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A DEGREE FORMULA FOR SECANT VARIETIES OF CURVES

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Abstract Using the Stückrad–Vogel self-intersection cycle of an irreducible and reduced curve in projective space, we obtain a formula that relates the degree of the secant variety, the degree and the genus of the curve and the self-intersection numbers, the multiplicities and the number of branches of the curve at its singular points. From this formula we deduce an expression for the difference between the genera of the curve. This result shows that the self-intersection multiplicity of a curve in projective N-space at a singular point is a natural generalization of the intersection multiplicity of a plane curve with its generic polar curve. In this approach, the degree of the secant variety (up to a factor 2), the self-intersection numbers and the multiplicities of the singular points are leading coefficients of a bivariate Hilbert polynomial, which can be computed by computer algebra systems.

Keywords: curves in projective space; self-intersection multiplicities; arithmetic genus; geometric genus; secant varieties

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1. Introduction

Let $C \subset \mathbb{P}^N_{\mathbb{C}}$ $(N \geqslant 4)$ denote an irreducible and reduced non-degenerate projective curve, where $\mathbb{C}$ is the field of complex numbers. The secant variety $\operatorname{Sec} C$ of C is a three-dimensional variety, the degree of which can be computed by intersecting it with a general $(N-3)$-dimensional linear subspace $\Gamma \subset \mathbb{P}^N$, and it is equal to the number of ordinary double points of the projection $\pi_\Gamma \colon C \to \mathbb{P}^2$ with centre Γ that are not images of singular points of C. This number can be obtained by subtracting from the arithmetic genus of the plane curve $\bar{C} = \pi_\Gamma(C)$ the geometric genus of C and the contributions of those singular points that are projections of singular points of C.

In this paper we present a formula for the degree of $\operatorname{Sec} C$ coming from the Stückrad–Vogel self-intersection cycle of C (see [**23**]) and involving the genus, the degree, the self-intersection numbers, the multiplicities and the number of branches of the curve at its singular points. This formula does not use a generic projection of the curve to $\mathbb{P}^2$, which might lead to too much computational complexity.

More precisely, by results of van Gastel [**26**] and Flenner and Manaresi [**10**], the self-intersection cycle $v(C,C)$ of Stückrad and Vogel [**23**] enables us to deduce the relation between the degree $d = \deg C$, the genus g, the degrees of the tangent variety $\operatorname{Tan} C$ and secant variety $\operatorname{Sec} C$ and the self-intersection numbers $j_i := j(C,C;P_i)$ at the singular points $P_1,\dots,P_s$ of $C \subset \mathbb{P}^N$, $N \geqslant 4$, as

$$(\deg C)^2 = \deg C + \sum_{i=1}^{s} j_i + \deg \operatorname{Tan} C + 2\deg \operatorname{Sec} C.$$

Combining this with a well-known formula for the degree of $\operatorname{Tan} C$ (see Proposition 3.3 and Remark 3.4), we get the following result.

Theorem 1.1. *With the preceding notation, if m_i and r_i denote the multiplicity and the number of branches of $C \subset \mathbb{P}^N$ $(N \geqslant 4)$ at P_i, respectively,*

$$\deg \operatorname{Sec} C = \binom{d-1}{2} - g - \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i).$$

The intersection multiplicities j_i and the ramification indices $m_i - r_i$ of P_i can be expressed in terms of the local parametrization of the curve (see Proposition 4.2 and Corollary 4.3 and [**12**, p. 264], respectively), but they can also be computed without knowing a local parametrization, since j_i and m_i are leading coefficients of a bivariate Hilbert polynomial (see [**3**, Theorem 4.1] applied to the local rings $\mathcal{O}_{C,P_i}$). In addition, $2\deg \operatorname{Sec} C$ is a leading coefficient of a bivariate Hilbert polynomial (see Theorems 2.2 and 3.5).

If $N = 3$, then in the preceding theorem the degree of the secant variety has to be replaced by the number ρ of secants of the curve C passing through a generic point of $\mathbb{P}^3 = \operatorname{Sec} C$, and a generalization to singular curves of the secant formula by Peters and Simonis [**18**] is obtained (see Proposition 3.7). Again, 2ρ is a leading coefficient of a bivariate Hilbert polynomial.

Using a generic plane projection, we give a second formula, which expresses the difference between the genera of the curve in terms of invariants of the singular points. This formula is obtained from Theorem 3.5 and Proposition 3.7, by expressing the degree of the secant variety of C (or the number ρ) as a function of the degree, the arithmetic genus and some local invariants of the singular points of C (see Corollary 3.13 and Remark 3.10).

Theorem 1.2. *If $C \subset \mathbb{P}^N$, $N \geqslant 2$, is a non-degenerate projective curve of arithmetic genus $p_a(C)$ and geometric genus $g(C)$, $\bar{C}$ is a generic plane projection of C, $\bar{P}_i$ is the image of the singular point P_i under the projection, $\mathfrak{c}_i$ is the conductor ideal for $\mathcal{O}_{C,P_i}$ in $\mathcal{O}_{\bar{C},\bar{P}_i}$, $\bar{\mu}_i$ is the Milnor number of $\bar{C}$ at $\bar{P}_i$ and μ_i is the Milnor number of C at P_i,*

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then

$$p_a(C) - g(C) = \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i - 2\dim_{\mathbb{C}}(\mathcal{O}_{\bar{C},\bar{P}_i}/\mathfrak{c}_i\mathcal{O}_{\bar{C},\bar{P}_i}))$$
$$= \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i - \bar{\mu}_i + \mu_i).$$

Moreover,

$$p_a(C) - g(C) = \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i)$$

holds if and only if the embedding dimension of C at each (singular) point P is at most 2.

This formula extends the classical genus formula of Max Noether for plane curves (see [**9**, p. 180]) to higher dimensions. Since for a plane curve C the self-intersection number $j(C, C; P_i) = j_i$ equals the intersection multiplicity of a generic polar curve with the curve at the point P_i (see [**9**, Appendix] and [**11**, Example 2.2.10]), the self-intersection multiplicity can be regarded as a natural generalization of the intersection multiplicity of a plane curve with its generic polar curve.

We also illustrate our results by a few examples.

