first condition is satisfied. We distinguish three cases.

Case 1 $n = 0$. In this case one of U, V, say U, equals $\{0\}$ and the other one equals $\mathcal{H}$. So, the second condition is fulfilled for $W = \mathcal{H}$ and $Z = \{0\}$.

Case 2 $n \in \mathbb{Z} \setminus \{0\}$. Without loss of generality assume that $n > 0$. Set $Z = V^\perp \in \mathrm{Lat}_n\mathcal{H}$. We have to prove that U and Z have a common complement. Indeed, if $U \cap Z$ is non-trivial, choose its basis $B = \{w_1, \dots, w_r\}$. Otherwise set $r = 0$ and $B = \varnothing$. Now extend the set B to a basis of U with $u_{r+1}, \dots, u_n \in U$ and analogously with $z_{r+1}, \dots, z_n \in Z$ to get a basis of Z. One can easily verify that $W = [u_{r+1} + z_{r+1}, \dots, u_n + z_n] \oplus (U + Z)^\perp$ is a common complement of U and Z, as desired.

Case 3 $n = \infty$. In this case the second condition is proved in [6, Theorem 1.4]. ■

For any pair of distinct $U, V \in \mathrm{Lat}\,\mathcal{H}$ the following equivalence holds:

$$\exists n \in \mathbb{Z}_\infty \setminus \{0\} : U, V \in \mathrm{Lat}_n\mathcal{H}, \tag{2.9}$$
$$\Updownarrow$$
$$\exists W, Z, Y \in \mathrm{Lat}\,\mathcal{H} : \{U, W\} \in \mathcal{C}_\mathcal{H}, \{W, Z\} \in \mathcal{C}_\mathcal{H}, \{Z, Y\} \in \mathcal{C}_\mathcal{H}, \{Y, V\} \in \mathcal{C}_\mathcal{H}.$$

The second condition again trivially implies the first one. To get the other direction we apply the previous equivalence for the pair $U \in \mathrm{Lat}_n\mathcal{H}$ and $Y = V^\perp \in \mathrm{Lat}_{-n}\mathcal{H}$.

Consequently, for every $n \in \mathbb{Z}_\infty \setminus \{0\}$ there exists $m \in \mathbb{Z}_\infty \setminus \{0\}$ such that $\phi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_m\mathcal{K}$ and $\psi(\mathrm{Lat}_{-n}\mathcal{H}) = \mathrm{Lat}_{-m}\mathcal{K}$. We assert that either $m = n$ or $m = -n$. In order to prove this, choose arbitrary $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_\infty \setminus \{0\}$.

We claim that the following two statements are equivalent:

(a) $m \in \{-n, n\}$.

(b) For every pair of distinct $U, V \in \mathrm{Lat}_m\mathcal{H}$ there exists an adjacent chain between U and V of length at most $|n|$. Moreover, there exists a pair of distinct $U, V \in \mathrm{Lat}_m\mathcal{H}$ such that any adjacent chain between U and V has length at least $|n|$.

Proof We distinguish four cases.

Case 1 $m = |n|$. Let $U, V \in \mathrm{Lat}_{|n|}\mathcal{H}$ be distinct. Then $\dim U \ominus (U \cap V) = \dim V \ominus (U \cap V) = |n| - \dim(U \cap V)$. By Lemma 2.2, there exists an adjacent chain between U and V of length $|n| - \dim(U \cap V) \le |n|$. By the same lemma, any adjacent chain between U and V has length at least $|n|$ if $U \cap V = \{0\}$.

Case 2 $m = -|n|$. For any pair of distinct $U, V \in \mathrm{Lat}_{-|n|}\mathcal{H}$ we have $\dim U \ominus (U \cap V) = \dim V \ominus (U \cap V) = |n| - \dim(U^\perp \cap V^\perp)$. We finish the proof in the same way as in the previous case.

Case 3 $|m| < |n|$. Let $U, V \in \mathrm{Lat}_m\mathcal{H}$ be distinct. Suppose that $m < 0$, as the other case is analogous. By Lemma 2.2 one can find an adjacent chain between U

and V of length $|m| - \dim(U^\perp \cap V^\perp) \le |m| < |n|$. Thus, the second statement in (b) does not hold.

Case 4 $m = \infty$ or $|m| > |n|$. One can find $U, V \in \mathrm{Lat}_m\mathcal{H}$ such that $|n| < \dim U \ominus (U \cap V) = \dim V \ominus (U \cap V) < \infty$. By Lemma 2.2, any adjacent chain between U and V has length greater than $|n|$, contradicting the first statement in (b) and proving our claim.

Let n be a positive integer. Using Corollary 2.12 we now conclude that $\phi(\mathrm{Lat}_n\mathcal{H})$ equals either $\mathrm{Lat}_n\mathcal{K}$ or $\mathrm{Lat}_{-n}\mathcal{K}$. In the first case we must have also $\phi(\mathrm{Lat}_{-n}\mathcal{H}) = \mathrm{Lat}_{-n}\mathcal{K}$, $\psi(\mathrm{Lat}_{-n}\mathcal{H}) = \mathrm{Lat}_{-n}\mathcal{K}$, and $\psi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$, while in the other we have $\phi(\mathrm{Lat}_{-n}\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$, $\psi(\mathrm{Lat}_{-n}\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$ and $\psi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_{-n}\mathcal{K}$. Hence, there exists a regular partition $\mathbb{Z}_\infty = \mathbb{Z}_\infty(\mathrm{id}) \sqcup \mathbb{Z}_\infty(\perp)$ such that $\phi(\mathrm{Lat}_n\mathcal{H}) = \psi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$ if $n \in \mathbb{Z}_\infty(\mathrm{id})$, while $\phi(\mathrm{Lat}_n\mathcal{H}) = \psi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_{-n}\mathcal{K}$ if $n \in \mathbb{Z}_\infty(\perp)$. Instead of a pair (ϕ, ψ), consider now a pair $(\widetilde{\phi}, \widetilde{\psi})$, where $\widetilde{\phi}(U) = \phi(U)$, $\widetilde{\psi}(U) = \psi(U)$ if $U \in \mathrm{Lat}_n\mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\mathrm{id})$ and $\widetilde{\phi}(U) = \phi(U^\perp)$, $\widetilde{\psi}(U) = \psi(U^\perp)$ otherwise. After doing this we may assume that $\phi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$ and $\psi(\mathrm{Lat}_n\mathcal{H}) = \mathrm{Lat}_n\mathcal{K}$ for every $n \in \mathbb{Z}_\infty$.

