FINITE DIMENSIONALITY, NILPOTENTS AND QUASINILPOTENTS IN BANACH ALGEBRAS

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1

In this note we present some rather loosely connected results on Banach algebras together with some illustrative examples. We consider various conditions on a Banach algebra which imply that it is finite dimensional. We also consider conditions which imply the existence of non-zero nilpotents, and hence the existence of finite dimensional subalgebras. In the setting of Banach algebras quasinilpotents figure more prominently than nilpotents. We give an example of a non-commutative Banach algebra in which 0 is the only quasinilpotent; this resolves a problem of Hirschfeld and Zelazko (**4**).

Throughout, all Banach algebras will be taken over the complex field. For the standard Banach algebra terms employed here we refer the reader to Rickart (**8**). We write $\mathrm{Sp}(A, x)$ for the spectrum of x in A, and $r(x)$ for the spectral radius of x.

2

Theorem 1 extends a result of LePage (**7**).

Theorem 1. *Let A be a Banach algebra with unit such that $Ax^2 = Ax$ $(x \in A)$. Then A is semi-simple, commutative and finite dimensional.*

Proof. A is semi-simple and commutative by (**7**).

Suppose that A has no proper idempotents. Given $x \in A$, $x \neq 0$, there exists $y \in A$ such that $yx^2 = x$. Then yx is idempotent by (**7**), $yx = 1$. Therefore A is a division algebra, $A = C1$.

Next, A cannot contain an infinite sequence of pairwise orthogonal idempotents. Suppose $\{e_n\}$ is such a sequence. Choose $\lambda_n > 0$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n e_n \in A,$$

and let $y \in A$ be such that $yx^2 = x$. Then

$$y \sum_{n=1}^{\infty} \lambda_n^2 e_n = \sum_{n=1}^{\infty} \lambda_n e_n.$$

Therefore, for each n, $\lambda_n y e_n = e_n$,

$$1 = r(e_n) = \lambda_n r(ye_n) \leqq \lambda_n r(y) r(e_n) = \lambda_n r(y),$$

which is impossible.

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Let $\{e_j\colon j = 1, \ldots, m\}$ be a family of pairwise orthogonal non-zero idempotents. For each j, either Ae_j has no proper idempotents or there exist non-zero idempotents $p_j, q_j \in Ae_j$ such that $e_j = p_j + q_j$, $p_j q_j = 0$. Since A cannot contain an infinite sequence of pairwise orthogonal idempotents, we may suppose that $\{e_j\colon j = 1, \ldots, m\}$ is chosen so that Ae_j has no proper idempotents for each j, and $1 = e_1 + \ldots + e_m$. For each j, Ae_j satisfies the given condition on principal ideals and so $Ae_j = \mathbf{C}e_j$. Therefore A has dimension m.

Remarks. (1) There is an alternative proof of Theorem 1 which uses Shilov's idempotent theorem to show that the open-and-closed subsets of the carrier space of A form a partition of the carrier space into singletons.

(2) The converse of Theorem 1 is immediate from Wedderburn's theorem.

(3) The assertion of Theorem 1 clearly fails without the assumption of a unit element.

(4) Let $A = C[0, 1]$, the algebra of continuous complex functions on $[0, 1]$. Then $Ax^2 = Ax$ for each invertible element x, and the set of invertible elements is dense in A. For this example we have the weaker condition that

$$\overline{Ax^2} = \overline{Ax}\ (x \in A). \tag{*}$$

It would be of interest to characterise those Banach algebras A which satisfy (*).

(5) Let A be an arbitrary Banach algebra and let $x \in A$ be quasinilpotent but not nilpotent. Grabiner (3) shows that the sequence $\{Ax^n\}$ is strictly decreasing.

Theorem 2. (Kaplansky (6)). *Let A be a semi-simple Banach algebra such that* $\mathrm{Sp}(A, x)$ *is finite for each $x \in A$. Then A is finite-dimensional.*

Corollary 3. *Let A be a Banach algebra such that* $\mathrm{Sp}(A, x)$ *is finite for each $x \in A$. Then A/R is finite dimensional, where R is the radical of A.*

Corollary 4. *Let X be a complex Banach space such that every compact operator on X has finite rank. Then X is finite dimensional.*

Remarks. (1) The proof of Theorem 2 as given by Kaplansky can be simplified. In particular, in the commutative case it is enough to show that the carrier space of A has only a finite number of components.

(2) A further proof of Theorem 2 is given in Dixon (2).

Let A be a Banach algebra with unit element and let B be a closed subalgebra of A with $1 \in B$. Given $x \in B$, we have

$$\mathrm{Sp}(A, x) \subset \mathrm{Sp}(B, x),\ \partial\mathrm{Sp}(A, x) = \partial\mathrm{Sp}(B, x).$$

In particular, if $\mathrm{Sp}(A, x)$ is finite, then $\mathrm{Sp}(A, x) = \mathrm{Sp}(B, x)$. Zelazko (9) shows for commutative A that

$$\mathrm{Sp}(A, x) = \mathrm{Sp}(B, x)\ (x \in B,\ B \text{ any closed subalgebra of } A)$$

if and only if $\mathrm{Sp}(A, x)$ is totally disconnected for each $x \in A$. In the opposite direction we may ask if it is possible to remove the topological interior of $\mathrm{Sp}(A, x)$ by considering $\mathrm{Sp}(C, x)$ for some superalgebra C of A.