2. A short review of the Stückrad–Vogel intersection cycle

Let X, Y be closed subvarieties of the projective space $\mathbb{P}^N = \mathbb{P}^N_K$, where K is an algebraically closed field of characteristic 0. Proving a Bézout theorem for improper intersections, Stückrad and Vogel [**23**] (see also [**11**]) introduced the cycles $v_k = v_k(X, Y)$ of dimension k on $X \cap Y$, which are obtained by a simple algorithm on the *ruled join*

$$J := J(X, Y) := \{(x : y) \in \mathbb{P}^{2N+1} \mid x \in X,\ y \in Y\}$$

in the following way. Let $\Delta := \{(x : x) \in \mathbb{P}^{2N+1} \mid x \in \mathbb{P}^N\}$ be the diagonal, so Δ is given by

$$x_0 - y_0 = \cdots = x_N - y_N = 0,$$

where $x_0, \dots, x_N$, $y_0, \dots, y_N$ are homogeneous coordinates in $\mathbb{P}^{2N+1}$. For the indeterminates $u_{ij} (0 \leqslant i, j \leqslant N)$, let L be the pure transcendental extension $K(u_{ij})_{0 \leqslant i,\, j \leqslant N}$ and for a subvariety $Z \subset \mathbb{P}^N_K$ set $Z_L := Z \otimes_K L$.

Let $H_i \subseteq J_L := J \otimes_K L$ $(i = 0, \dots, N)$ be the divisor given by

$$l_i := \sum_{j=0}^{N} u_{ij}(x_j - y_j) = 0.$$

Then, one defines the cycles β_k and v_k inductively by setting

$$\beta_{\dim J} := [J].$$

If β_k is already defined, decompose the intersection

$$\beta_k \cap H_{\dim J-k} = v_{k-1} + \beta_{k-1} \quad (\dim J - N \leqslant k \leqslant \dim J),$$

where the support of v_{k-1} lies in Δ and where no component of β_{k-1} is contained in Δ. It follows that v_k is a k-dimensional cycle on $X_L \cap Y_L \cong J_L \cap \Delta_L$. In general, $v = \sum_k v_k$ is a cycle defined over L, which can also be written in the form

$$v = \sum_C j_C[C],$$

where C are the subvarieties of $X_L \cap Y_L$ that appear in the cycle v and $j_C = j(X, Y; C)$ is a positive integer called the *intersection number* or the *intersection multiplicity* of X and Y along C.

The part of the cycle v_k that is defined over the base field K, the so-called *K-rational part* or *fixed part*, will be denoted by $\operatorname{rat}(v_k)$ and the remaining part, the so-called *irrational* or *movable part*, will be denoted by $\operatorname{mov}(v_k)$, that is,

$$v_k = \operatorname{rat}(v_k) + \operatorname{mov}(v_k).$$

Using the theory of residual intersections, Flenner and Manaresi [**10**] gave a geometric interpretation of the β_k as the cycles of double points of generic linear projections and of the non-K-rational components of v, at least in the case when X, Y are smooth and meet smoothly (see [**10**] or [**11**, Chapter 8] for precise definitions and statements). In the case of self-intersection, i.e. $X = Y$, from the more general result [**10**, Theorem 4.6] one has the following.

Theorem 2.1 (Flenner and Manaresi [10, Corollary 4.9]). *Let $X \subseteq \mathbb{P}^N$ be an algebraic variety of dimension n. Let k be an integer such that $0 \leqslant k < n$ and $N \geqslant 2n - k - 1$. Let $p\colon X_L \to \mathbb{P}_L^{2n-k-1}$ be the generic linear projection and let $R(p)$ be its ramification locus. Then, $\dim R(p) \leqslant k$, and the associated k-cycle $[R(p)]_k$ is just v_k on the smooth locus $\operatorname{Sm}(X)$.*

The degree of v_k can be calculated as follows. Let $I(J)$ and $I(\Delta)$ be the ideals of J and Δ, respectively, in the ring $L[x_0, \dots, x_N, y_0, \dots, y_N]$ and denote by $\mathfrak{m}$ the ideal $(x_0, \dots, x_N, y_0, \dots, y_N)$. Let $A := (L[x_0, \dots, x_N, y_0, \dots, y_N]/I(J))_{\mathfrak{m}}$, let $I := I(\Delta)A$ and let $G_I(A) := \bigoplus_{j \geqslant 0} I^j/I^{j+1}$ be the associated graded ring of A with respect to I. Consider the bigraded ring

$$R = \bigoplus_{i,j \geqslant 0} R_{i,j} = \bigoplus_{i,j \geqslant 0} G_{\mathfrak{m}}^i(G_I^j(A)) = \bigoplus_{i,j \geqslant 0} (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1}),$$

where $R_{00} = A/\mathfrak{m} = L$. Then, for i and j sufficiently large, the twofold sum transform $\sum_{q=0}^{j} \sum_{p=0}^{i} \dim R_{pq}$ of the bivariate Hilbert function of R can be written in the form

$$\sum_{k=0}^{d} c_k \binom{i+k}{k} \binom{j+d-k}{d-k} + \text{lower-order terms},$$

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where the non-negative integers c_k are the *generalized Samuel multiplicities* of [**3**]. For $k = 0, \dots, d$ we set

$$c_k(X, Y) := c_k(I) := c_k.$$

Theorem 2.2 (Achilles and Manaresi [3, Corollary 4.2]). *With the previous notation, defining* $d = \dim A = \dim J + 1$,

$$c_0(X, Y) = \deg \beta_0,\ c_1(X, Y) = \deg v_0,\ c_2(X, Y) = \deg v_1,\ \dots,\ c_d(X, Y) = \deg v_{d-1}.$$

Moreover, if $k > \dim(X \cap Y) + 1$, *then* $c_k(X, Y) = 0$.

The integers c_k can be computed by using computer algebra systems (e.g. Cali [**1**]), in which the calculation of the Hilbert series of a multi-graded ring has been implemented.

The main result of [**23**] is the following generalized Bézout theorem.

Theorem 2.3 (Stückrad and Vogel [23]).

$$\deg X \deg Y = \sum_{i \geqslant 0} \deg v_i + \deg \beta_0.$$

In [**23**] the number $\deg \beta_0$ was called the *multiplicity of the empty set*. It admits the following geometric interpretation in terms of the *embedded join* XY. Recall that the embedded join variety XY is, by definition, the closure of the image of J under the rational map

$$\pi\colon J \dashrightarrow \mathbb{P}^N, \qquad (x : y) \mapsto x - y.$$

Theorem 2.4 (van Gastel [26]). *We have that*

$$\deg \beta_0 = \deg XY \deg(J/XY),$$

where $\deg(J/XY)$ *is the mapping degree of* π.

