53

Light and heavy quark masses, chiral condensates and weak leptonic decay constants

We review the present status for the determinations of the light and heavy quark masses, the light quark chiral condensate and the decay constants of light and heavy-light (pseudo)scalar mesons from QCD spectral sum rules (QSSR). Bounds on the light quark running masses at 2 GeV are found to be (see Tables 53.1 and 53.2): 6 MeV $< (\bar{m}_d + \bar{m}_u)(2) <$ 11 MeV and 71 MeV$< \bar{m}_s(2) <$ 148 MeV. The agreement of the ratio $m_s/(m_u + m_d) = 24.2$ in Eq. (53.45) from pseudoscalar sum rules with the one (24.4 $\pm$ 1.5) from ChPT indicates the consistency of the pseudoscalar sum rule approach. QSSR predictions from different channels for the light quark running masses lead to (see Section 53.9.3): $\bar{m}_s(2) = (117.4 \pm 23.4)$ MeV, $(\bar{m}_d + \bar{m}_u)(2) = (10.1 \pm 1.8)$ MeV, $(\bar{m}_d - \bar{m}_u)(2) = (2.8 \pm 0.6)$ MeV with the corresponding values of the RG invariant masses. The different QSSR predictions for the heavy quark masses lead to the running masss values: $\bar{m}_c(\bar{m}_c) = (1.23 \pm 0.05)$ GeV and $\bar{m}_b(\bar{m}_b) = (4.24 \pm 0.06)$ GeV (see Tables 53.5 and 53.6), from which one can extract the scale independent ratio $m_b/m_s = 48.8 \pm 9.8$. Runned until M_Z, the b-quark mass becomes: $\bar{m}_b(M_Z) = (2.83 \pm 0.04)$ GeV in good agreement with the average of direct measurements (2.82 ± 0.63) GeV from three-jet heavy quark production at LEP, and then supports the QCD running predictions based on the renormalization group equation. As a result, we have updated our old predictions of the weak decay constants $f_{\pi'(1.3)}$, $f_{K'(1.46)}$, $f_{a_0(0.98)}$ and $f_{K_0^*(1.43)}$ [see Eqs. (53.75) and (53.77)]. We obtain from a global fit of the light (pseudo)scalar and B_s mesons, the flavour breakings of the *normal ordered* chiral condensate ratio: $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle = 0.66 \pm 0.10$ [see Eq. (53.100)]. The last section is dedicated to the QSSR determinations of $f_{D_{(s)}}$ and $f_{B_{(s)}}$.

53.1 Introduction

One of the most important parameters of the standard model and chiral symmetry is the light and heavy quark masses. Light quark masses and chiral condensates are useful for a much better understanding of the realizations of chiral symmetry breaking [55–57] and for some eventual explanation of the origin of quark masses in unified models of interactions [664]. Within some popular parametrizations of the hadronic matrix elements [665], the strange quark mass can also largely influence the Standard Model prediction of the CP violating

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parameters ϵ'/ϵ, which have been measured recently [599]. However, contrary to the QED case where leptons are observed, and then the physical masses can be identified with the pole of the propagator (on-shell mass value),[1] the quark masses are difficult to define because of confinement which does not allow us to observe free quarks. However, despite this difficulty, one can consistently treat the quark masses in perturbation theory like the QCD coupling constant.They obey a differential equation, where its boundary condition can be identified with the renormalized mass of the QCD Lagrangian. The corresponding solution is the running mass, which is gauge invariant but renormalization scheme and scale dependent, and the associated renormalization group-invariant mass. To our knowledge, these notions have been introduced for the first time in [28]. In practice, these masses are conveniently defined within the standard $\overline{MS}$ scheme discussed in previous chapters. In addition to the determination of the ratios of light quark masses (which are scale independent) from current algebra [55], and from chiral perturbation theory (ChPT), its modern version [498–502], a lot of effort reflected in the literature [16] has been put into extracting directly from the data the running quark masses using the SVZ [1] QCD spectral sum rules (QSSR) [3], LEP experiments and lattice simulations. The content of these notes is:

- a review of the light and heavy quark mass determinations from the different QCD approaches;
- a review of the direct determinations of the quark vacuum condensate using QSSR and an update of the analysis of its flavour breakings using a global fit of the meson systems;
- an update of the determinations of the light (pseudo)scalar decay constants, which, in particular, are useful for understanding the $\bar{q}q$ contents of the light scalar mesons; and
- a review of the determinations of the weak leptonic decay of the heavy-light pseudoscalar mesons $D_{(s)}$ and $B_{(s)}$.

This review develops and updates the review papers [54,364] and some parts of the book [3]. It also updates previous results from original works.

53.2 Quark mass definitions and ratios of light quark masses

Let us remind ourselves of the meaning of quark masses in QCD. One starts from the mass term of the QCD Lagrangian:

$$\mathcal{L}_m = m_i \bar{\psi}_i \psi_i \ , \tag{53.1}$$

where m_i and ψ_i are respectively the quark mass and field. The renormalized mass will be improved by the uses of the RGE leading to the running mass, for which a definition is given in Section 11.11. We shall also use the short-distance pole masses defined in Section 11.12 and the alternative definition in Section 11.13. Finally, we often use the value of the ratios of quark masses from ChPT given in Eq. (42.5.4).

[1] For a first explicit definition of the perturbative quark pole mass in the $\overline{MS}$ scheme, see [147,133] (renormalization-scheme invariance) and [148] (regularization-scheme invariance).