Let again n be a positive integer. Then Proposition 2.13 guarantees the existence of $A_n \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ for which $\phi(U) = A_n(U)$, $U \in \mathrm{Lat}_n\mathcal{H}$, and $\psi(V) = A_n(V)$, $V \in \mathrm{Lat}_{-n}\mathcal{H}$. By (1.3), we can also apply Proposition 2.13 to a pair of maps $U \mapsto \phi(U^\perp)^\perp$, $U \in \mathrm{Lat}_n\mathcal{H}$, and $V \mapsto \psi(V^\perp)^\perp$, $V \in \mathrm{Lat}_{-n}\mathcal{H}$. Thus, there exists $B_n \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ such that $\phi(V) = B_n(V^\perp)^\perp$, $V \in \mathrm{Lat}_{-n}\mathcal{H}$, and $\psi(U) = B_n(U^\perp)^\perp$, $U \in \mathrm{Lat}_n\mathcal{H}$. By setting $A_{-n} := (B_n^*)^{-1}$, we therefore get $\phi(V) = A_{-n}(V)$, $V \in \mathrm{Lat}_{-n}\mathcal{H}$, and $\psi(U) = A_{-n}(U)$, $U \in \mathrm{Lat}_n\mathcal{H}$, as desired.

It remains to describe the behavior of ϕ and ψ on the set $\mathrm{Lat}_\infty \mathcal{H}$. Corollary 2.12 and Lemma 2.4 yield that for every $W \in \mathrm{Lat}_\infty \mathcal{H}$ there exists $W' \in \mathrm{Lat}_\infty \mathcal{K}$ so that either $\phi(\mathcal{A}_W) = \mathcal{A}_{W'}$ or $\phi(\mathcal{A}_W) = \mathcal{B}_{W'}$. We define an equivalence relation on the set of maximal adjacent subsets of $\mathrm{Lat}_\infty \mathcal{H}$ by $\mathcal{S}_1 \sim \mathcal{S}_2$ if and only if there exist $W_1, W_2 \in \mathrm{Lat}_\infty \mathcal{H}$ such that either $\{\mathcal{S}_1, \mathcal{S}_2\} = \{\mathcal{A}_{W_1}, \mathcal{A}_{W_2}\}$ or $\{\mathcal{S}_1, \mathcal{S}_2\} = \{\mathcal{B}_{W_1}, \mathcal{B}_{W_2}\}$. Our next aim is to show that ϕ and ψ preserve this relation; that is

$$\phi(\mathcal{S}_1) \sim \phi(\mathcal{S}_2) \iff \mathcal{S}_1 \sim \mathcal{S}_2 \iff \psi(\mathcal{S}_1) \sim \psi(\mathcal{S}_2).$$

In order to do this we start with some observations. We claim that for $W, Z \in \mathrm{Lat}_\infty \mathcal{H}$ we have

(a) $\{W, Z\} \in \mathcal{C}_\mathcal{H} \iff \forall U \in \mathcal{A}_W \exists V \in \mathcal{B}_Z : \{U, V\} \in \mathcal{C}_\mathcal{H}$,
(b) $\{W, Z\} \in \mathcal{C}_\mathcal{H} \iff \forall U \in \mathcal{B}_W \exists V \in \mathcal{A}_Z : \{U, V\} \in \mathcal{C}_\mathcal{H}$,
(c) $\exists U \in \mathcal{A}_W \forall V \in \mathcal{A}_Z : \{U, V\} \notin \mathcal{C}_\mathcal{H}$,
(d) $\exists U \in \mathcal{B}_W \forall V \in \mathcal{B}_Z : \{U, V\} \notin \mathcal{C}_\mathcal{H}$.

Let us prove (a). The first condition implies the second in the same way as (2.8) in Proposition 2.13. Assume now that the second condition holds. We can again copy the proof of (2.8), with the only difference being that we have to change the proof of the existence of $z \in Z \setminus W$. Assume that such a z does not exist, that is $Z \subset W$. Then for any $U \in \mathcal{A}_W$ and $V \in \mathcal{B}_Z$ we have $V \subset U$. Thus, for such a pair we can have $\{U, V\} \in \mathcal{C}_\mathcal{H}$ only if $V = \{0\}$ and $U = \mathcal{H}$, a contradiction with $W, Z \in \mathrm{Lat}_\infty \mathcal{H}$.

It follows from (a) and (1.2) that

$$\begin{aligned}\{W,Z\}\in\mathcal{C}_{\mathcal{H}} &\iff \{W^{\perp},Z^{\perp}\}\in\mathcal{C}_{\mathcal{H}}\\ &\iff \forall U\in\mathcal{A}_{W^{\perp}}\,\exists V\in\mathcal{B}_{Z^{\perp}}:\{U^{\perp},V^{\perp}\}\in\mathcal{C}_{\mathcal{H}}.\end{aligned}$$

To complete the proof of (b) we use (2.2). We further choose $U\in\mathcal{A}_W$ with $U\cap Z\neq\{0\}$ to prove (c), and $U\in\mathcal{B}_W$ with $U+Z\neq\mathcal{H}$ to prove (d).

It follows from (a)–(d) that $\phi(\mathcal{A}_{W_1})\not\sim\psi(\mathcal{B}_{W_2})$ for any pair $W_1,W_2\in\operatorname{Lat}_\infty\mathcal{H}$ with $\{W_1,W_2\}\in\mathcal{C}_{\mathcal{H}}$. Now let $W_1,W_2\in\operatorname{Lat}_\infty\mathcal{H}$ be arbitrary. By (2.9) and the previous statement, there exist $W_3,W_4,W_5\in\operatorname{Lat}_\infty\mathcal{H}$ such that

$$\phi(\mathcal{A}_{W_1})\not\sim\psi(\mathcal{B}_{W_3})\not\sim\phi(\mathcal{A}_{W_4})\not\sim\psi(\mathcal{B}_{W_5})\not\sim\phi(\mathcal{A}_{W_2}).$$

Hence, $\phi(\mathcal{A}_{W_1})\sim\phi(\mathcal{A}_{W_2})$, and similarly $\phi(\mathcal{B}_{W_1})\sim\phi(\mathcal{B}_{W_2})$, $\psi(\mathcal{A}_{W_1})\sim\psi(\mathcal{A}_{W_2})$, and $\psi(\mathcal{B}_{W_1})\sim\psi(\mathcal{B}_{W_2})$, as desired.