As usual, the mapping degree of π is defined to be 0 if π has fibres of positive dimension. It is equal to the number of points in the generic fibre if π is a finite map. When the embedded join has the expected dimension, then $\deg \beta_0 \neq 0$ and, by the previous theorem, one can easily compute the product $\deg XY \deg(J/XY)$ by computing the integer $c_0(X, Y) = \deg \beta_0$.

In the case $X = Y$, the embedded join is the *secant variety* $\operatorname{Sec} X$. It is well known that for a non-plane curve $C \subset \mathbb{P}^N$ ($N \geqslant 3$) the secant variety has the expected dimension $3 = 2 \dim C + 1 = \dim J(C, C)$ (see, for example, [**13**, Exercise 11.25]).

In the case of curves, from [**11**, Proposition 8.2.12] (in which 1 is erroneously printed instead of 2) and its proof, one has the following result.

Proposition 2.5. *Let* $C, D \subset \mathbb{P}^N$, $N \geqslant 4$, *be irreducible, reduced, non-degenerate distinct curves. Then,* $\deg(J(C, D)/CD) = 1$ *and* $\deg(J(C, C)/\operatorname{Sec} C) = 2$.

Corollary 2.6. *Let* $C, D \subset \mathbb{P}^N$, $N \geqslant 4$, *be irreducible, reduced, non-degenerate distinct curves. Then,*

$$\deg CD = \deg C \deg D - \sum_{P \in C \cap D} j_P = \deg C \deg D - \deg v_0.$$

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Proof. From the refined Bézout theorem we get that

$$\deg C \deg D = \sum_{P \in C \cap D} j_P + \deg(J(C,D)/CD) \deg CD,$$

and by Proposition 2.5 we get that $\deg(J(C,D)/CD) = 1$, from which we complete the proof. □

3. Self-intersections of curves and secant formulae

In this section we will use the self-intersection cycle for projective curves in order to derive secant formulae. Our formulae hold for arbitrary (possibly singular) curves. We will start by fixing the notation for this section.

Notation 3.1. Let $C \subset \mathbb{P}^N_{\mathbb{C}}$ ($N \geqslant 2$) be an *irreducible* and *reduced* curve over the complex numbers $\mathbb{C}$ of degree d. Let $P_1, \dots P_s$ be the singular points of C, that is, $\operatorname{Sing} C = \{P_1, \dots, P_s\}$, and let m_i denote the multiplicity of C at P_i.

Let $\tau\colon \tilde{C} \to C$ be the normalization of C. For each $i = 1, \dots, s$ let $Q_{i1}, \dots, Q_{ir_i}$ be the points of $\tilde{C}$ over P_i, that is, $\tau^{-1}(P_i) = \{Q_{i1}, \dots, Q_{ir_i}\}$. We note that $r_i = r_{P_i}(C)$ is the number of branches of C at P_i ($i = 1, \dots, s$), which is known to be equal to the number of minimal primes of the completion of the local ring $\mathcal{O}_{C,P_i}$.

For a point $P \in C$, we denote by $\tilde{\mathcal{O}}_{C,P}$ the integral closure of the local ring $\mathcal{O}_{C,P}$ in its field of fractions and by

$$\delta_P(C) = \dim_{\mathbb{C}} \tilde{\mathcal{O}}_{C,P}/\mathcal{O}_{C,P} = \operatorname{length}_{\mathcal{O}_{C_P}} \tilde{\mathcal{O}}_{C,P}/\mathcal{O}_{C,P}$$

the *delta invariant* or *order of singularity* of C at P (see, for example, [**22**, Chapter IV.2]). We remark that $\delta_P(C) > 0$ if and only if $P \in \operatorname{Sing} C$, and by Milnor's formula

$$2\delta_P(C) = \mu_P(C) + r_P(C) - 1, \tag{3.1}$$

where $\mu_P(C)$ is the *(generalized) Milnor number* in the sense of Buchweitz and Greuel [**6**, Proposition 1.2.1], which is zero if and only if the curve C is smooth at P.

If we define by

$$\delta(C) = \sum_{i=1}^{s} \delta_{P_i}(C)$$

the *total delta invariant* of C and by $p_a(C)$ and $g(C) = p_a(\tilde{C})$ the *arithmetic* and *geometric genus* of C, respectively, then (see [**20**, Theorem 8] or [**22**, Chapter IV.7, Proposition 3])

$$\delta(C) = p_a(C) - g(C). \tag{3.2}$$

If $N \geqslant 3$, following [**21**, Chapter IX, § 1] we denote by $\mu_1(C)$ the *rank of the curve* C, that is, the number of points $P \in C$ such that the tangent line $T_{C,P}$ intersects a generic linear subspace Λ of $\mathbb{P}^N$ of dimension $N-2$, or, equivalently, the degree of the image of C under the Gauss map $C \dashrightarrow G(1,N)$ (the Grassmannian of lines in $\mathbb{P}^N$) with respect to the Plücker embedding (see, for example, [**12**, Chapter 2, § 4]). It is immediate that

$$\mu_1(C) = \deg R(\pi_\Lambda), \tag{3.3}$$

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where $R(\pi_\Lambda)$ denotes the ramification locus of the linear projection $\pi_\Lambda\colon C \to \mathbb{P}^1$ of the curve C along the generic $(N-2)$-plane $\Lambda \subset \mathbb{P}^N$.

We denote by $\operatorname{Tan} C$ the *tangent variety* of C, that is, the closure of the union of all projective tangent lines to C at smooth points.

Proposition 3.2. *For any non-plane curve C,*

$$\mu_1(C) = \deg \operatorname{Tan} C$$

holds.