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53.3 Bounds on the light quark masses

In QSSR, the estimate and lower bounds of the sum of the light quark masses from the pseudoscalar sum rule were first found in [167,626], while a bound on the quark mass difference was first derived in [666]. The literature in this subject of light quark masses increases with time.[2] However, it is in some sense quite disappointing that in most of the published papers no noticeable progress has been made since the early pioneering studies. The most impressive progress comes from the QCD side of the sum rules where new calculations have become available both on the perturbative radiative corrections known to order α_s^3 [167,667,442] and on the non-perturbative corrections [1,3].[3] Another new contribution is due to the inclusion of the tachyonic gluon mass as a manifestation of the resummation of the pQCD series [162,161,394]. Alas, no sharp result is available on the exact size of direct instanton contributions advocated to be important in this channel [383], while [385] claims the opposite. Though the instanton situation remains controversial, recent analysis [668,669] using the results of [670] based on the Instanton Liquid Model (ILM) of [386] indicates that this effect is negligible justifying the neglect of this effect in different analysis of this channel. However, it might happen that adding together the effect of the tachyonic gluon to that of the direct instanton might also lead to a double counting in a sense that there can be two alternative ways for parametrizing the non-perturbative vacuum [394]. In the absence of precise control of the origin and size of these effects, we shall consider them as new sources of errors in the sum rule analysis.

53.3.1 Bounds on the sum of light quark masses from pseudoscalar channels

Lower bounds for $(\bar{m}_u + \bar{m}_d)$ based on moments inequalities and the positivity of the spectral functions have been obtained, for the first time, in [167,626]. These bounds have been rederived recently in [671,672] to order α_s. As checked in [54] for the lowest moment and redone in [669] for higher moments, the inclusion of the α_s^3 term decrease by about 10 to 15% the strength of these bounds, which is within the expected accuracy of the result.

For definiteness, we shall discuss in details the pseudoscalar two-point function in the $\bar{u}s$ channel. The analysis in the $\bar{u}d$ channel is equivalent. It is convenient to start from the second derivative of the two-point function which is superficially convergent:

$$\Psi''(Q^2) = \int_0^\infty dt \frac{2}{(t+Q^2)^3} \frac{1}{\pi} \mathrm{Im}\Psi_5(t) \,. \tag{53.2}$$

The bounds follow from the restriction of the sum over all possible hadronic states which can contribute to the spectral function to the state(s) with the lowest invariant mass. The lowest hadronic state which contributes to the corresponding spectral function is the K–pole.

[2] Previous works are reviewed in [54,3].
[3] See also Part VIII on two-point functions where more references to original works are given.

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From Eq. (53.2) we then have:

$$\Psi_5''(Q^2) = \frac{2}{\left(M_K^2 + Q^2\right)^3} 2 f_K^2 M_K^4 + \int_{t_0}^{\infty} dt \frac{2}{(t+Q^2)^3} \frac{1}{\pi} \mathrm{Im}\Psi_5(t) \,, \tag{53.3}$$

where $t_0 = (M_K + 2m_\pi)^2$ is the threshold of the hadronic continuum.

It is convenient to introduce the moments $\Sigma_N(Q^2)$ of the hadronic continuum integral:

$$\Sigma_N(Q^2) = \int_{t_0}^{\infty} dt \frac{2}{(t+Q^2)^3} \times \left(\frac{t_0 + Q^2}{t + Q^2}\right)^N \frac{1}{\pi} \mathrm{Im}\Psi_5(t) \,. \tag{53.4}$$

One is then confronted with a typical moment problem (see e.g. [673].) The positivity of the continuum spectral function $\frac{1}{\pi}\mathrm{Im}\Psi_5(t)$ constrains the moments $\Sigma_N(Q^2)$ and hence the LHS of Eq. (53.3) where the light quark masses appear. The most general constraints among the first three moments for $N = 0, 1, 2$ are:

$$\Sigma_0(Q^2) \geq 0, \quad \Sigma_1(Q^2) \geq 0, \quad \Sigma_2(Q^2) \geq 0\,; \tag{53.5}$$

$$\Sigma_0(Q^2) - \Sigma_1(Q^2) \geq 0, \quad \Sigma_1(Q^2) - \Sigma_2(Q^2) \geq 0\,; \tag{53.6}$$

$$\Sigma_0(Q^2)\Sigma_2(Q^2) - (\Sigma_1(Q^2))^2 \geq 0\,. \tag{53.7}$$

The inequalities in Eq. (53.6) are in fact trivial unless $2Q^2 < t_0$, which constrains the region in Q^2 to too small values for pQCD to be applicable. The other inequalities lead however to interesting bounds which we next discuss.

The inequality $\Sigma_0(Q^2) \geq 0$ results in a first bound on the running masses:

$$[m_s(Q^2) + m_u(Q^2)]^2 \geq \frac{16\pi^2}{N_c} \frac{2 f_K^2 M_K^4}{Q^4} \times \frac{1}{\left(1 + \frac{M_K^2}{Q^2}\right)^3} \frac{1}{\left[1 + \frac{11}{3}\frac{\alpha_s(Q^2)}{\pi} + \cdots\right]}\,, \tag{53.8}$$

where the dots represent higher order terms which have been calculated up to $\mathcal{O}(\alpha_s^3)$, as well as non-perturbative power corrections of $\mathcal{O}(1/Q^4)$ and strange quark mass corrections of $\mathcal{O}(m_s^2/Q^2)$ and $\mathcal{O}(m_s^4/Q^4)$ including $\mathcal{O}(\alpha_s)$ terms. Notice that this bound is non-trivial in the large–N_c limit ($f_K^2 \sim \mathcal{O}(N_c)$) and in the chiral limit ($m_s \sim M_K^2$). The bound is of course a function of the choice of the Euclidean Q-value at which the RHS in Eq. (53.8) is evaluated. For the bound to be meaningful, the choice of Q has to be made sufficiently large. In [671] it is shown that $Q \geq 1.4$ GeV is already a safe choice to trust the pQCD corrections as such. The lower bound which follows from Eq. (53.8) for $m_u + m_s$ at a renormalization scale $\mu^2 = 4$ GeV^2 results in the solid curves shown in Fig. 53.1.