We may now assume that for any $W,Z\in\operatorname{Lat}_\infty\mathcal{H}$ there exist $W',Z'\in\operatorname{Lat}_\infty\mathcal{K}$ such that $\phi(\mathcal{A}_W)=\mathcal{A}_{W'}$ and $\psi(\mathcal{B}_Z)=\mathcal{B}_{Z'}$, respectively (otherwise we consider a pair of maps $(U,V)\mapsto(\phi(U^{\perp}),\psi(V^{\perp}))$, $U,V\in\operatorname{Lat}_\infty\mathcal{H}$, instead of a pair (ϕ,ψ)). Consequently, ϕ and ψ induce bijective maps $\tau,\rho\colon\operatorname{Lat}_\infty\mathcal{H}\to\operatorname{Lat}_\infty\mathcal{K}$, determined by

$$\phi(\mathcal{A}_W)=\mathcal{A}_{\tau(W)},W\in\operatorname{Lat}_\infty\mathcal{H},\quad\text{and}\quad\psi(\mathcal{B}_Z)=\mathcal{B}_{\rho(Z)},Z\in\operatorname{Lat}_\infty\mathcal{H},\tag{2.10}$$

respectively. It follows from (a) that

$$\{W,Z\}\in\mathcal{C}_{\mathcal{H}}\iff\{\tau(W),\rho(Z)\}\in\mathcal{C}_{\mathcal{K}},\quad W,Z\in\operatorname{Lat}_\infty\mathcal{H}.\tag{2.11}$$

Choose any $W,Z\in\operatorname{Lat}_\infty\mathcal{H}$ with $\{W,Z\}\in\mathcal{C}_{\mathcal{H}}$. Then any element of $\mathcal{A}_W$ can be uniquely written as $W\oplus[z]$, where $[z]\in\operatorname{Lat}_1 Z$. By (2.11), we also have $\{\tau(W),\rho(Z)\}\in\mathcal{C}_{\mathcal{K}}$. Therefore, it follows from (2.10) that any element of $\phi(\mathcal{A}_W)$ can be uniquely written as $\tau(W)\oplus[z']$, where $[z']\in\operatorname{Lat}_1\rho(Z)$. Hence, ϕ induces a bijective map $\xi\colon\operatorname{Lat}_1 Z\to\operatorname{Lat}_1\rho(Z)$, determined by $\phi(W\oplus[z])=\tau(W)\oplus\xi([z])$, $[z]\in\operatorname{Lat}_1 Z$. One can easily show that (2.11) implies

$$\{[z],V\}\in\mathcal{C}_Z\iff\{\xi([z]),\psi(V)\}\in\mathcal{C}_{\rho(Z)},\quad[z]\in\operatorname{Lat}_1 Z,\ V\in\mathcal{B}_Z.$$

We use Proposition 2.13 for $n=1$ and Hilbert spaces Z and $\rho(Z)$ to establish that there exists $A_{W,Z}\in\operatorname{BCI}(Z,\rho(Z))$ such that

$$\xi([z])=A_{W,Z}([z]),\ [z]\in\operatorname{Lat}_1 Z,\quad\text{and}\quad\psi(V)=A_{W,Z}(V),\ V\in\mathcal{B}_Z.$$

We claim that the last equation yields that $A_{W,Z}$ is independent of W up to a multiplication with nonzero scalars. Indeed, let $A\in\operatorname{BCI}(Z,\rho(Z))$ be some other operator satisfying $\psi(V)=A(V)$, $V\in\mathcal{B}_Z$. Because any $[z]\in\operatorname{Lat}_1 Z$ is an intersection of closed hyperplanes in Z, it follows that $A_{W,Z}([z])=A([z])$, $[z]\in\operatorname{Lat}_1 Z$. A rather

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easy argument (see *e.g.*, [4, Lemma 2.4]) shows that $A = \lambda A_{W,Z}$ for some $\lambda \in \mathbb{F} \setminus \{0\}$, as desired.

Thus, for every $Z \in \operatorname{Lat}_\infty \mathcal{H}$ there exists $A_Z \in \operatorname{BCI}(Z, \rho(Z))$ such that for any $W \in \operatorname{Lat}_\infty \mathcal{H}$ with $\{W, Z\} \in \mathcal{C}_\mathcal{H}$ we have

$$\begin{gathered} \phi(W \oplus [z]) = \tau(W) \oplus A_Z([z]),\ [z] \in \operatorname{Lat}_1 Z, \\ \psi(V) = A_Z(V),\ V \in \mathcal{B}_Z. \end{gathered} \tag{2.12}$$

Our next aim is to show that for any pair of distinct $Z_1, Z_2 \in \operatorname{Lat}_\infty \mathcal{H}$ with $Z_1 \cap Z_2 \neq \{0\}$ there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that

$$A_{Z_2}|_{Z_1 \cap Z_2} = \lambda A_{Z_1}|_{Z_1 \cap Z_2}. \tag{2.13}$$

Let $z \in (Z_1 \cap Z_2) \setminus \{0\}$. Using the same argument as when proving that $A_{W,Z}$ is independent of W, we conclude that it is enough to show that $[A_{Z_1}(z)] = [A_{Z_2}(z)]$. We will distinguish two cases.

Case 1 There exists $W \in \operatorname{Lat}_\infty \mathcal{H}$ such that $\{W, Z_1\}, \{W, Z_2\} \in \mathcal{C}_\mathcal{H}$. By (2.12), $\tau(W) \oplus A_{Z_1}([z]) = \tau(W) \oplus A_{Z_2}([z])$. Consequently, there exists $\alpha \in \mathbb{F} \setminus \{0\}$ such that $w := A_{Z_1}(z) + \alpha A_{Z_2}(z) \in \tau(W)$. This, together with (2.11), shows that for every $U \in \operatorname{Lat}_\infty \mathcal{K}$ with

$$\{U, \rho(Z_1)\} \in \mathcal{C}_\mathcal{K} \quad \text{and} \quad \{U, \rho(Z_2)\} \in \mathcal{C}_\mathcal{K} \tag{2.14}$$

there exists $\alpha_U \in \mathbb{F} \setminus \{0\}$ such that $A_{Z_1}(z) + \alpha_U A_{Z_2}(z) \in U$.

If $w = 0$, we are done, so assume the opposite. We will show that this assumption yields a contradiction. We start by providing some technical tools.

Note that $\rho(Z_j) \oplus (\tau(W) \ominus [w])$, $j = 1, 2$, are hyperplanes in $\mathcal{K}$. We claim that there exist linearly independent $x, y \in \mathcal{K}$ such that

(i) $x, y \notin \rho(Z_j) \oplus (\tau(W) \ominus [w])$, $j = 1, 2$,
(ii) $[x, y] \cap \tau(W) = \{0\}$.