Proof. By the definition of the rank we have that

$$\mu_1(C) = \gamma \deg \operatorname{Tan} C, \tag{3.4}$$

where γ denotes the number of tangent lines to C passing through a generic point of $\operatorname{Tan} C$. We observe that if C is not a plane curve, then $\dim \operatorname{Sec} C = 3$. By [**11**, Corollary 4.3.3] it follows that the tangent lines to C at generic points do not intersect; hence, $\gamma = 1$ and the proof is complete. $\square$

The self-intersection cycle of C is

$$v(C, C) = [C] + \operatorname{rat}(v_0) + \operatorname{mov}(v_0) + \beta_0. \tag{3.5}$$

For a point $P \in C$ we denote by j_P the self-intersection number $j(C, C; P)$. Then, by [**2**, Corollary 2.5] or [**11**, Corollary 5.4.13],

$$\operatorname{rat}(v_0) = \sum_{P \in \operatorname{Sing} C} j_P[P] = \sum_{i=1}^{s} j_i[P_i];$$

hence, by taking degrees,

$$\deg(\operatorname{rat}(v_0)) = \sum_{i=1}^{s} j_i.$$

Proposition 3.3. *Let $C \subset \mathbb{P}^N$ $(N \geqslant 3)$ be a non-plane curve. With Notation 3.1 we have that*

$$\deg(\operatorname{mov}(v_0)) = \mu_1(C) = \deg \operatorname{Tan} C = 2d + 2g - 2 - \sum_{i=1}^{s} (m_i - r_i).$$

Proof. By [**2**, Corollary 2.5] or [**11**, Corollary 5.4.13], $\operatorname{mov}(v_0)$ is the part of v_0 supported on $\operatorname{Sm}(C)$ and, by [**10**, Theorem 4.6], it is the restriction to $\operatorname{Sm}(C)$ of the ramification locus of a projection $\pi_\Lambda\colon C \to \mathbb{P}^1$ of C along a generic $(N-2)$-plane $\Lambda \subset \mathbb{P}^N$. By (3.3) we have the first equality and by Proposition 3.2 we have the second one. The last equality can be deduced from the general Plücker formulae (see, for example, [**12**, p. 273]). With the notation of [**12**], for $k = 0$ one has that $d_{-1} + d_1 = \deg \operatorname{Tan} C$ and the total ramification index of C (called β_0 in [**12**, p. 268]) is $\sum_{i=1}^{s} (m_i - r_i)$. $\square$

Remark 3.4. The last equality of Proposition 3.3 can also be obtained in the following way. With Notation 3.1, let $\tau\colon \tilde{C} \to C$ be the normalization of C and let $\tau \circ \pi\colon \tilde{C} \to \mathbb{P}^1$ be the composition of τ and π. By Hurwitz's formula (see, for example, [**12**, p. 216])

$$2d + 2g - 2 = \sum_{i=1}^{s}\sum_{j=1}^{r_i}(v_{Q_{ij}} - 1) + \sum_{P\in\tau^{-1}(\operatorname{Sm} C)}(v_P - 1), \tag{3.6}$$

where v_R denotes the ramification index of $\tau \circ \pi$ at the point $R \in \tilde{C}$; hence,

$$\deg \operatorname{mov}(v_0) = \sum_{P\in\tau^{-1}(\operatorname{Sm} C)}(v_P - 1) = 2d + 2g - 2 - \sum_{i=1}^{s}\sum_{j=1}^{r_i}(v_{Q_{ij}} - 1). \tag{3.7}$$

For $P_i \in \operatorname{Sing} C$ and $\lambda \in \mathcal{O}_{\mathbb{P}^N,P_i}$ a local equation for the hyperplane $\langle \Lambda, P_i\rangle$, we have that

$$v_{Q_{ij}} = \operatorname{length}(\mathcal{O}_{\tilde{C},Q_{ij}}/\lambda\mathcal{O}_{\tilde{C},Q_{ij}});$$

hence,

$$\begin{aligned}\sum_{i=1}^{s}\sum_{j=1}^{r_i}(v_{Q_{ij}} - 1) &= \sum_{i=1}^{s}\left(\sum_{j=1}^{r_i}\operatorname{length}(\mathcal{O}_{\tilde{C},Q_{ij}}/\lambda\mathcal{O}_{\tilde{C},Q_{ij}}) - r_i\right)\\ &= \sum_{i=1}^{s}(\operatorname{length}(\mathcal{O}_{C,P_i}/\lambda\mathcal{O}_{C,P_i}) - r_i) = \sum_{i=1}^{s}(m_i - r_i).\end{aligned}$$

Therefore,

$$\deg \operatorname{mov}(v_0) = 2d + 2g - 2 - \sum_{i=1}^{s}(m_i - r_i),$$

as required.

Taking degrees in (3.5) and using Notation 3.1, from the above proposition and Theorem 2.4 we have that

$$(\deg C)^2 = \deg C + \sum_{i=1}^{s} j_i + \deg \operatorname{Tan} C + \deg \operatorname{Sec} C \deg(J/\operatorname{Sec} C), \tag{3.8}$$

that is,

$$d^2 = d + \sum_{i=1}^{s} j_i + 2d + 2g - 2 - \sum_{i=1}^{s}(m_i - r_i) + \deg \operatorname{Sec} C \deg(J/\operatorname{Sec} C), \tag{3.9}$$

from which we can deduce the following theorem for the degree of the secant varieties of non-degenerate curves in $\mathbb{P}^N$ (see also [**25**, Chapter 3] for ideas in this direction).

Theorem 3.5. *Let $C \subset \mathbb{P}^N$ $(N \geqslant 4)$ be a non-degenerate curve. With Notation 3.1, we have that*

$$\deg \operatorname{Sec} C = \frac{c_0(C,C)}{2} = \binom{d-1}{2} - g - \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i).$$

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Proof. First note that $\dim \operatorname{Sec} C = 3$. Hence, by Proposition 2.5 the rational map $J(C,C) \dashrightarrow \operatorname{Sec} C$ is finite of degree 2. Since, by Theorems 2.2 and 2.4,

$$\deg \beta_0 = c_0(C,C) = \deg \operatorname{Sec} C \deg(J(C,C)/\operatorname{Sec} C),$$

by (3.9) we complete the proof. □

Remark 3.6. With the notation of the preceding theorem, Dale proved in [**8**, Theorem 4.3] that

$$\deg \operatorname{Sec} C = \tfrac{1}{2}(d^2 - 3d - p_1),$$

where d and p_1 are the degrees of the Segre classes of the curve. Combining our formula with Dale's result, we obtain that

$$p_1 = g + 1 - \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i).$$

Applying (3.9) to singular curves in $\mathbb{P}^3$ we obtain a generalization of the secant formula of Peters and Simonis, whose result holds for n-dimensional non-singular projective varieties in $\mathbb{P}^{2n+1}$ (see [**18**, Theorem 3.4]). The formula we present in the next proposition also holds for possibly singular curves.

Proposition 3.7. *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve. Under Notation 3.1, the number ρ of secants of C passing through a generic point of $\mathbb{P}^3$ is given by*

$$\rho = \binom{d-1}{2} - g - \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i).$$

Proof. We remark that $\operatorname{Sec} C = \mathbb{P}^3$ and the number 2ρ is the degree of the finite map $J(C,C) \dashrightarrow \mathbb{P}^3$, that is, $\deg \beta_0$. By (3.9) the proof is complete. □

As a trivial consequence of Theorem 3.5 and Proposition 3.7 there exists the following bound on the geometric genus of a space curve $C \subset \mathbb{P}^N$, $N \geqslant 3$.