The resulting value of the bound at $Q = 1.4$ GeV is:

$$(m_s + m_u)(2) \geq 80 \text{ MeV} \quad \Longrightarrow \quad (m_u + m_d)(2) \geq 6.6 \text{ MeV}\,, \tag{53.9}$$

if one uses either ChPT and the previous SR analysis for the mass ratios. Radiative corrections tend to decrease the strengths of these bounds. Their contributions to the second

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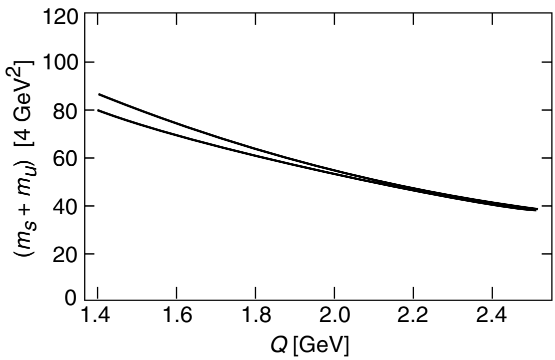

Fig. 53.1. Lower bound in MeV to order α_s for $(m_s + m_u)(2)$ versus Q in GeV from Eq. (53.8) for $\Lambda_3 = 290$ MeV (upper curve) and 380 MeV (lower curve).

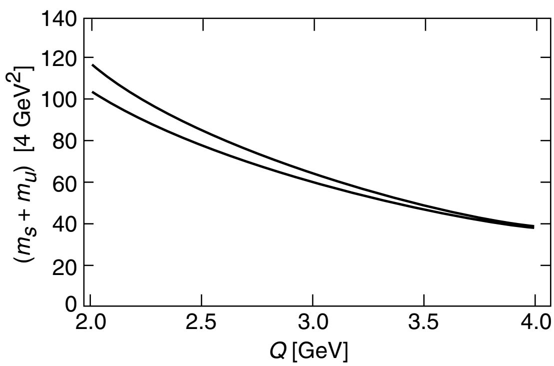

Fig. 53.2. The same curves as those in Fig. 53.1 but from the quadratic inequality to order α_s.

moment of the two-point function are (see previous part of the book):

$$\Psi_5''(q^2) = \frac{3}{8\pi^2}\frac{(\bar{m}_u + \bar{m}_s)^2}{Q^2}\left[1 + \frac{11}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right) + 14.179\left(\frac{\bar{\alpha}_s}{\pi}\right)^2 + 77.368\left(\frac{\bar{\alpha}_s}{\pi}\right)^3\right]. \quad (53.10)$$

At this scale, the PT series converges quite well and behaves as:

$$\text{Parton}[1 + 0.45 + 0.22 + 0.15]\,. \quad (53.11)$$

Including these higher order corrections, the bounds become:

$$(m_s + m_u)(2) > (71.4 \pm 3.7)\,\text{MeV} \quad \Longrightarrow \quad (m_u + m_d)(2) > (5.9 \pm 0.3)\,\text{MeV}\,, \quad (53.12)$$

The bound will be saturated in the extreme limit where the continuum contribution to the spectral function is neglected.

The quadratic inequality in Eq. (53.7) results in improved lower bounds for the quark masses which we show in Fig. 53.2.

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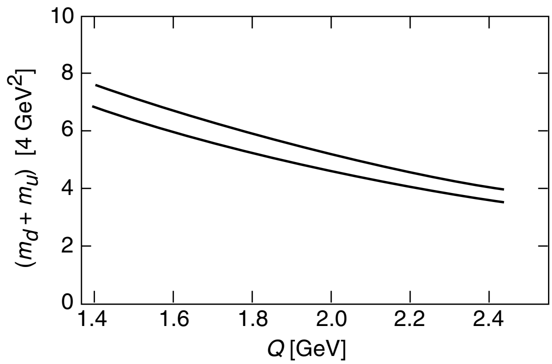

Fig. 53.3. Lower bound in MeV for $(m_d + m_u)(2)$ from the quadratic inequality to order α_s.

The quadratic bound is saturated for a δ–like spectral function representation of the hadronic continuum of states at an arbitrary position and with an arbitrary weight. This is certainly less restrictive than the extreme limit with the full hadronic continuum neglected, and it is therefore not surprising that the quadratic bound happens to be better than those from $\Sigma_N(Q^2)$ for $N = 0, 1,$ and 2. Notice however that the quadratic bound in Fig. 53.2 is plotted at higher Q-values than the bound in Fig. 53.1. This is due to the fact that the coefficients of the perturbative series in $\alpha_s(Q^2)$ become larger for the higher moments. In [671] it is shown that for the evaluation of the quadratic bound $Q \geq 2$ GeV is already a safe choice.