Indeed, note that $\rho(Z_1)$ and $\rho(Z_2)$ are different and have a common complement $\tau(W)$. Hence, they are not comparable. Choose $z_1 \in \rho(Z_1) \setminus \rho(Z_2)$ and $z_2 \in \rho(Z_2) \setminus \rho(Z_1)$ such that z_1, z_2, w are linearly independent. Then we have $z_j + w \notin \rho(Z_j) \oplus (\tau(W) \ominus [w])$, $j = 1, 2$. Furthermore, we can choose z_1 in such a way that $z_1 + w \notin \rho(Z_2) \oplus (\tau(W) \ominus [w])$. Indeed, if this was not true, $\rho(Z_2) \oplus (\tau(W) \ominus [w])$ would contain $z_1 + w$, $-z_1 + w$, and consequently w. This would contradict $\rho(Z_2) \oplus \tau(W) = \mathcal{K}$. Symmetrically, we can choose z_2 such that $z_2 + w \notin \rho(Z_1) \oplus (\tau(W) \ominus [w])$. Thus, $x = z_1 + w$ and $y = z_2 + w$ satisfy (i).

Turn now to the proof of (ii). Let z_1 and z_2 be as before. We can find $z_1' \in \rho(Z_1) \setminus \rho(Z_2)$ such that z_1 and z_1' are linearly independent and so are z_1', z_2, w. After multiplying z_1' with -1, if necessary, we may again assume that a pair $z_1' + w$, $z_2 + w$ satisfies (i). Then one can easily check that not both, $[z_1, z_2]$ and $[z_1', z_2]$ can intersect $\tau(W)$ non-trivially. So, we may suppose that $[z_1, z_2] \cap \tau(W) = \{0\}$, otherwise we replace z_1 by z_1'. Since $w \in \tau(W)$, we also have $[z_1 + w, z_2 + w] \cap \tau(W) = \{0\}$, as desired.

Let x, y be as in (i) and (ii). Set $U = (\tau(W) \ominus [w]) \oplus [x]$ and $V = (\tau(W) \ominus [w]) \oplus [y]$. Then $U, V \in \operatorname{Lat}_\infty \mathcal{K}$ are distinct and both satisfy (2.14). Hence,

$$A_{Z_1}(z) + \beta A_{Z_2}(z) \in U \quad \text{and} \quad A_{Z_1}(z) + \gamma A_{Z_2}(z) \in V$$

for some $\beta, \gamma \in \mathbb{F} \setminus \{0\}$. Therefore, there exist $w_1, w_2 \in \tau(W) \ominus [w]$ and $\lambda, \mu \in \mathbb{F}$ such that

$$w + (\beta - \alpha) A_{Z_2}(z) = A_{Z_1}(z) + \beta A_{Z_2}(z) = w_1 + \lambda x$$

and

$$w + (\gamma - \alpha) A_{Z_2}(z) = A_{Z_1}(z) + \gamma A_{Z_2}(z) = w_2 + \mu y.$$

Furthermore,

$$(\gamma - \beta) w - (\gamma - \alpha) w_1 + (\beta - \alpha) w_2 = (\gamma - \alpha)\lambda x - (\beta - \alpha)\mu y \in [x, y] \cap \tau(W).$$

By (ii),

$$(\gamma - \beta) w = (\gamma - \alpha) w_1 - (\beta - \alpha) w_2 \in \tau(W) \ominus [w] \subset [w]^\perp.$$

Thus, $\gamma = \beta$, which yields $w_1 + \lambda x = w_2 + \mu y$. Hence, $w_1 - w_2 = \mu y - \lambda x \in [x, y] \cap \tau(W) = \{0\}$. Consequently, $w_1 = w_2$ and $\lambda = \mu = 0$, because x and y are linearly independent. Moreover, $(\beta - \alpha) A_{Z_2}(z) = w_1 - w \in \rho(Z_2) \cap \tau(W) = \{0\}$. Finally, $w = w_1 \in [w]^\perp$ implies $w = 0$, a contradiction.

Case 2 Z_1 and Z_2 are arbitrary. Equation (2.9), applied to the Hilbert space $[z]^\perp$ yields that there exist $W_1, W_2, W_3 \in \operatorname{Lat}_\infty \mathcal{H}$ such that $(Z_1 \ominus [z]) \oplus W_1 = W_1 \oplus W_2 = W_2 \oplus W_3 = W_3 \oplus (Z_2 \ominus [z]) = [z]^\perp$. For $W = W_2 \oplus [z]$ we therefore have $\{Z_1, W_1\}, \{W_1, W\}, \{W, W_3\}, \{W_3, Z_2\} \in \mathcal{C}_\mathcal{H}$. It follows from Case 1 that $[A_{Z_1}(z)] = [A_W(z)] = [A_{Z_2}(z)]$, as desired.

We now define the operator $A_\infty \colon \mathcal{H} \to \mathcal{K}$ in the following way. Fix $Z_0 \in \operatorname{Lat}_\infty \mathcal{H}$ and an operator A_{Z_0} from (2.12). For $x \in Z_0$ we define $A_\infty x := A_{Z_0} x$. For $x \notin Z_0$ set $A_\infty x := A_Z x$, where we choose Z and A_Z such that $x \in Z$, $Z \cap Z_0 \neq \{0\}$, and $A_Z|_{Z \cap Z_0} = A_{Z_0}|_{Z \cap Z_0}$ (we achieve the latter by scalar multiplication if neccessary). It follows from (2.13) that the definition is independent of the choice of Z and that either all the operators A_Z, $Z \in \operatorname{Lat}_\infty \mathcal{H}$, are linear or all are conjugate linear. Consequently, A_∞ is linear or conjugate linear. It is easy to see that A_∞ is bijective, and it is bounded because the restriction of A_∞ to any $Z \in \operatorname{Lat}_\infty \mathcal{H}$ is bounded. It remains to show that $\phi(U) = \psi(U) = A_\infty(U)$, $U \in \operatorname{Lat}_\infty \mathcal{H}$. So, choose an arbitrary $U \in \operatorname{Lat}_\infty \mathcal{H}$. Then one can find $Z \in \operatorname{Lat}_\infty \mathcal{H}$ such that $U \in \mathcal{B}_Z$ and $Z \cap Z_0 \neq \{0\}$. By (2.12), $\psi(U) = A_Z(U) = A_\infty(U)$. Therefore,