Corollary 3.8. *Let $C \subset \mathbb{P}^N$, $N \geqslant 3$, be a non-degenerate curve. Under Notation 3.1, the bounds*

$$g < \binom{d-1}{2} - \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i) \leqslant \binom{d-1}{2} - \frac{s}{2}$$

exist.

Proof. For the second inequality, observe that in the formulae of Theorem 3.5, Proposition 3.7 and Corollary 3.8 it holds that

$$j_i - m_i + r_i > 0.$$

In fact, by the local version of Bézout's theorem (see [**4**, Corollary 3.8] or [**11**, Corollary 5.4.10]), for each singular point $P_i \in C$ it holds that

$$j_i \geqslant m_i^2 - m_i \geqslant m_i.$$

Therefore,

$$\sum_{i=1}^{s}(j_i - m_i + r_i) \geqslant s.$$

□

Now, with Notation 3.1 and using a generic plane projection of the curve, we deduce from Theorem 3.5 and Proposition 3.7 a formula for the difference between the arithmetic genus $p_a(C)$ and the geometric genus $g(C)$ of a curve $C \subset \mathbb{P}^N$, $N \geqslant 2$, in terms of the singularities of the curve (see Corollary 3.15). Our formula extends the classical genus formula of Max Noether for plane curves to curves in $\mathbb{P}^N$, $N > 2$. This is possible by expressing the degree of the secant variety of C as a function of the degree and the arithmetic genus of C (see Corollary 3.13). We begin with a lemma for which we need to introduce the following notation.

For each $P \in \operatorname{Sing} C$ we consider an affine chart containing P and we regard P as a point in $\mathbb{C}^N$. There, we take the cone $C_5(C, P)$ (in the sense of Whitney [**27**, p. 212]) of limits of secant vectors to the germ (C, P). We remark that this cone was called the *limit of secant variety* of C at P in [**11**]. More precisely, a vector v belongs to $C_5(C, P)$ if and only if there are sequences of points $\{q_n\}$, $\{r_n\}$ of C converging to P and complex numbers λ_n such that $\lambda_n(q_n - r_n) \to v$. By [**5**, Propostion IV.1] it is known that the affine cone $P + C_5(C, P)$ consists of a finite number of 2-planes, each of them passing through a tangent line to C at P.

For each $(N-3)$-dimensional subspace Γ of $\mathbb{P}^N$ we denote by $\pi_\Gamma : C \to \mathbb{P}^2$ the restriction of the linear projection $\mathbb{P}^N \setminus \Gamma \to \mathbb{P}^2$ to C. We set $C^\Gamma := \pi_\Gamma(C)$ and $P^\Gamma := \pi_\Gamma(P)$. We recall that $r_P(C)$ denotes the number of branches of C at P.

Lemma 3.9. *For each non-degenerate curve $C \subset \mathbb{P}^N$, $N \geqslant 4$, there exists a dense open subset A of the Grassmannian $G(N-3, N)$ of $(N-3)$-dimensional subspaces of $\mathbb{P}^N$ such that, for all $\Gamma \in A$, the following hold.*

- *The singularities of the plane curve C^Γ are precisely the s distinct points $P_1^\Gamma, \dots, P_s^\Gamma$ that are images of the singular points of C and $\deg \operatorname{Sec}(C)$ ordinary double points.*
- *For each $P \in \operatorname{Sing} C$, $r_{P^\Gamma}(C^\Gamma) = r_P(C)$ and the Milnor number $\mu_{P^\Gamma}(C^\Gamma)$ is constant as Γ varies in A.*

Proof. Let A be the open subset of $G(N-3, N)$ such that any $\Gamma \in A$ intersects the secant variety $\operatorname{Sec}(C)$ at $\deg \operatorname{Sec}(C)$ distinct points that are not on the projective closures in $\mathbb{P}^N$ of the affine cones $P_i + C_5(C, P_i)$, $i = 1, \dots, s$, and such that through each of the points of $\operatorname{Sec}(C) \cap \Gamma$ there passes exactly one secant line to C, which is not a tangent line, not a trisecant line and not a line meeting C in two points with intersecting tangent lines. The open subset A satisfies the statement of the lemma. For the smooth case see, for example, [**14**, Chapter IV, Propositions 3.4, 3.7 and Theorem 3.10]; for the singular case see [**5**, Chapter IV]. □

Remark 3.10. Note that

$$\tfrac{1}{2}(\mu_{P^\Gamma}(C^\Gamma) - \mu_P(C)) = \delta_{P^\Gamma}(C^\Gamma) - \delta_P(C) =: \ell_P(C)$$

is constant as Γ varies in A (the first equality follows by Milnor's formula (3.1)). The local analytic invariant $\ell_P(C)$ has been studied by Briançon *et al*. [**5**, Chapter IV, (b)].

If we denote by $\mathfrak{c}_P$ the conductor ideal of the local ring $\mathcal{O}_{C^\Gamma,P^\Gamma}$ in the local ring $\mathcal{O}_{C,P}$, the invariant $\ell_P(C)$ can also be described as (see [**17**, Proposition 1.5 and Theorem 3.6])

$$\ell_P(C) = \dim_{\mathbb{C}}(\mathcal{O}_{C^\Gamma,P^\Gamma}/\mathfrak{c}_P\mathcal{O}_{C^\Gamma,P^\Gamma}) = \dim_{\mathbb{C}}(\mathcal{O}_{C,P}/\mathcal{O}_{C^\Gamma,P^\Gamma}).$$

From this description and by the local criterion for isomorphism (see, for example, [**13**, Theorem 14.9 and Corollary 14.10]) it is clear that $\ell_P(C) = 0$ if and only if the *embedding dimension of C at P*, which we will denote by $\operatorname{emdim}_P(C)$, is at most 2.

Lemma 3.11. *For each non-degenerate curve $C \subset \mathbb{P}^N$, $N \geqslant 4$, there exists a dense open subset A of $G(N-3, N)$ such that for each $\Gamma \in A$ we have that*

$$\delta(C^\Gamma) - \delta(C) = \deg \operatorname{Sec}(C) + \sum_P \ell_P(C),$$

where P runs over all (singular) points of C with $\operatorname{emdim}_P(C) \geqslant 3$.