Similar bounds can be obtained for $(m_u + m_d)$ when one considers the two–point function associated with the divergence of the axial current:

$$\partial_\mu A^\mu(x) = (m_d + m_u) :\bar{d}(x) i\gamma_5 u(x): . \tag{53.13}$$

The method to derive the bounds is exactly the same as the one discussed above and therefore we only show, in Fig. 53.3 below, the results for the corresponding lower bounds which one obtains from the quadratic inequality. At $Q = 2$ GeV, one can deduce the lower bounds from the quadratic inequality:

$$(m_s + m_u)(2) > 105 \text{ MeV} , \qquad (m_u + m_d)(2) > 7 \text{ MeV} . \tag{53.14}$$

The convergence of the QCD series is less good here than in the lowest moment. It behaves as [669]:

$$\text{Parton}\left[1 + \frac{25}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right) + 61.79\left(\frac{\bar{\alpha}_s}{\pi}\right)^2 + 517.15\left(\frac{\bar{\alpha}_s}{\pi}\right)^3\right], \tag{53.15}$$

which numerically reads:

$$\text{Parton}[1 + 0.83 + 0.61 + 0.51] . \tag{53.16}$$

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Table 53.1. *Lower bounds on $\bar{m}_{u,d,s}(2)$ in MeV*

Observables	Sources	Authors
$\bar{m}_u + \bar{m}_d$		
6	π	LRT97 [671], Y97 [672] (updated here to orderα_s^3)
6.8	$\langle\bar{\psi}\psi\rangle$ + GMOR	DN98 [423] (leading order)
$\bar{m}_d - \bar{m}_u$		
1.1	$K\pi$	Y97 [672] (updated here to orderα_s^3)
$\bar{m}_s$		
71.4	K	LRT97 [671] (updated here to orderα_s^3)
90	$\langle\bar{\psi}\psi\rangle$ + ChPT	DN98 [423] (leading order)

This leads to the radiatively corrected lower bound to order α_s^3:

$$(m_s + m_u)\,(2) > (82.7 \pm 13.3)\ \text{MeV}\ , \qquad (m_u + m_d)(2) > (6 \pm 1)\ \text{MeV}\ , \quad (53.17)$$

where the error is induced by the truncation of the QCD series which we have estimated to be about the contribution of the last known α_s^3 term of the series.[4] From the previous analysis, and taking into account the uncertainties induced by the higher order QCD corrections, *the best lower bound* comes from the lowest inequality and is given in Eq. (53.12). The result is summarized in Table 53.1.

53.3.2 Lower bound on the light quark mass difference from the scalar sum rule

As in [666], one can extract lower bound on the light quark mass difference $(m_u - m_d)$ and $(m_u - m_s)$ working with the two-point function associated to the divergence of the vector current:

$$\partial_\mu V^\mu_{\bar{u}q} = (m_u - m_q) : \bar{\psi}_u(i)\psi_q : \ . \qquad (53.18)$$

The most recent analysis has been done in [672]. We have updated the result by including the α_s^3-term. It is given in Table 53.1.

53.3.3 Bounds on the sum of light quark masses from the quark condensate and $e^+e^- \to I = 0$ hadrons data.

Among the different results in [423], we shall use the range of the chiral $\langle\bar{\psi}\psi\rangle \equiv \langle\bar{u}u\rangle \simeq \langle\bar{d}d\rangle$ condensate from the vector form factor of $D \to K^* l\nu$. Using three-point function sum

[4] In [668], alternative bound has been derived using a Hölder type inequality. The lower bound obtained from this method, which is about 4.2 MeV is weaker than the one obtained previously.

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rules, the form factor reads to leading order:

$$V(0)=\frac{m_c(m_D+m_{K^*})}{4f_D f_{K^*}m_D^2 m_{K^*}}\exp\left[\left(m_D^2-m_c^2\right)\tau_1+m_{K^*}^2\tau_2\right] \tag{53.19}$$
$$\times\langle\bar\psi\psi\rangle\Bigg\{-1+M_0^2\left(-\frac{\tau_1}{3}+\frac{m_c^2}{4}\tau_1^2+\frac{2m_c^2-m_c m_s}{6}\tau_1\tau_2\right)$$
$$-\frac{16\pi}{9}\alpha_s\rho\langle\bar\psi\psi\rangle\left(\frac{2m_c}{9}\tau_1\tau_2-\frac{m_c^3}{36}\tau_1^3\right.$$
$$\left.-\frac{2m_c^3-m_c^2m_s}{36}\tau_1^2\tau_2+\frac{-m_c}{9}\tau_1^2+\frac{2m_s}{9}\tau_2^2+\frac{2}{9}m_s\tau_1\tau_2+\frac{4}{9}\frac{\tau_2}{m_c}\right)$$
$$+\frac{e^{m_c^2\tau_1}}{\langle\bar\psi\psi\rangle}\int_0^{s_{20}}ds_2\int_{s_2+m_c^2}^{s_{10}}ds_1\,\rho_v(s_1,s_2)e^{-s_1\tau_1-s_2\tau_2}\Bigg\}$$

$$\text{with}\rho_v(s_1,s_2)=\frac{3}{4\pi^2(s_1-s_2)^3}\left\{m_s\left((s_1+s_2)\left(s_1-m_c^2\right)-2s_1s_2\right)\right.$$
$$\left.+\,m_c\left((s_1+s_2)s_2-2s_2\left(s_1-m_c^2\right)\right)\right\}. \tag{53.20}$$

