Mathematical Notes.

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Coordinates, Conics and Conjugate Points.

1. A fundamental result in the theory of conics is the following:

Two opposite vertices of a complete quadrilateral are conjugate points for any conic which passes through the four remaining vertices.

Let $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$. Then we may call the function

$$ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + z_1y_2) + g(z_1x_2 + x_1z_2) + h(x_1y_2 + y_1x_2)$$

the *polar function* for the two points (x_1, y_1, z_1), (x_2, y_2, x_2), and denote it by ϕ_{12}.

In this notation, therefore,

$$\phi_{rr} = \phi(x_r, y_r, z_r).$$

Let 1, 4; 2, 5; 3, 6 be the three pairs of opposite vertices of a quadrilateral. The above theorem asserts that $\phi_{36} = 0$ if ϕ_{11}, ϕ_{22}, ϕ_{33}, ϕ_{44} are all zero. It occurs to one to enquire what relation connects ϕ_{36}, ϕ_{11}, ϕ_{22}, ϕ_{33}, ϕ_{44} in the general case; *i.e.* when the vertices 1, 2, 3, 4 are not restricted to lie on the conic $\phi = 0$.

It turns out that *the relation is a linear one.* For the proof, the following method seems as simple as any, and it has interesting features. A few cognate results—all well known—are pointed out in passing.

2. In two-dimensional geometry a point may be represented by three coordinates x, y, z, and a line by a homogeneous equation of the first degree, say $lx + my + nz = 0$. Ordinary plane Cartesian coordinates are most simply included by taking $z \equiv 1$.

Three points 1, 2, 3 lie on a line if

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = 0.$$

But this is exactly the condition for the coexistence of three equations of the form

$$px_1 + qx_2 + rx_3 = 0,$$
$$py_1 + qy_2 + ry_3 = 0,$$
$$pz_1 + qz_2 + rz_3 = 0.$$

Now we may say that the condition of collinearity of the three points is simply

$$px_1 + qx_2 + rx_3 = 0,$$

on the implied understanding that x is, as it were, a *typical coordinate*, and that the typical equation holds for the coordinate of every type.

Note that none of the three coefficients in the equation $px_1 + qx_2 + rx_3 = 0$ can be zero; for if r, say, were 0, the points 1, 2 would coincide. From this point of view the condition of collinearity of three points is the same whatever be the number of dimensions of the space considered.

If x, y, z are trilinear or areal or Cartesian plane coordinates then we have $p + q + r = 0$. But we need not impose this condition the analysis will then apply (with slight changes of nomenclature to a more general geometry—to spherical geometry, for example with great circles taking the place of lines, and opposite extremities of a diameter counting as one point.

If the line joining the points 1, 2 meets the line 34 in the point 5, we have typical relations

$$ax_1 + bx_2 = x_5,$$
$$cx_3 + dx_4 = x_5.$$

Hence a relation of the form

$$ax_1 + bx_2 = cx_3 + dx_4$$

holds for any four points in a plane.

Conversely, if we are given the relation for four points

$$lx_1 + mx_2 + nx_3 + px_4 = 0,$$

we can find at once the point where the join of any two of the points meets the join of the other two. Thus we have

$$lx_1 + mx_2 = -nx_3 - px_4;$$

and putting each side $= x_5$,

$$lx_1 + mx_2 = x_5,$$

and $\quad -nx_3 - px_4 = x_5.$

Hence 1, 2, 5 are collinear, also 3, 4, 5; so that 5 is the point where 12 and 34 intersect.

In the pair of equations last written we might equally well put

$$lx_1 + mx_2 = \lambda x_5,$$
$$-nx_3 - px_4 = \lambda x_5.$$

This, however, is a matter of indifference, since the point $(\lambda x_5, \lambda y_5, \lambda z_5)$ is the same point as (x_5, y_5, z_5).

For four points in a plane there is *only one* relation such as $lx_1 + mx_2 + nx_3 + px_4 = 0$, unless all four points are collinear. For if there were two such relations we could, by eliminating one of the coordinates, as x_4, derive a linear relation among the three others. Again, if three of the points are collinear, say 1, 2, 3, then *the* relation among 1, 2, 3, 4 will not involve 4.

It is sometimes convenient to change the coordinates of a point to other values which must, of course, be proportional to the original values. Thus, if we take x_1/a, y_1/a, z_1/a as coordinates of a point in place of x_1, y_1, z_1; and make similar changes for the points 2, 3, 4; the relation

$$ax_1 + bx_2 + cx_3 + dx_4 = 0$$

assumes the simplified form

$$x_1 + x_2 + x_3 + x_4 = 0.$$

The change is equivalent to a sort of *projection* of the figure.

The following proof of a familiar theorem will illustrate the method.