Proof. Let $A \subset G(N-3, N)$ be the open subset of the previous lemma. We note that if $\Gamma \in A$, then the linear projection $\pi_\Gamma \colon C \to C^\Gamma \subset \mathbb{P}^2$ is one-to-one except in the preimages of the $\deg \operatorname{Sec}(C)$ ordinary double points of C^Γ, which correspond to the $\deg \operatorname{Sec}(C)$ bisecants of C intersecting Γ.

Moreover, if P is a singular point with $\operatorname{emdim}_P(C) = 2$, then the curve germs (C, P) and (C^Γ, P^Γ) are isomorphic (see, for example, [**13**, p. 179]). Hence, in this case the invariant $\ell_P(C) = 0$ (see [**5**, Remarque IV.5]). □

For space curves we have an analogous result.

Lemma 3.12. *Maintaining the above notation, let C be a non-degenerate irreducible and reduced curve in $\mathbb{P}^3_{\mathbb{C}}$ and let $\rho(C)$ be the number of secants of the curve passing through a generic point of $\mathbb{P}^3_{\mathbb{C}}$. Then,*

$$\delta(C^\Gamma) - \delta(C) = \rho(C) + \sum_P \ell_P(C),$$

where P runs over all (singular) points of C with $\operatorname{emdim}_P(C) = 3$.

Proof. Let $\Gamma \in \mathbb{P}^3$ be a point not on the projective closures in $\mathbb{P}^3$ of the cones $C_5(C, P)$ at the singular points P of C and such that through Γ there pass exactly $\rho(C)$ secant lines to C, which are not tangent lines, not trisecant lines and not lines meeting C in two points with intersecting tangent lines.

If we apply the same argument used in the proof of Lemma 3.9 to the linear projection $\pi_\Gamma \colon C \to \mathbb{P}^2$, the proof is complete. □

From the above lemmas, we have the following corollaries.

Corollary 3.13. *With the preceding notation, if $N \geqslant 4$, the following holds:*

$$\deg \operatorname{Sec}(C) = \binom{d-1}{2} - p_a(C) - \sum_P \ell_P(C),$$

where P runs over all (singular) points of C with $\operatorname{emdim}_P(C) \geqslant 3$.

Proof. By (3.2) and taking into account that $g(C) = g(C^\Gamma)$, we obtain that

$$\begin{aligned}\delta(C^\Gamma) - \delta(C) &= p_a(C^\Gamma) - g(C^\Gamma) - (p_a(C) - g(C)) \\ &= p_a(C^\Gamma) - p_a(C) \\ &= \binom{d-1}{2} - p_a(C).\end{aligned}$$

Now the proof follows by Lemma 3.11. □

Corollary 3.14. *With the preceding notation, if $N = 3$, the following holds:*

$$\rho(C) = \binom{d-1}{2} - p_a(C) - \sum_P \ell_P(C),$$

where P runs over all (singular) points of C with $\operatorname{emdim}_P(C) = 3$.

The following result extends the classical genus formula of Max Noether for plane curves to curves in $\mathbb{P}^N$, $N \geqslant 3$.

Theorem 3.15. *If $N \geqslant 2$, with the preceding notation, setting $\ell_i := \ell_{P_i}(C)$, then*

$$p_a(C) - g(C) = \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i - 2\ell_i).$$

Moreover,

$$p_a(C) - g(C) = \frac{1}{2}\sum_{i=1}^{s}(j_i - m_i + r_i)$$

holds if and only if for each point $P \in C$ the embedding dimension is at most 2.

Proof. The formula is known in the case $N = 2$ (see [**9**, Noether's genus formula on p. 180]), since for a plane curve C the self-intersection number $j(C, C; P_i) = j_i$ equals the intersection multiplicity of a generic polar curve with the curve at the point P_i (see, for example, [**11**, Example 2.2.10] and [**9**, Appendix 5]).

For $N \geqslant 3$ it is an immediate consequence of our main results, Theorem 3.5 and Proposition 3.7, taking into account the previous two corollaries. □

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Remark 3.16. The preceding proposition shows that for curves $C \subset \mathbb{P}^N$ the self-intersection multiplicity $j(C, C; P_i)$ is a natural generalization of the intersection multiplicity of a generic polar curve with the curve at the point P_i, which makes sense only for plane curves. In the plane, both multiplicities coincide.

We illustrate our results with two examples of curves $C \subset \mathbb{P}^n$, $n > 3$, with a singular point of embedding dimension greater than 2 (Example 3.17) and embedding dimension 2 (Example 3.18). In Example 3.17 it is interesting to see how the δ-invariant of the singular point grows under projection to $\mathbb{P}^2$. Finally, we give an example of a curve in $\mathbb{P}^3$ (Example 3.19) in order to illustrate the computation of the number of secants passing through a general point of $\mathbb{P}^3$ according to Proposition 3.7. In all these examples the curves have precisely one singular point P, for which j_P and m_P can be easily obtained by a single computer calculation as in [**1**, sample file Segre4.txt]. Then the Stückrad–Vogel intersection cycle $v(C, C)$ is

$$v(C, C) = [C] + j_P[P] + [P_1] + \cdots + [P_t],$$

where $t = \deg \operatorname{Tan} C$ and $P_1, \ldots, P_t$ are movable points on $C \setminus \{P\}$. We also remark that the contribution of the empty set ($\deg \beta_0$ in Theorem 2.3) to the Bézout number is $2 \deg \operatorname{Sec} C$ (Examples 3.17 and 3.18) or 2ρ (Example 3.19).

Example 3.17. Let $C \subset \mathbb{P}^4$ be the curve defined by the homogenous ideal

$$(x_0x_4 - x_3^2, 4x_2^2 - 17x_3^2 - 191x_4^2, 4x_1^2 - x_3^2 - 15x_4^2)$$

in $K[x_0, \ldots, x_4]$. Then, C is a curve of degree $d = 8$, which is singular only at the point $P = (1 : 0 : 0 : 0 : 0)$. By a computer calculation we obtain that $j_P = 12$, $m_P = 4$ and $\frac{1}{2}c_0(C, C) = \deg \operatorname{Sec}(C) = 14$. Moreover, we have $p_a(C) = 5$, $r_P(C) = 4$, $\operatorname{emdim}_P(C) = 3$ and $g(C) = 1$. Finally, by (3.8) and Proposition 2.5, $\deg \operatorname{Tan}(C) = 16$.