The factor $\rho\simeq 2\sim 3$ expresses the uncertainty in the factorization of the four quark condensate. In our numerical analysis, we start from standard values of the QCD parameters and use $f_{K^*}=0.15$ GeV($f_\pi=93.3$ MeV). The value of $f_D\simeq(1.35\pm0.07)f_\pi$ is consistently determined by a two-point function sum rule including radiative corrections as we shall see in the next chapter, where the sum rule expression can, for example, be found in [3]. The following parameters enter only marginally: $m_s(1\text{ GeV})=(0.15\sim0.19)$ GeV, $s_{10}=(5\sim7)$ GeV^2, $s_{20}=(1.5\sim2)$ GeV^2. Using the conservative range of the charm quark mass: m_c(pole) between 1.29 and 1.55 GeV (the lower limit comes from the estimate in [3] and the upper limit is one-half of the J/Ψ mass), one can deduce the running condensate value at 1 GeV [423]:

$$0.6\le\langle\bar\psi\psi\rangle/[-225\text{ MeV}]^3\le1.5\,. \tag{53.21}$$

This result has been confirmed by the lattice [674]. Using the GMOR relation:

$$2m_\pi^2f_\pi^2=-(m_u+m_d)\langle\bar uu+\bar dd\rangle+\mathcal{O}\left(m_q^2\right). \tag{53.22}$$

one can translate the upper bound into a lower bound on the sum of light quark masses. The lower bound on the chiral condensate can be used in conjunction with the positivity of the m_q^2 correction in order to give an upper bound to the quark mass value. In this way, one obtains:

$$6.8\text{ MeV}\le(\bar m_u+\bar m_d)(2\text{ GeV})\le11.4\text{ MeV}\,. \tag{53.23}$$

The resulting values are quoted in Tables 53.1 and 53.2. We expect that these bounds are satisfied within the typical 10% accuracy of the sum rule approach.

We also show in Table 53.2 the upper bound obtained in [354] by using the positivity of the spectral function from the analysis of the $e^+e^-\to I=0$ hadrons data where the determination will be discussed in the next section.

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Table 53.2. *Upper bounds on $\bar{m}_{u,d,s}$ (2) in MeV*

Observables	Sources	Authors
$\bar{m}_u + \bar{m}_d$		
11.4	$\langle\bar{\psi}\psi\rangle$ + GMOR	DN98 [423] (leading order)
$\bar{m}_s$		
148	$\langle\bar{\psi}\psi\rangle$ + ChPT	DN98 [423] (leading order)
147 ± 21	$e^+e^- + \tau$-decay	SN99 [354] (to orderα_s^3)

53.4 Sum of light quark masses from pseudoscalar sum rules

53.4.1 The (pseudo)scalar Laplace sum rules

The Laplace sum rule for the (pseudo)scalar two-point correlator reads (see [3,167,376, 400,582]:

$$\int_0^{t_c} dt \exp(-t\tau)\frac{1}{\pi}\text{Im}\Psi_{(5)}(t) \simeq (\bar{m}_u \pm \bar{m}_d)^2 \frac{3}{8\pi^2}\tau^{-2}\left[(1-\rho_1)\left(1+\delta_\pm^{(0)}\right) + \sum_{n=2}^{6}\delta_\pm^{(n)}\right], \tag{53.24}$$

where the indices 5 and + refer to the pseudoscalar current. Here, τ is the Laplace sum rule variable, t_c is the QCD continuum threshold and $\bar{m}_i$ is the running mass to three loops:

$$\rho_1 \equiv (1+t_c\tau)\exp(-t_c\tau)\,. \tag{53.25}$$

Using the results compiled in the previous chapter, the perturbative QCD corrections read for n flavours:

$$\begin{aligned}
\delta_\pm^{(0)} &= \left(\frac{\bar{\alpha}_s}{\pi}\right)\left[\frac{11}{3} - \gamma_1\gamma_E\right] \\
&\quad + \left(\frac{\bar{\alpha}_s}{\pi}\right)^2\left[\frac{10801}{144} - \frac{39}{2}\zeta(3) - \left(\frac{65}{24} - \frac{2}{3}\zeta(3)\right)n\right. \\
&\quad - \frac{1}{2}\left(1-\gamma_E^2\right)\left[\frac{17}{3}(2\gamma_1 - \beta_1) + 2\gamma_2\right] \\
&\quad \left. + \left(3\gamma_E^2 - 6\gamma_E - \frac{\pi^2}{2}\right)\frac{\gamma_1}{12}(2\gamma_1 - \beta_1)\right] \\
\delta_\pm^{(2)} &= -2\tau\left[\left[1 + \left(\frac{\bar{\alpha}_s}{\pi}\right)C_F(4+3\gamma_E)\right]\left(\bar{m}_i^2 + \bar{m}_j^2\right)\right. \\
&\quad \left. \mp \left[1 + \left(\frac{\bar{\alpha}_s}{\pi}\right)C_F(7+3\gamma_E)\right]\bar{m}_i\bar{m}_j\right],
\end{aligned} \tag{53.26}$$

where $C_F = 4/3$ and $\gamma_E = 0.5772\ldots$ is the Euler constant; $\gamma_1,\ \gamma_2$ and $\beta_1,\ \beta_2$ are respectively the mass-anomalous dimensions and β-function coefficients defined in a previous chapter. For three colours and three flavours, they read:

$$\gamma_1 = 2\,,\quad \gamma_2 = 91/12\,,\quad \beta_1 = -9/2\,,\quad \beta_2 = -8\,. \tag{53.27}$$

In practice, the perturbative correction to the sum rule simplifies as:

$$\delta_\pm^{(0)} = 4.82 a_s + 21.98 a_s^2 + 53.14 a_s^3 + \mathcal{O}\left(a_s^4\right) \quad : \quad a_s \equiv \left(\frac{\bar\alpha_s}{\pi}\right). \tag{53.28}$$