Example 5. Let A be the algebra of all bounded linear operators on l^2, and let t be the unilateral shift operator. Then $\mathrm{Sp}(A, t) = \{\lambda \in C: |\lambda| \leq 1\}$ Since every singular element of A is a topological divisor of zero, $\lambda - t$ is a topological divisor of zero, and hence permanently singular, for each

$$\lambda \in \mathrm{Sp}(A, t)$$

(see (**8**) p. 185, and p. 20).

3

We consider now the existence of non-zero nilpotents and quasinilpotents in Banach algebras. Theorem 6 is due to Kaplansky (see Dixmier (**1**), p. 58).

Theorem 6. *A C^*-algebra is commutative if and only if* 0 *is the only nilpotent.*

The next result is entirely algebraic.

Theorem 7. *An algebra of operators on a complex vector space which contains a non-central operator of finite rank also contains a non-zero nilpotent.*

Proof. Let A be an algebra of operators on a complex vector space and let b be a non-central finite rank operator in A. Since b has finite rank, the subalgebra bAb is finite dimensional and hence its radical consists of nilpotents. Suppose bAb is semi-simple. By Wedderburn's theorem, bAb is isomorphic to a finite direct sum of full matrix algebras and hence contains non-zero nilpotents unless it is commutative. Suppose now that bAb is also commutative. Let $\{e_i: i = 1, \ldots, k\}$ be a spanning subset of minimal idempotents of bAb, and let $t \in A$. For each i, $e_i t - e_i t e_i$ and $t e_i - e_i t e_i$ are nilpotent. If these are all zero then $e_i t = t e_i$ $(i = 1, \ldots, k)$, and so

$$ct = tc \quad (t \in A,\ c \in bAb),$$

$$(bt)^n = (tb)^n \quad (t \in A,\ n = 2, 3, \ldots).$$

Then $bt^2b^2t = b^2t^2bt = tb^2t^2b$. It follows that $(bt - tb)^3 = 0$, and we have $bt - tb$ non-zero for some $t \in A$.

Corollary 8. *Let A be an irreducible Banach algebra of operators and let A contain a non-zero finite rank operator. Then A contains a non-zero nilpotent.*

Proof. The centre of A is either (0) or the scalar multiples of the identity.

LePage (**7**), and Hirschfeld and Zelazko (**4**) show that if A is a complex Banach algebra such that

$$\inf \{r(x): \|x\| = 1\} > 0$$

then A is commutative. This raises the question whether every non-commutative Banach algebra contains non-zero quasinilpotents. Hirschfeld and Rolewicz (5) give an example in which 0 is the only divisor of zero.

Theorem 9. *There exists a non-commutative Banach algebra in which* 0 *is the only quasinilpotent.*

Proof. Let A be the free algebra on two symbols u, v, i.e. the algebra of all finite linear combinations of words in u and v. The set of all such words is denumerable, $\{w_n\}$, and we take the standard enumeration given by

$$u,\ v,\ u^2,\ uv,\ vu,\ v^2,\ u^3,\ u^2v,\ \ldots$$

Let B be the algebra of all infinite series $x = \Sigma\alpha_n w_n$ where $\| x \| = \Sigma \mid \alpha_n \mid < \infty$. Then B is a non-commutative Banach algebra. Let $x \in B$, $x \neq 0$, and let α_p be the first non-zero coefficient in the series $\Sigma\alpha_n w_n$. Then the coefficient of w_p^m in x^m is precisely α_p^m and so

$$\| x^m \| \geqq \mid \alpha_p \mid^m \quad (m = 1, 2, 3, \ldots),$$

$$r(x) \geqq \mid \alpha_p \mid > 0.$$

Observe that B is an infinite dimensional non-commutative Banach algebra in which the set of quasinilpotents coincides with the set of nilpotents. A slight modification of the construction gives another interesting example.

Example 10. Let A be as in Theorem 9, let $\nu(w_n)$ denote the length of the word w_n, and let C be the algebra of all infinite series $x = \Sigma\alpha_n w_n$ where

$$\| x \| = \Sigma \mid \alpha_n \mid / \nu(w_n)! < \infty.$$

Then C is a non-commutative radical Banach algebra which is an integral domain.

Proof. It is straightforward that C is a non-commutative Banach algebra. Let $x \in C$ and let k be a positive integer. Then

$$\begin{aligned} \| x^k \| &\leqq \Sigma_{n_i} \frac{\mid \alpha_{n_1} \mid \mid \alpha_{n_2} \mid \ldots \mid \alpha_{n_k} \mid}{\nu(w_{n_1} w_{n_2} \ldots w_{n_k})!} \\ &= \Sigma_{n_i} \frac{\nu(w_{n_1})! \ldots \nu(w_{n_k})!}{\{\nu(w_{n_1}) + \ldots + \nu(w_{n_k})\}!} \frac{\mid \alpha_{n_1} \mid}{\nu(w_{n_1})!} \cdots \frac{\mid \alpha_{n_k} \mid}{\nu(w_{n_k})!} \\ &\leqq \frac{1}{k!} \| x \|^k \end{aligned}$$

and so $r(x) = 0$. It is clear that C is an integral domain.

Observe that C can be regarded as an algebra of compact operators on A.

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