$$\begin{aligned}\{V' \in \operatorname{Lat}_\infty \mathcal{K} : \{\phi(U), V'\} \in \mathcal{C}_\mathcal{K}\} &= \{A_\infty(V) : V \in \operatorname{Lat}_\infty \mathcal{H}, \{U, V\} \in \mathcal{C}_\mathcal{H}\} \\ &= \{V' \in \operatorname{Lat}_\infty \mathcal{K} : \{A_\infty(U), V'\} \in \mathcal{C}_\mathcal{K}\},\end{aligned}$$

and consequently, $\phi(U) = A_\infty(U)$. ■

3 Maps on $\mathcal{I}(\mathcal{H})$

Let n be an integer. Set

$$\mathcal{I}_n(\mathcal{H}) = \begin{cases} \{P \in \mathcal{I}(\mathcal{H}) : \operatorname{rank} P = n\} & \text{if } n > 0, \\ \{P \in \mathcal{I}(\mathcal{H}) : \dim \ker P = -n\} & \text{if } n < 0, \\ \{0, I\} & \text{if } n = 0, \end{cases}$$

and

$$\mathcal{I}_\infty(\mathcal{H}) = \{P \in \mathcal{I}(\mathcal{H}) : \dim \operatorname{im} P = \infty \text{ and } \dim \ker P = \infty\}.$$

Theorem 3.1 *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces and let $\varphi\colon \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{K})$ be a bijective map such that for all $P, Q \in \mathcal{I}(\mathcal{H})$ we have*

$$\operatorname{im} P = \operatorname{im} Q \iff \operatorname{im} \varphi(P) = \operatorname{im} \varphi(Q),$$
$$\ker P = \ker Q \iff \ker \varphi(P) = \ker \varphi(Q).$$

Then there exist a sequence $(A_n)_{n \in \mathbb{Z}_\infty} \subset \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ and a regular partition $\mathbb{Z}_\infty = \mathbb{Z}_\infty(\mathrm{id}) \sqcup \mathbb{Z}_\infty(\perp)$ such that

- *if $P \in \mathcal{I}_n(\mathcal{H})$ for $n \in \mathbb{Z}_\infty(\mathrm{id})$, then $\varphi(P) = A_n P A_n^{-1}$;*
- *if $P \in \mathcal{I}_n(\mathcal{H})$ for $n \in \mathbb{Z}_\infty(\perp)$, then $\varphi(P) = A_n (I - P^*) A_n^{-1}$.*

Proof For $U \in \operatorname{Lat} \mathcal{H}$ let P_U be the projection onto U. We will show that the pair of maps $\phi, \psi\colon \operatorname{Lat} \mathcal{H} \to \operatorname{Lat} \mathcal{K}$, $\phi(U) = \operatorname{im} \varphi(P_U)$, $\psi(U) = \ker \varphi(I - P_U)$, satisfies the assumptions of Theorem 2.14. Indeed, it is easy to verify that they are bijective. Let $\mathcal{H} = U \oplus V$ for some $U, V \in \operatorname{Lat} \mathcal{H}$. Denote by P the idempotent operator whose image is U and whose kernel is V. It follows from $\operatorname{im} P_U = \operatorname{im} P$ and $\ker(I - P_V) = \operatorname{im} P_V = \ker P$ that $\phi(U) = \operatorname{im} \varphi(P_U) = \operatorname{im} \varphi(P)$ and $\psi(V) = \ker \varphi(I - P_V) = \ker \varphi(P)$, and consequently, $\mathcal{K} = \phi(U) \oplus \psi(V)$. If on the other hand, $\mathcal{K} = \phi(U) \oplus \psi(V)$ for some $U, V \in \operatorname{Lat} \mathcal{H}$, Q is the idempotent operator whose image is $\phi(U)$ and whose kernel is $\psi(V)$, and $P = \varphi^{-1}(Q)$, then $\operatorname{im} \varphi(P) = \operatorname{im} Q = \phi(U) = \operatorname{im} \varphi(P_U)$ and $\ker \varphi(P) = \ker Q = \psi(V) = \ker \varphi(I - P_V)$, and consequently, $\operatorname{im} P = U$ and $\ker P = \ker(I - P_V) = V$. Thus, $\mathcal{H} = U \oplus V$.

Therefore, by Theorem 2.14 there exist a sequence $(A_n)_{n \in \mathbb{Z}_\infty} \subset \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ and a regular partition $\mathbb{Z}_\infty = \mathbb{Z}_\infty(\mathrm{id}) \sqcup \mathbb{Z}_\infty(\perp)$ such that

- if $U \in \operatorname{Lat}_n \mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\mathrm{id})$, then $\phi(U) = A_n(U)$ and $\psi(U) = A_{-n}(U)$;
- if $U \in \operatorname{Lat}_n \mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\perp)$, then $\phi(U) = A_n(U^\perp)$ and $\psi(U) = A_{-n}(U^\perp)$.

First let P be an idempotent with $P \in \mathcal{I}_n(\mathcal{H})$ for some $n \in \mathbb{Z}_\infty(\mathrm{id})$, or equivalently, its image U belongs to $\operatorname{Lat}_n \mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\mathrm{id})$, and then its kernel V belongs to $\operatorname{Lat}_{-n} \mathcal{H}$. On one hand we know that $\operatorname{im} \varphi(P) = \operatorname{im} \varphi(P_U) = \phi(U)$ and $\ker \varphi(P) = \ker \varphi(I - P_V) = \psi(V)$. On the other hand, for every $y \in \phi(U) = A_n(U)$ we have $y = A_n u$ for some $u \in U$, and therefore, $A_n P A_n^{-1} y = y$, while for every $z \in \psi(V) = A_n(V)$ we have $z = A_n v$ for some $v \in V$, and consequently, $A_n P A_n^{-1} z = A_n P v = 0$. Thus, $\varphi(P) = A_n P A_n^{-1}$, as desired.

Assume finally that P is an idempotent whose image U belongs to $\operatorname{Lat}_n \mathcal{H}$ for some $n \in \mathbb{Z}_\infty(\perp)$. Then for every $y \in \operatorname{im} \varphi(P) = \phi(U) = A_n(U^\perp)$ we have $y =$

$A_n u(\perp)$ for some $u(\perp) \in U^\perp$, and therefore, $A_n(I-P^*)A_n^{-1}y = A_n(I-P^*)u(\perp) = A_n u(\perp) = y$. And for every $z \in \ker\varphi(P) = \psi(V) = A_n(V^\perp)$ there is $v(\perp) \in V^\perp$ such that $z = A_n v(\perp)$, and consequently, $A_n(I-P^*)A_n^{-1}z = 0$. It follows that in this case we have $\varphi(P) = A_n(I-P^*)A_n^{-1}$. ∎

Of course, the dual form of this statement reads as follows.