Introducing the RGI condensates defined in the previous chapter, the non-perturbative contributions are [325]:

$$\begin{aligned}
\delta_\pm^{(4)} = \frac{4\pi^2}{3}\tau^2 \Bigg[& \frac{1}{4}\overline{\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle} - \frac{\gamma_1}{\beta_1}\left(\frac{\bar\alpha_s}{\pi}\right)\sum_i \overline{\langle m_i\bar\psi_i\psi_i\rangle} - \frac{3}{8\pi^2}\frac{1}{(4\gamma_1+\beta_1)}\sum_i \bar m_i^4 \\
& + \left[1 + \left(\frac{\bar\alpha_s}{\pi}\right) C_F\left(\frac{11}{4} + \frac{3}{2}\gamma_E\right)\right]\left(\overline{\langle m_j\bar\psi_j\psi_j\rangle} + \overline{\langle m_i\bar\psi_i\psi_i\rangle}\right) \\
& \mp \left[2 + \left(\frac{\bar\alpha_s}{\pi}\right) C_F\left(7 + 3\gamma_E\right)\right]\left(\overline{\langle m_i\bar\psi_j\psi_j\rangle} + \overline{\langle m_j\bar\psi_i\psi_i\rangle}\right) \\
& - \frac{3}{2\pi^2}\Bigg[\frac{1}{(4\gamma_1+\beta_1)}\left[\frac{\pi}{\bar\alpha_s} + C_F\left(\frac{11}{4}+\frac{3}{2}\gamma_E\right) + \frac{1}{6}(4\gamma_1+\beta_1) \right.\\
& \left. - \frac{1}{4\gamma_1}(4\gamma_2+\beta_2)\right] - \frac{1}{4}(1-2\gamma_E)\Bigg]\left(\bar m_i^4 + \bar m_j^4\right) - \frac{3}{2\pi^2}\bar m_j^2 \bar m_i^2 \\
& \pm \Bigg[\frac{1}{(4\gamma_1+\beta_1)}\left[\frac{2\pi}{\bar\alpha_s} + \frac{1}{3}(4\gamma_1+\beta_1) - \frac{1}{2\gamma_1}(4\gamma_2+\beta_2) + C_F(7+3\gamma_E)\right] \\
& + \gamma_E\Bigg]\left(\bar m_j^3 \bar m_i + \bar m_i^3 \bar m_j\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\delta_\pm^{(6)} = \mp\frac{8\pi^2}{3}\tau^3 \Bigg[& \frac{1}{2}\left[m_j\left\langle\bar\psi_i\sigma^{\mu\nu}\frac{\lambda_a}{2}G^a_{\mu\nu}\psi_i\right\rangle + m_i\left\langle\bar\psi_j\sigma^{\mu\nu}\frac{\lambda_a}{2}G^a_{\mu\nu}\psi_j\right\rangle\right] \\
& - \frac{16}{27}\pi\bar\alpha_s[\langle\bar\psi_j\psi_j\rangle^2 + \langle\bar\psi_i\psi_i\rangle^2 \mp 9\langle\bar\psi_j\psi_j\rangle\langle\bar\psi_i\psi_i\rangle]\Bigg].
\end{aligned} \tag{53.29}$$

Beyond the SVZ expansion, one can have two contributions:

- The direct instanton contribution can be obtained from [386] and reads:

$$\delta_+^{\text{inst}} = \frac{\rho_c^2}{\tau^3}\exp\left(-r_c\right)\left[K_0(r_c) + K_1(r_c)\right] \tag{53.30}$$

  with: $r_c \equiv \rho_c^2/(2\tau)$; $\rho_c \approx 1/600\ \text{MeV}^{-1}$ being the instanton radius; K_i is the MacDonald function. However, one should notice that analogous contribution in the scalar channel leads to some contradictions ([386] and private communication from Valya Zakharov).

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- The tachyon gluon mass contribution can be deduced from [161]:

$$\delta_{\pm}^{\text{tach}} = -4\left(\frac{\bar{\alpha}_s}{\pi}\right)\lambda^2 , \tag{53.31}$$

where $(\alpha_s/\pi)\lambda^2 \simeq -0.06$ GeV^2 [161],

which completes the different QCD contributions to the two-point correlator.

53.4.2 The $\bar{u}d$ channel

From the experimental side, we do not still have a complete measurement of the pseudoscalar spectral function. In the past [3], one has introduced the radial excitation π' of the pion using a NWA where the decay constant has been fixed from chiral symmetry argument [57] and from the pseudoscalar sum rule analysis itself [422,675,3], through the quantity:

$$r_\pi \equiv \frac{M_{\pi'}^4 f_{\pi'}^2}{m_\pi^4 f_\pi^2} . \tag{53.32}$$

Below the QCD continuum t_c, the spectral function is usually saturated by the pion pole and its first radial excitation and reads:

$$\int_0^{t_c} dt \exp(-t\tau)\frac{1}{\pi}\mathrm{Im}\Psi_5(t) \simeq 2m_\pi^4 f_\pi^2 \exp\left(-m_\pi^2\tau\right)\left[1 + r_\pi \exp\left[\left(m_\pi^2 - M_{\pi'}^2\right)\tau\right]\right] . \tag{53.33}$$