In this example, since C is a complete intersection, we can compute $\delta(C) = \delta_P(C)$ by using Lê's formula (see [**16**]) for the Milnor number $\mu_P(C)$ of the curve germ at P and taking into account Milnor's formula $\mu_P(C) = 2\delta_P(C) - r_P(C) + 1$. We apply Lê's result [**16**] to the germ of the holomorphic map $f = (f_1, f_2, f_3)\colon (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ with $f_1 = 4x_1^2 + 4x_2^2 - 18x_3^2 - 206x_4^2$, $f_2 = 4x_2^2 - 17x_3^2 - 191x_4^2$ and $f_3 = x_4 - x_3^2$. Then, denoting by $\mathcal{O}$ the ring of germs of holomorphic functions at $0 \in \mathbb{C}^4$ and by J_l, $l = 1, 2, 3$, the ideal generated by $f_1, \ldots, f_{l-1}$ and all the $(l \times l)$-minors of the matrix $(\partial f_i / \partial x_j)$, $i = 1, \ldots, l$, we get that

$$\mu_P(C) = \dim_{\mathbb{C}} \mathcal{O}/J_1 - \dim_{\mathbb{C}} \mathcal{O}/J_2 + \dim_{\mathbb{C}} \mathcal{O}/J_3 = 1 - 8 + 12 = 5;$$

hence, $\delta_P(C) = 4$.

If we consider a generic plane projection $\bar{C}$ of C and if we denote by $\bar{P}$ the image of P, we conclude, by Lemma 3.11 and taking into account that

$$\delta(C^{\Gamma}) - \delta(C) = \binom{d-1}{2} - p_a(C)$$

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(see the proof of Corollary 3.13), that

$$\delta_{\bar{P}}(\bar{C}) = \binom{d-1}{2} - p_a(C) - \deg \operatorname{Sec}(C) + \delta_P(C) = 6.$$

Such a generic projection can be obtained as the composition of the linear projections from the point $(0:0:0:-1:1)$ into the $\mathbb{P}^3$ given by $x_4 = 0$ and from the point $(0:-1:1:1) \in \mathbb{P}^3$ into the $\mathbb{P}^2$ of $x_2 = 0$. The resulting plane curve $\bar{C}$ has, in fact, 15 distinct singular points, say $Q_1, \dots, Q_{14}$ with $\delta_{Q_1}(\bar{C}) = \dots = \delta_{Q_{14}}(\bar{C}) = 1$ and $\bar{P}$ with $\delta_{\bar{P}}(\bar{C}) = 6$.

Example 3.18. Let $C \subset \mathbb{P}^5$ be the curve defined by the homogenous ideal

$$(x_0x_3 - x_1^2, x_0x_4 - x_1x_2, x_1x_4 - x_2x_3, x_0x_5 - x_2^2,\\ x_1x_5 - x_2x_4, x_3x_5 - x_4^2, 2x_0^2 + x_0x_2 + x_0x_5 + x_2x_5 - x_3x_5)$$

in $K[x_0, \dots, x_5]$. Then, C is a curve of degree $d = 8$ and genus $g = 1$, which is singular only at the point $P = (0:0:0:1:0:0)$ with $\operatorname{emdim}_P(C) = 2$. Furthermore, $j_P = 4$, $m_P = 2$ and $r_P = 2$, so $\deg \operatorname{Sec} C = 18$. In fact, $\operatorname{Sec} C$ is a complete intersection of two hypersurfaces of degrees 3 and 6, respectively.

Example 3.19. Let $C \subset \mathbb{P}^3$ be the curve defined by the homogenous ideal

$$(x_1^2 + x_3^2 - x_0x_2, x_2^2 - x_1x_3) \subset K[x_0, \dots, x_3].$$

Then, C is a curve of degree $d = 4$ and genus $g = 0$, which is singular only at the point $P = (1:0:0:0)$ with $\operatorname{emdim}_P(C) = 2$. Furthermore, $j_P = 2$, $m_P = 2$ and $r_P = 2$, so $\rho = 2$ by Proposition 3.7. In this example the projection from the point $(0:0:1:0)$ gives a generic projection of C into the plane of $x_2 = 0$. The image of C is the plane curve of

$$x_0^2x_1x_3 - x_1^4 + 2x_1^2x_3^2 - x_3^4 = 0,$$

which has three ordinary double points, one of which is the image of the singular point P.

4. Intersection numbers for curves

In view of Theorem 3.5 and Proposition 3.7 it is important to know how to calculate for two curves $C, D \subset \mathbb{P}^N_{\mathbb{C}}$ $(N \geqslant 3)$ the intersection multiplicities $j_P = j(C, D; P)$ of C and D at a point $P \in C \cap D$. We recall that in the case of a self-intersection of C only the singular points $P_1, \dots, P_s$ of C have to be considered. Of course, the intersection multiplicities j_P can be calculated using their algorithmic definition, but this requires primary decomposition and is quite hard to do; see [**24**, Example 2], where the self-intersection of the curve $C \subset \mathbb{P}^3$ given parametrically by $(s^6, s^4t^2, s^3t^3, t^6)$ is computed. This computation can also be done by using computer algebra systems and the generalized Samuel multiplicities of [**3**] (see [**1**]; in particular the sample file Segre4.txt).

A better way is to use a formula of Chądzyński *et al.* [**7**], which expresses $j(C, D; P)$ in terms of local parametrizations of C and D near P, or the natural extension of this formula to the case of self-intersection given by Krasiński [**15**]. We use this result below to compute the self-intersection numbers j_i in the case of monomial curves. Ranestad's result [**19**, Proposition 3.1] on the degree of secant varieties of monomial curves is then a corollary of Theorem 3.5.

In order to calculate $j(C, D; P)$, we may consider C and D as analytic curves locally in a neighbourhood Ω of $P = 0 \in \mathbb{C}^N$ and we may assume that $C = C_1 \cup \cdots \cup C_m$ and $D = D_1 \cup \cdots \cup D_n$, where C_i, D_j are analytic curves in Ω having irreducible germs at 0. Then, by the bilinearity of the intersection cycle $v(C, D)$ we have that $j(C, D; 0) = \sum_{i,j} j(C_i, D_j; 0)$. Hence, we may restrict our considerations to analytic curves C and D with irreducible germs at 0. By Puiseux's theorem such curves have local parametrizations. To formulate the result of [**7**], we need the order of a holomorphic map $\lambda = (\lambda_2, \ldots, \lambda_N)\colon \mathbb{C} \to \mathbb{C}^{N-1}$, which is defined as $\operatorname{ord}\lambda := \min\{\operatorname{ord}\lambda_i \mid i = 2, \ldots, N\}$, where $\operatorname{ord}\lambda_i$ means the order of λ_i at 0.