Theorem 3.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces and let $\varphi\colon \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{K})$ be a bijective map such that for all $P, Q \in \mathcal{I}(\mathcal{H})$ we have*

$$\operatorname{im} P = \operatorname{im} Q \iff \ker\varphi(P) = \ker\varphi(Q)$$

and

$$\ker P = \ker Q \iff \operatorname{im}\varphi(P) = \operatorname{im}\varphi(Q).$$

Then there exist a sequence $(A_n)_{n\in\mathbb{Z}_\infty} \subset \mathrm{BCI}(\mathcal{H},\mathcal{K})$ and a regular partition $\mathbb{Z}_\infty = \mathbb{Z}_\infty(\mathrm{id}) \sqcup \mathbb{Z}_\infty(\perp)$ such that

- *if $P \in \mathcal{I}_n(\mathcal{H})$ for $n \in \mathbb{Z}_\infty(\mathrm{id})$, then $\varphi(P) = A_n P^* A_n^{-1}$;*
- *if $P \in \mathcal{I}_n(\mathcal{H})$ for $n \in \mathbb{Z}_\infty(\perp)$, then $\varphi(P) = A_n(I-P)A_n^{-1}$.*

The proof is almost the same as that of Theorem 3.1. Another possible proof is to compose φ with the map $P \mapsto P^*$ and then apply Theorem 3.1.

It is now easy to reprove the main result from [11]. The first part of the proof is based on the same arguments as in [11].

Theorem 3.3 *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces and let $\varphi\colon \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{K})$ be a bijective map such that for all $P, Q \in \mathcal{I}(\mathcal{H})$ we have*

$$P \le Q \iff \varphi(P) \le \varphi(Q).$$

Then there exists $A \in \mathrm{BCI}(\mathcal{H},\mathcal{K})$ such that either

$$\varphi(P) = APA^{-1}, \quad P \in \mathcal{I}(\mathcal{H}), \quad \text{or} \quad \varphi(P) = AP^*A^{-1}, \quad P \in \mathcal{I}(\mathcal{H}).$$

Proof We first introduce some notation. For every pair of nonzero vectors $x, y \in \mathcal{H}$ we denote by $x \otimes y^*$ the rank one operator from $\mathcal{H}$ into itself defined by $(x\otimes y^*)z = \langle z, y\rangle x$. Every rank one operator can be written in this form. Clearly, the image of this operator is the one-dimensional subspace $[x]$, and its kernel is the orthogonal complement of y, $y^\perp = \{z \in \mathcal{H} : \langle z, y\rangle = 0\}$. It is trivial to verify that $x \otimes y^*$ is an idempotent if and only if $\langle x, y\rangle = 1$.

Two rank one idempotents $x\otimes y^*$ and $u \otimes w^*$ have the same image if and only if x and u are linearly dependent, while they have the same kernel if and only if y and w are linearly dependent. We write $x\otimes y^* \approx u\otimes w^*$ if and only if these two rank one idempotents have the same image or the same kernel. Let x and y be nonzero vectors in $\mathcal{H}$. We denote by $L(x)$ the set of all rank one idempotents whose image is $[x]$, that is,

$$L(x) = \{x\otimes w^* : w \in \mathcal{H} \text{ and } \langle x, w\rangle = 1\},$$

and by $R(y)$ the set of all rank one idempotents whose kernel is $y^\perp$, that is,

$$R(y) = \{u \otimes y^* : u \in \mathcal{H} \text{ and } \langle u, y\rangle = 1\}.$$

Let $\mathcal{M}$ be a subset of $\mathcal{I}_1(\mathcal{H})$. We say that $\mathcal{M}$ is a $\approx$-set if for any two elements $P, Q \in \mathcal{M}$ we have $P \approx Q$. We claim that if $\mathcal{M}$ is a $\approx$-set, then either there exists a nonzero $x \in \mathcal{H}$ such that $\mathcal{M} \subset L(x)$, or there exists a nonzero $y \in \mathcal{H}$ such that $\mathcal{M} \subset R(y)$. There is nothing to prove if $\mathcal{M}$ is a singleton. So, assume that $\mathcal{M}$ contains at least two distinct elements $x \otimes y^*$ and $u \otimes w^*$. They have either the same image or the same kernel. We will consider just the second possibility. Then, after absorbing the constant in the tensor product, if necessary, we may assume that $y = w$. Because $x \otimes y^* \neq u \otimes y^*$ and $\langle x, y\rangle = \langle u, y\rangle = 1$, the vectors x and u are linearly independent. In other words, $x \otimes y^*$ and $u \otimes y^*$ have distinct images. It follows that any projection $P \in \mathcal{M}$ has the same kernel as $x \otimes y^*$ and $u \otimes y^*$, and therefore $P \in R(y)$. We have proved that $\mathcal{M} \subset R(y)$.

Clearly, $L(x)$ and $R(y)$ are $\approx$-sets. Hence, for every pair of nonzero vectors $x, y \in \mathcal{H}$ the sets $L(x)$ and $R(y)$ are maximal $\approx$-sets, and each $\approx$-set is contained either in some $L(x)$, or in some $R(y)$. A direct consequence is that every bijective map $\tau\colon \mathcal{I}_1(\mathcal{H}) \to \mathcal{I}_1(\mathcal{K})$ with the property that for every pair $P, Q \in \mathcal{I}_1(\mathcal{H})$ we have $P \approx Q \iff \tau(P) \approx \tau(Q)$ has the following property: for each nonzero $x \in \mathcal{H}$ either there exists a nonzero $u \in \mathcal{K}$ such that $\tau(L(x)) = L(u)$, or there exists a nonzero $y \in \mathcal{K}$ such that $\tau(L(x)) = R(y)$.

Since the zero idempotent is the least element of $\mathcal{I}(\mathcal{H})$ we have $\varphi(0) = 0$. Clearly, rank one idempotents are minimal nonzero elements, and, therefore, φ maps the set of all rank one idempotents on $\mathcal{H}$ bijectively onto the set of all rank one idempotents on $\mathcal{K}$, and similarly, $\varphi(\mathcal{I}_2(\mathcal{H})) = \mathcal{I}_2(\mathcal{K})$. In fact, we have $\varphi(\mathcal{I}_n(\mathcal{H})) = \mathcal{I}_n(\mathcal{K})$ for every $n \in \mathbb{Z}_\infty$.