The theoretical estimate of the spectral function enters through the not yet measured ratio r_π. Detailed discussions of the sum rule analysis can be found in [3,420,422]. However, this channel is quite peculiar due to the Goldstone nature of the pion, where the value of the sum rule scale ($1/\tau$ for Laplace and t_c for FESR) is relatively large, being about 2 GeV^2, compared with the pion mass where the duality between QCD and the pion is lost. Hopefully, this paradox can be cured by the presence of the new $1/q^2$ [162,161,394] due to the tachyonic gluon mass, which enlarges the duality region to lower scale and then minimizes the role of the higher states into the sum rule. This naïve NWA parametrization has been improved in [676] by the introduction of threshold effect and finite width corrections. Within the advent of ChPT, one has been able to improve the previous parametrization by imposing constraints consistent with the chiral symmetry of QCD [677]. In this way, the spectral function reads:

$$\frac{1}{\pi}\mathrm{Im}\Psi_5(t) \simeq 2m_\pi^4 f_\pi^2 \left[\delta\left(t - m_\pi^2\right) + \theta\left(t - 9m_\pi^2\right)\frac{1}{\left(16\pi^2 f_\pi^2\right)^2}\frac{t}{18}\rho^{3\pi}(t)\right] , \tag{53.34}$$

with:

$$\begin{aligned}\rho^{3\pi}(t) = &\int_{4m_\pi^2}^{(\sqrt{t}-m_\pi)^2} \frac{du}{t}\sqrt{\lambda\left(1, \frac{u}{t}, \frac{m_\pi^2}{t}\right)}\sqrt{1 - \frac{4m_\pi^2}{u}}\left\{5 + \frac{1}{2\left(t - m_\pi^2\right)^2}\right.\\ &\times\left[\frac{4}{3}\left[t - 3\left(u - m_\pi^2\right)\right]^2 + \frac{8}{3}\lambda\left(t, u, m_\pi^2\right)\left(1 - \frac{4m_\pi^2}{u}\right) + 10m_\pi^4\right]\\ &\left. + \frac{1}{\left(t - m_\pi^2\right)}\left[3\left(u - m_\pi^2\right) - t + 10m_\pi^2\right]\right\} ,\end{aligned} \tag{53.35}$$

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where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$ is the usual phase space factor. Based on this parametrization but including finite width corrections, a recent re-analysis of this sum rule has been given to order α_s^2 [677]. Result from the LSR is, in general, expected to be more reliable than the one from the FESR due to the presence of the exponential factor which suppresses the high-energy tail of the spectral function, although the two analysis are complementary. In [677], FESR has been used for matching by duality the phenomenological and theoretical parts of the sum rule. This matching has been achieved in the energy region around 2 GeV^2, where the optimal value of $m_u + m_d$ has been extracted. In [54], the LSR analysis has been updated by including the α_s^3 correction obtained in [442]. In this way, we get:

$$(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (9.3 \pm 1.8)\ \text{MeV}\ , \tag{53.36}$$

where we have converted the original result obtained at the *traditional* 1 GeV to the lattice choice of scale of 2 GeV through:

$$\bar{m}_i(1\ \text{GeV}) \simeq (1.38 \pm 0.06)\ \bar{m}_i(2\ \text{GeV})\ , \tag{53.37}$$

for running, to order α_s^3, the results from 1 to 2 GeV. This number corresponds to the average value of the QCD scale $\Lambda_3 \simeq (375 \pm 50)$ MeV from PDG [16] and [139]. Analogous value of (9.8 ± 1.9) MeV for the quark mass has also been obtained in [678] to order α_s^3 as an update of the [677] result. We take as a final result the average from [54] and [678]:

$$(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (9.6 \pm 1.8)\ \text{MeV}\ . \tag{53.38}$$

The inclusion of the tachyonic gluon mass term reduces this value to [161]:

$$\Delta^{\text{tach}}(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) \simeq -0.5\ \text{MeV}\ . \tag{53.39}$$

As already mentioned, adding to this effect the one of direct instanton might lead to a double counting in a sense that they can be alternative ways for parametrizing the non-perturbative QCD vacuum. Considering this contribution as another source of errors, it gives:

$$\Delta^{\text{inst}}(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) \simeq -0.5\ \text{MeV}\ . \tag{53.40}$$

Therefore, adding different sources of errors, we deduce from the analysis:

$$(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (9.6 \pm 1.8 \pm 0.4 \pm 0.5 \pm 0.5)\ \text{MeV}\ , \tag{53.41}$$

leading to the conservative result for the sum of light quark masses:

$$(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (9.6 \pm 2.0)\ \text{MeV}\ . \tag{53.42}$$

The first error comes from the SR analysis, the second one comes from the running mass evolution and the two last errors come, respectively, from the (eventual) tachyonic gluon and direct instanton contributions. This result is in agreement with previous determinations [3,419–423,679,680], although we expect that the errors given there have been underestimated. One can understand that the new result is lower than the old result [3,420] obtained

Table 53.3. *QSSR determinations of $\bar{m}_s$ (2) in MeV to order α_s^3. Some older results have been updated by the inclusion of the higher-order terms. The error contains the evolution from 1 to 2 GeV. In addition, the errors in the (pseudo)scalar channels contain those due to the small size instanton and tachyonic gluon mass. Their quadratic sum increases the original errors by 8.9%. The estimated error in the average comes from an arithmetic average of the different errors*

Channels	$\bar{m}_s$ (2)	Comments	Authors
Pion SR + ChPT	117.1 ± 25.4	$\mathcal{O}\left(\alpha_s^3\right)$	SN99 [54]Eq. (53.43)
$\langle\bar{\psi}\psi\rangle$ + ChPT	129.3 ± 23.2	$N,\ B - B^*$ (l.o)	DN98 [423] Eq. (53.52)
	117.1 ± 49.0	$D \to K^* l\nu$ (l.o)	DN98 [423] Eq. (53.53)
Kaon SR	119.6 ± 18.4	updated to $\mathcal{O}\left(\alpha_s^3\right)$	SN89 [420,3]
	112.3 ± 23.2	$\mathcal{O}\left(\alpha_s^3\right)$	DPS99[681]
	116 ± 12.8	”	KM01 [669]
Scalar SR	148.9 ± 19.2	$\mathcal{O}\left(\alpha_s^3\right)$	CPS97 [443]
	103.6 ± 15.4	”	CFNP97 [682]
	115.9 ± 24.0	”	J98 [683]
	115.2 ± 13.0	”	M99 [684]
	99 ± 18.3	”	JOP01 [685]
τ-like ϕ SR: e^+-e^- + τ-decay	129.2 ± 25.6	average: $\mathcal{O}\left(\alpha_s^3\right)$	SN99[354]
$\Delta S = -1$ part of τ-decay	$169.5^{+46.7}_{-57}$	$\mathcal{O}\left(\alpha_s^2\right)$	ALEPH99* [348]
	144.9 ± 38.4	”	CKP98 [349]
	114 ± 23	”	PP99 [350]
	125.7 ± 25.4	”	KKP00 [351]
	115 ± 21	”	KM01 [352]
	116^{+20}_{-25}	”	CDGHKK01[353]
Average	117.4 ± 23.4		