3. *Desargues' Theorem. If the lines joining corresponding vertices of two triangles are concurrent, then the meeting points of corresponding sides are collinear.*

Let the vertices be 1, 2, 3 and 4, 5, 6; and let 14, 25, 36 meet at 7. Then we may take

$$x_1 + x_4 = x_7,$$
$$x_2 + x_5 = x_7,$$
$$x_3 + x_6 = x_7,$$

Hence $$x_1 + x_4 = x_2 + x_5,$$

or $$x_1 - x_2 = x_5 - x_4.$$

If we put each member of this last equation $= x_8$, then 8 is the meeting point of 12 and 45. If 23, 56 meet at 9, and 31, 64 at 10 we thus have

$$x_1 - x_2 = x_8,$$
$$x_2 - x_3 = x_9,$$
$$x_3 - x_1 = x_{10},$$

so that $x_8 + x_9 + x_{10} = 0$, and 8, 9, 10 are collinear.

Converse. The dual theorem may be regarded as proved by the above analysis. We have only to suppose that x, y, z are *line* coordinates. If, however, we prefer to restrict ourselves to point coordinates, we may take

$$x_8 = a_1x_1 + A_2x_2 = a_4x_4 + A_5x_5,$$
$$x_9 = a_2x_2 + A_3x_3 = a_5x_5 + A_6x_6,$$
$$x_{10} = a_3x_3 + A_1x_1 = a_6x_6 + A_4x_4,$$

with $$x_8 + x_9 + x_{10} = 0.$$

Thus $(a_1x_1 + A_2x_2) + (a_2x_2 + A_3x_3) + (a_3x_3 + A_1x_1) = 0$, which must be an identity, since 1, 2, 3 are not collinear. Hence

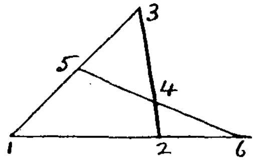
$$a_1 + A_1 = 0, \ a_2 + A_2 = 0, \text{ and } a_3 + A_3 = 0.$$

Similarly $$a_4 + A_4 = 0, \ a_5 + A_5 = 0, \text{ and } a_6 + A_6 = 0.$$

Then $a_1x_1 - a_4x_4 = a_2x_2 - a_5x_5 = a_3x_3 - a_6x_6$; and if we put each of these $= x_7$, we see that 14, 25, 36 are concurrent at 7.

4. *The complete quadrilateral.*

Let 1, 4; 2, 5; 3, 6 be the pairs of opposite vertices, the sides 12 and 54 meeting at 6, and 15, 24 at 3.



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Let the coordinate relation for 1, 2, 4, 5 be

$$x_1 - x_4 = x_5 + x_2.$$

Then we have $x_1 - x_2 = x_4 + x_5$; and each $= x_6$, say.

Also $\quad x_5 - x_1 = -x_2 - x_4$; and each $= x_3$.

Hence

$$\left.\begin{aligned} x_1 - x_4 &= x_5 + x_2, \\ x_5 - x_2 &= x_6 + x_3, \\ x_6 - x_3 &= x_1 + x_4. \end{aligned}\right\} \quad \text{......................(1)}$$

Then

$$\left.\begin{aligned} (x_6 + x_3)^2 - (x_6 - x_3)^2 &= (x_5 - x_2)^2 - (x_1 + x_4)^2, \\ \text{and} \quad 0 &= (x_5 + x_2)^2 - (x_1 - x_4)^2. \end{aligned}\right\} \quad \text{..........(2)}$$

From (2), by addition,

$$2x_3x_6 = x_5^2 + x_2^2 - x_1^2 - x_4^2; \quad \text{................(3)}$$

and, by subtraction,

$$2x_3x_6 = -2x_2x_5 - 2x_1x_4,$$

or

$$x_1x_4 + x_2x_5 + x_3x_6 = 0. \quad \text{....................(4)}$$

Now, just as x is a typical coordinate, so x^2 *may be regarded as a typical quadratic function, and x_1x_4 as a typical corresponding polar function.* To see that this is so, observe that the equations (1), and those derived from them, continue to hold if we replace x by y, or by z, or by any linear function, as $lx + my + nz$. And obviously any quadratic function can be built up by addition from functions of the types x^2, y^2, z^2, $(lx + my + nz)^2$. By simple addition, we thus derive from (3) and (4) the general relations

$$2\phi_{36} = \phi_{55} + \phi_{22} - \phi_{11} - \phi_{44}, \quad \text{......................(5)}$$

and

$$\phi_{14} + \phi_{25} + \phi_{36} = 0. \quad \text{......................(6)}$$

5. The equation (5) proves the theorem which we set out to prove. We may derive it in a more conventional way from the equations of type (1) as follows:

From (1)

$$\phi(x_6 + x_3,\ y_6 + y_3,\ z_6 + z_3) - \phi(x_6 - x_3,\ y_6 - y_3,\ z_6 - z_3)$$
$$= \phi(x_5 - x_2,\ y_5 - y_2,\ z_5 - z_2) - \phi(x_1 + x_4,\ y_1 + y_4,\ z_1 + z_4);$$

and $\quad 0 = \phi(x_5 + x_2,\ y_5 + y_2,\ z_5 + z_2) - \phi(x_1 - x_4,\ y_1 - y_4,\ z_1 - z_4).$

But $\phi(x_6 + x_3,\ y_6 + y_3,\ z_6 + z_3) \equiv \phi_{66} + \phi_{33} + 2\phi_{36}$,

and similarly in other cases.

Hence by addition and subtraction as before,

$$2\phi_{36} = \phi_{55} + \phi_{22} - \phi_{11} - \phi_{44},$$

and
$$\phi_{36} = -\phi_{55} - \phi_{14}.$$

6. For the sake of economy of symbols, the proof has been stated in the contracted notation explained near the end of Art. 2 The two theorems obtained are of course similarly contracted This is immaterial for *projective* applications; if *metrical* results are wanted, the extended forms of the relations can easily be written down.

7. The second of the two relations found, viz.,

$$\phi_{14} + \phi_{25} + \phi_{36} = 0,$$

shows that if $\phi_{14} = 0$ and $\phi_{25} = 0$, then also $\phi_{36} = 0$. In words: *If two pairs of opposite vertices of a complete quadrilateral are conjugate for a conic, then so is the third pair.* This important theorem is perhaps not so well known as it deserves to be. It is sometimes stated in the form: *The polars of the vertices of a triangle with respect to a conic meet the opposite sides in collinear points.*

The reciprocal theorem, viz: *If two pairs of opposite sides of a quadrangle are conjugate for a conic, then so the third pair* is, from the standpoint of general projective theory, exactly equivalent to the orthocentre property of a spherical triangle, since great circles at right angles may be regarded as lines conjugate with respect to the circle at infinity.

A few special cases may be noticed.

(*a*) Take the lines 14, 25 for the conic; then 1, 4 and also 2, 5 are conjugate points. Hence so are 3, 6, *i.e. A pair of opposite vertices are harmonic conjugates of a pair of vertices of the diagonal triangle.*

(*b*) Take for the conic the line joining the mid points of 14 and 25, along with the line at infinity. Then 1, 4 as also 2, 5 are conjugate; therefore 3, 6 are also conjugate, *i e. The mid points of the three diagonals are collinear.*

(*c*) Take for the conic the point circle with centre at any point *P*. If the lines *P*1 and *P*4 are at right angles, then 1, 4

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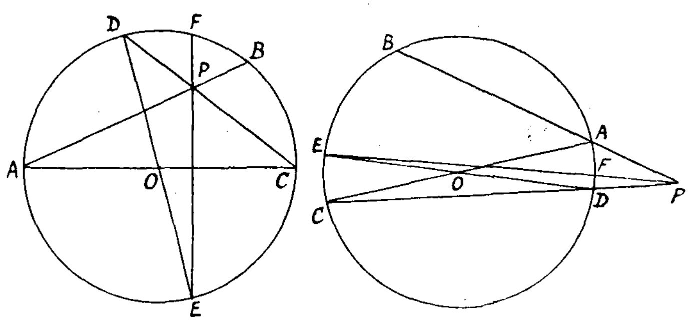
are conjugate, for the polar of either passes through the other. Similarly if $P2$ and $P5$ are at right angles, 2, 5 are conjugate. Hence 3, 6 are conjugate, and $P3$, $P6$ are at right angles, *i.e* *The circles on the three diagonals as diameters are coaxial.*

(*d*) Take for the conic the pair of lines through any point Q, harmonically conjugate both to $Q1$, $Q4$ and to $Q2$, $Q5$. Then $Q3$, $Q6$ are also harmonically conjugate to the same pair of lines, *i e.* *The lines from any point to the six vertices are in involution.*

The dual of this is: *The three pairs of opposite sides of a quadrangle are cut in involution by any transversal.*

JOHN DOUGALL.

A Limit Proof of the Theorem Euc. III, 35.



Let AB be any chord passing through the fixed point P. Draw the diameter AOC, join CP and let it meet the circumference at D. Then in triangle ACP, $AC^2 = AP^2 + CP^2$ together with *either* $2\,AP\,.\,PB$ *or* $2\,CP\,.\,PD$ according as CP is projected on AP or AP on CP. Hence $AP\,.\,PB = CP\,.\,PD$. Now draw the diameter DOE, join EP and let it meet the circumference at F. Then the theorem is true for chords CD, EF.

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