Proposition 4.1 (Chądzyński *et al.* [7, Theorem 1]; Krasiński [15, Theorem 4]). *Let Ω be a neighbourhood of $0 \in \mathbb{C}^N$ $(N \geqslant 2)$ and let C and D be analytic curves in Ω with irreducible germs at 0. In addition, let*

$$U \ni t \mapsto (t^p, \varphi(t)) \in C \quad \text{and} \quad V \ni \tau \mapsto (\tau^q, \psi(\tau)) \in D$$

be parametrizations of C and D in Ω, respectively (U and V are neighbourhoods of 0 in $\mathbb{C}$), let $\operatorname{ord}\varphi > p$, let $\operatorname{ord}\psi > q$, and let η and ε be primitive roots of unity of degree q and p, respectively.

- *If 0 is an isolated point of intersection of C and D, then*

$$j(C, D; 0) = \frac{1}{q}\sum_{i=1}^{q} \operatorname{ord}(\varphi(t^q) - \psi(\eta^i t^p)) = \frac{1}{p}\sum_{i=1}^{p} \operatorname{ord}(\psi(t^p) - \varphi(\varepsilon^i t^q)).$$

- *If 0 is a singular point of C, then*

$$j(C, C; 0) = \sum_{i=1}^{p-1} \operatorname{ord}(\varphi(t) - \varphi(\varepsilon^i t)).$$

Recall that a monomial curve $C \subset \mathbb{P}^N$ $(N \geqslant 2)$ is defined to be the image of an injective morphism $\mathbb{P}^1 \to \mathbb{P}^N$ defined by monomials. After ordering the monomials by ascending degree it is, therefore, given by

$$(s : t) \mapsto (s^d : s^{d-a_1}t^{a_1} : \cdots : s^{d-a_{N-1}}t^{a_{N-1}} : t^d),$$

where $a_1 < a_2 < \cdots < a_N = d$ are positive integers. Without loss of generality we may assume that $\gcd(a_1, \ldots, a_N) = 1$, so $d = \deg C$. The only possible singular points of C are the two points $P = (1 : 0 : \cdots : 0)$ and $Q = (0 : \cdots : 0 : 1)$. The curve C is singular at P if and only if $a_1 > 1$ and it is singular at Q if and only if $d - a_{N-1} > 1$.

In order to apply Theorem 3.5 to monomial curves we need an expression for the intersection numbers $j(C, C; P)$ and $j(C, C; Q)$ in terms of the numerical data of the curve.

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Proposition 4.2. *Let $C \subset \mathbb{P}^N_{\mathbb{C}}$ $(N \geqslant 2)$ be a monomial curve of degree d defined by the sequence of positive integers $a_1 < a_2 < \cdots < a_N = d$. For $i = 1, \dots, N-1$ set $b_i = d - a_{N-i}$, $g_i = \gcd(a_1, \dots, a_i)$, $h_i = \gcd(b_1, \dots, b_i)$, $b_N = d$ and $g_N = h_N = 1$. Then,*

$$j_P = j(C, C; P) = \sum_{i=1}^{N-1} a_{i+1}(g_i - g_{i+1}) \quad \text{and} \quad j_Q = j(C, C; Q) = \sum_{i=1}^{N-1} b_{i+1}(h_i - h_{i+1}).$$

Proof. By symmetry it is enough to prove the formula for $j(C, C; P)$. A local parametrization of C in a neighbourhood of P is given by $(t^{a_1}, \varphi(t))$ with $\varphi(t) = (t^{a_2}, \dots, t^{a_N})$. If ε denotes an a_1th primitive root of unity, then by Proposition 4.1 one has that

$$\begin{aligned} j_P &= \sum_{k=1}^{a_1-1} \operatorname{ord}(\varphi(t) - \varphi(\varepsilon^k t)) && (4.1\,a) \\ &= \sum_{k=1}^{a_1-1} \operatorname{ord}((t^{a_2}, \dots, t^{a_N}) - ((\varepsilon^k t)^{a_2}, \dots, (\varepsilon^k t)^{a_N})) \\ &= \sum_{k=1}^{a_1} \operatorname{ord}((1 - \varepsilon^{ka_2}) t^{a_2}, \dots, (1 - \varepsilon^{ka_N}) t^{a_N}). && (4.1\,b) \end{aligned}$$

The orders appearing in (4.1 b) can be $a_2, \dots, a_N$ or 0. The order will not be a_2 if and only if ka_2 is a multiple of a_1, and this happens $g_2 = \gcd(a_1, a_2)$ times. Taking away from (4.1 b) all orders equal to a_2 we get the contribution $a_2(a_1 - g_2) = a_2(g_1 - g_2)$ to j_P, which is just the first term of

$$\sum_{k=1}^{N-1} a_{k+1}(g_i - g_{k+1}).$$

Now collect among the g_2 remaining terms of (4.1 b) the orders equal to a_3. Such an order is not equal to a_3 if and only if both ka_3 and ka_2 are multiples of a_1, and this happens $g_3 = \gcd(a_1, a_2, a_3)$ times. Therefore, we have $g_2 - g_3$ orders equal to a_3, which we take away as before. Their contribution to j_P is just the second term of

$$\sum_i a_{i+1}(g_i - g_{i+1}).$$

Proceeding in this way we obtain

$$j_P = \sum_{i=1}^{N-1} a_{i+1}(g_i - g_{i+1}).$$

Note that $g_N = 1$ and, therefore, the number of as in

$$\sum_{i=1}^{N-1} a_{i+1}(g_i - g_{i+1})$$

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is

$$(g_1 - g_2) + \cdots + (g_{N-1} - g_N) = g_1 - g_N = a_1 - 1,$$

which is just the number of terms in (4.1 a). □

As an application of Theorem 3.5, Proposition 4.2 yields the following result of Ranestad (see [**19**, Proposition 3.1]) for the degree of the secant variety of a monomial curve.

Corollary 4.3. *With the notation of Proposition 4.2 and under the assumption that* $N \geqslant 4$, *one has that*

$$\deg \operatorname{Sec} C = \binom{d-1}{2} - \frac{1}{2}\left(\sum_{i=1}^{N-1} a_{i+1}(g_i - g_{i+1}) - a_1 + \sum_{i=1}^{N-1} b_{i+1}(h_i - h_{i+1}) - b_1\right) - 1.$$

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