In the next step we will show that for any pair $P, Q \in \mathcal{I}_1(\mathcal{H})$ the following two statements are equivalent:

- $P \approx Q$;
- there exist infinitely many idempotents $R \in \mathcal{I}_2(\mathcal{H})$ such that $P \leq R$ and $Q \leq R$.

Indeed, assume first that $P \approx Q$. Then either P and Q have the same images, or they have the same kernels. We will treat only one of these two cases, say the first one. Then $P = x \otimes y^*$ and $Q = x \otimes w^*$ for some $x, y, w \in \mathcal{H}$ with $\langle x, y\rangle = 1$ and $\langle x, w\rangle = 1$. Obviously, there are infinitely many idempotents $R \in \mathcal{I}_2(\mathcal{H})$ such that $P \leq R$ and $Q \leq R$ when $P = Q$. So, assume that $P \neq Q$. Then $w = y + u$ for some nonzero $u \in x^\perp$. In particular, y and u are linearly independent, and therefore we can find infinitely many vectors $z \in H$ satisfying

$$\langle z, y\rangle = 0 \quad \text{and} \quad \langle z, u\rangle = 1.$$

Because $\langle x, y\rangle = 1$, the vectors x and z are linearly independent. It follows that $R_z = x \otimes y^* + z \otimes u^*$ is of rank two. Straightforward computations show that

$$R_z^2 = R_z, \quad R_zP = PR_z = P, \quad \text{and} \quad R_zQ = QR_z = Q,$$

which completes the one direction in the proof of the above equivalence.

To prove the other direction assume that $P \not\approx Q$ and that $P \leq R$, $Q \leq R$, $P \leq S$, and $Q \leq S$ for some rank two idempotents R and S. In particular, we have $\operatorname{im} P \subset \operatorname{im} R$, $\operatorname{im} Q \subset \operatorname{im} R$, $\operatorname{im} P \subset \operatorname{im} S$, and $\operatorname{im} Q \subset \operatorname{im} S$. As $\operatorname{im} P$ and $\operatorname{im} Q$ are linearly independent one-dimensional subspaces of a two-dimensional subspace $\operatorname{im} R$, we have $\operatorname{im} R = \operatorname{im} P \oplus \operatorname{im} Q$, and, of course, $\operatorname{im} S = \operatorname{im} P \oplus \operatorname{im} Q$. Similarly, $\ker R = \ker P \cap \ker Q = \ker S$. As R and S have the same images and the same kernels, we conclude that $R = S$, as desired.

The above characterization of $\approx$-pairs of idempotents of rank one and the above statement concerning the map τ imply that for each nonzero $x \in \mathcal{H}$ either there exists a nonzero $u \in \mathcal{K}$ such that $\varphi(L(x)) = L(u)$, or there exists a nonzero $y \in \mathcal{K}$ such that $\varphi(L(x)) = R(y)$. Assume that there is a nonzero $x \in \mathcal{H}$ such that $\varphi(L(x)) = R(y)$ for some nonzero $y \in \mathcal{K}$ (the case when $\varphi(L(x)) = L(u)$ for some nonzero $u \in \mathcal{K}$ can be treated in the same way). We claim that then for each nonzero $u \in \mathcal{H}$ there is a nonzero $w \in \mathcal{K}$ such that

$$\varphi(L(u)) = R(w) \tag{3.1}$$

and for each nonzero $z \in \mathcal{H}$ there is a nonzero $v \in \mathcal{K}$ such that

$$\varphi(R(z)) = L(v). \tag{3.2}$$

Clearly, we may assume that u and x are linearly independent. We know that either $\varphi(L(u)) = R(w)$ for some nonzero w, or $\varphi(L(u)) = L(w)$ for some nonzero w. We need to show that the second possibility cannot occur. Assume on the contrary that we have the second possibility. We choose a vector $a \in \mathcal{H}$ such that $\langle a, x\rangle = 1 = \langle a, u\rangle$. Then $L(x) \cap R(a) = \{x \otimes a^*\}$ and $L(u) \cap R(a) = \{u \otimes a^*\}$. It follows that both sets

$$R(y) \cap \phi(R(a)) \quad \text{and} \quad L(w) \cap \phi(R(a))$$

are singletons. And we further know that either $\phi(R(a)) = L(b)$ for some nonzero $b \in \mathcal{K}$, or $\phi(R(a)) = R(b)$ for some nonzero $b \in \mathcal{K}$. In the first case we have $L(w) \cap \phi(R(a)) = L(w) \cap L(b)$, and $L(w) \cap L(b)$ is either empty (when w and b are linearly independent), or $L(w) = L(b)$ (when w and b are linearly dependent). This contradicts the fact that $L(w) \cap \phi(R(a))$ is a singleton. Similarly, we get a contradiction in the second case. This proves (3.1). The proof of (3.2) is now trivial.

We next claim that for an arbitrary nonzero idempotent $P \in \mathcal{I}(\mathcal{H})$ and an arbitrary nonzero $x \in \mathcal{H}$ the following are equivalent:

- $x \in \operatorname{im} P$;
- there exists $Q \in L(x)$ such that $Q \leq P$.

Indeed, if $x \in \operatorname{im} P$, then $x \notin (\operatorname{im} P^*)^{\perp} = \ker P$. Therefore, we can find $y \in \operatorname{im} P^*$ such that $\langle x, y\rangle = 1$. It follows that $Q = x \otimes y^* \leq P$. If, on the other hand, $Q = x \otimes u^* \in \mathcal{I}_1(\mathcal{H})$ satisfies $Q \leq P$, then $PQ = Q$ yields that $x \in \operatorname{im} P$. Similarly, we see that for an arbitrary nonzero idempotent $P \in \mathcal{I}(\mathcal{H})$ and arbitrary nonzero $y \in \mathcal{H}$ we have $y \in (\ker P)^{\perp}$ if and only if there exists $Q \in R(y)$ such that $Q \leq P$.

This, together with the above obtained properties of the map φ, yields that for all $P, Q \in \mathcal{I}(\mathcal{H})$ we have

$$\begin{aligned}\operatorname{im} P = \operatorname{im} Q &\iff \ker \varphi(P) = \ker \varphi(Q),\\ \ker P = \ker Q &\iff \operatorname{im} \varphi(P) = \operatorname{im} \varphi(Q).\end{aligned}$$

Hence, we can apply Theorem 3.2. Using also the fact that $\varphi(\mathcal{I}_n(\mathcal{H})) = \mathcal{I}_n(\mathcal{K})$ for every $n \in \mathbb{Z}$ we conclude that there exists a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ such that $\varphi(P) = A_n P^* A_n^{-1}$ for every $P \in \mathcal{I}_n(\mathcal{H})$, $n \in \mathbb{Z}$. After composing φ by the map from $\mathcal{I}(\mathcal{K})$ onto $\mathcal{I}(\mathcal{H})$ defined by $P \mapsto A_1^{-1} P^* A_1$, $P \in \mathcal{I}(\mathcal{K})$, we may assume that φ maps $\mathcal{I}(\mathcal{H})$ bijectively onto itself and $\varphi(P) = P$ for every operator of rank one. It is then trivial to conclude that $\varphi(P) = P$ for every $P \in \mathcal{I}(\mathcal{H})$. ■

Many other structural results for maps on idempotents can be deduced from our description of image equality and kernel equality preserving maps. As another example we reprove one of the results from [12].

Theorem 3.4 *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces and let $\varphi\colon \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{K})$ be a bijective map such that for all $P, Q \in \mathcal{I}(\mathcal{H})$ we have*

$$PQ \text{ is a nonzero idempotent} \iff \varphi(P)\varphi(Q) \text{ is a nonzero idempotent.}$$

Then there exists $A \in \mathrm{BCI}(\mathcal{H}, \mathcal{K})$ such that

$$\varphi(P) = APA^{-1}, \quad P \in \mathcal{I}(\mathcal{H}).$$

Proof Clearly, 0 is the only idempotent operator having the property that its product with each idempotent is not a nonzero idempotent. And PQ is a nonzero idempotent for each nonzero idempotent Q if and only if $P = I$. Hence, $\varphi(0) = 0$ and $\varphi(I) = I$. We next claim that for every idempotent $P \neq 0, I$ the following are equivalent:

- P is of rank one;
- for each $Q \in \mathcal{I}(\mathcal{H})$ the product PQ is a nonzero idempotent if and only if QP is a nonzero idempotent.

Indeed, if P is of rank one, $P = x \otimes y^*$, and Q an arbitrary idempotent, then $PQ = x \otimes (Q^* y)^*$ is a nonzero idempotent if and only if $\langle x, Q^* y\rangle = 1$, while QP is a nonzero idempotent if and only if $\langle Qx, y\rangle = 1$. Thus, the nonzero idempotency of PQ is equivalent to the nonzero idempotency of QP. If, on the other hand, P is a non-trivial idempotent that is not of rank one, then after applying a similarity transformation we may assume that P is a projection whose image is at least two-dimensional and whose kernel is nonzero. Take orthonormal vectors $x, y \in \operatorname{im} P$ and a unit vector $u \in \ker P$ and set $Q = x \otimes x^* + (y + u) \otimes u^*$. Clearly, Q is an idempotent, $PQ = x \otimes x^* + y \otimes u^*$ is not an idempotent, while $QP = x \otimes x^*$ is an idempotent. This proves that the above two statements are equivalent, and consequently, $\varphi(\mathcal{I}_1(\mathcal{H})) = \mathcal{I}_1(\mathcal{K})$.

It was proved in [12, Lemma 2.8] that for two rank one idempotents $P, Q \in \mathcal{I}_1(\mathcal{H})$, $P \neq Q$, the following statements are equivalent:

- $P \approx Q$;
- there exists $R \in \mathcal{I}_1(\mathcal{H}) \setminus \{P, Q\}$ such that for every $T \in \mathcal{I}_1(\mathcal{H})$, if PT and QT are both nonzero idempotents, then RT is a nonzero idempotent.

It follows that for every pair $P, Q \in \mathcal{I}_1(\mathcal{H})$ we have $P \approx Q$ if and only if $\varphi(P) \approx \varphi(Q)$. As in the proof of the previous theorem we conclude that either

- for every pair of nonzero vectors $x, y \in \mathcal{H}$ there exist nonzero vectors $u, v \in \mathcal{K}$ such that $\varphi(L(x)) = L(u)$ and $\varphi(R(y)) = R(v)$, or
- for every pair of nonzero vectors $x, y \in \mathcal{H}$ there exist nonzero vectors $u, v \in \mathcal{K}$ such that $\varphi(L(x)) = R(u)$ and $\varphi(R(y)) = L(v)$.

We will consider just the first possibility.

Let $P \in \mathcal{I}(\mathcal{H})$ and let $x \in \mathcal{H}$ be a nonzero vector. Then $x \in \operatorname{im} P$ if and only if PQ is a nonzero idempotent for every $Q \in L(x)$. Indeed, if $x \in \operatorname{im} P$ and $Q = x \otimes y^*$ with $\langle x, y\rangle = 1$, then $PQ = Q$. If, on the other hand, $x \notin \operatorname{im} P$, then either $Px = 0$ or Px and x are linearly independent. In both cases we can find $y \in \mathcal{H}$ such that $\langle x, y\rangle = 1$ and $\langle Px, y\rangle = 0$. Consequently, $Q = x \otimes y^* \in L(x)$ but PQ is not a nonzero idempotent.

It follows that for every pair $P, Q \in \mathcal{I}(\mathcal{H})$ we have $\operatorname{im} P = \operatorname{im} Q \iff \operatorname{im} \varphi(P) = \operatorname{im} \varphi(Q)$. And similarly, for every pair $P, Q \in \mathcal{I}(\mathcal{H})$ we have $\ker P = \ker Q \iff \ker \varphi(P) = \ker \varphi(Q)$. Hence, we can apply Theorem 3.1. We also know that rank one idempotents are mapped into rank one idempotents. Hence, after composing φ with an appropriate isomorphism (see the proof of the previous theorem) we may assume that φ maps $\mathcal{I}(\mathcal{H})$ onto itself and satisfies $\varphi(P) = P$ for every idempotent P of rank one. We once again use the fact that for an arbitrary $P \in \mathcal{I}(\mathcal{H})$ we have $x \in \operatorname{im} P$ if and only if PQ is a nonzero idempotent for every $Q \in L(x)$, together with the fact that for every rank one idempotent Q the product PQ is a nonzero idempotent if and only if $\varphi(P)Q$ is a nonzero idempotent, to conclude that $\operatorname{im} \varphi(P) = \operatorname{im} P$. Similarly, $\ker \varphi(P) = \ker P$. The proof is complete. ∎

Let us conclude with the remark that Theorem 2.14 plays an essential role in the proof of Theorems 3.1 and 3.2, while the remaining two theorems in this section could be derived directly from Proposition 2.13.

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