* Not included in the average.

without the α_s^2 and α_s^3 terms as both corrections enter with a positive sign in the LSR analysis. However, it is easy to check that the QCD perturbative series converge quite well in the region where the optimal result from LSR is obtained. Combining the previous value in Eq. (53.42) with the ChPT mass ratio, one can also deduce:

$$\bar{m}_s(2\ \text{GeV}) = (117.1 \pm 25.4)\ \text{MeV}\ . \tag{53.43}$$

53.4.3 The $\bar{u}s$ channel and QSSR prediction for the ratio $m_s/(m_u + m_d)$

Doing analogous analysis for the kaon channel, one can also derive the value of the sum $(m_u + m_s)$. The results obtained from [420] updated to order α_s^3 and from [681] are shown in Table 53.3 given in [54] but updated. We add to the original errors the one from the tachyonic gluon (5.5%), from the direct instanton (5.5%) and the one due to the evolution

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from 1 to 2 GeV (4.4%), which altogether increases the original errors by 8.9%. Therefore, we deduce the (arithmetic) average from the kaon channel:

$$\bar{m}_s(2\text{ GeV}) = (116.0 \pm 18.1)\text{ MeV}\ , \tag{53.44}$$

One should notice here that, unlike the case of the pion, the result is less sensitive to the contribution of the higher states continuum due to the relatively higher value of M_K, although the parametrization of the spectral function still gives larger errors than the QCD series. It is interesting to deduce from Eqs. (53.42) and (53.44), the sum rule prediction for the scale invariant quark mass ratios:

$$r_3 \equiv \frac{2m_s}{m_u + m_d} \simeq 24.2\ , \tag{53.45}$$

where we expect that the ratio is more precise than the absolute values due to the cancellation of the systematics of the SR method. This ratio compares quite well with the ChPT ratio [57]:

$$r_3^{CA} = 24.4 \pm 1.5\ , \tag{53.46}$$

and confirms the self-consistency of the pseudoscalar SR approach. This is a non-trivial test of the SR method used in this channel and may confirm a posteriori the neglect of less controlled contributions like direct instantons for example.

53.5 Direct extraction of the chiral condensate $\langle \bar{u}u \rangle$

As mentioned in previous section, the chiral $\bar{u}u$ condensate can be extracted directly from the nucleon, B^*-B splitting and vector form factor of $D \to K^* l\nu$, which are particularly sensitive to it and to the mixed condensate $\langle \bar{\psi}\sigma^{\mu\nu}(\lambda_a/2)G^a_{\mu\nu}\psi \rangle \equiv M_0^2 \langle \bar{\psi}\psi \rangle$ [423]. We have already used the result from the $D \to K^* l\nu$ form factor in order to derive upper and lower bounds on $(m_u + m_d)$. Here, we shall use information from the nucleon and from the B^*-B splitting in order to give a more accurate estimate. In the nucleon sum rules [424–430,3], which seem, at first sight, a very good place for determining $\langle \bar{\psi}\psi \rangle$, we have two form factors for which spectral sum rules can be constructed, namely the form factor F_1 which is proportional to the Dirac matrix $\gamma\, p$ and F_2 which is proportional to the unit matrix. In F_1 the four quark condensates play an important role, but these are not chiral symmetry breaking and are related to the condensate $\langle \bar{\psi}\psi \rangle$ only by the factorization hypothesis [1] which is known to be violated by a factor of two to three [424,404,3]. The form factor F_2 is dominated by the condensate $\langle \bar{\psi}\psi \rangle$ and the mixed condensate $\langle \bar{\psi}\sigma G\psi \rangle$, such that the baryon mass is essentially determined by the ratio M_0^2 of the two condensates:

$$M_0^2 = \langle \bar{\psi}\sigma G\psi \rangle / \langle \bar{\psi}\psi \rangle\ . \tag{53.47}$$

Therefore, from the nucleon sum rules one gets quite a reliable determination of M_0^2 [430,426]:

$$M_0^2 = (.8 \pm .1)\text{ GeV}^2. \tag{53.48}$$

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A sum rule based on the ratio F_2/F_1 would in principle be ideally suited for a determination of $\langle\bar\psi\psi\rangle$ but this sum rule is completely unstable [426] due to fact that odd parity baryonic excitations contribute with different signs to the spectral functions of F_1 and F_2. In the correlators of heavy mesons (B, B^* and D, D^*) the chiral condensate gives a significant direct contribution in contrast to the light meson sum rules [3], since, here, it is multiplied by the heavy quark mass. However, the dominant contribution to the meson mass comes from the heavy quark mass and therefore a change of a factor two in the value of $\langle\bar\psi\psi\rangle$ leads only to a negligible shift of the mass. However, from the B-B^* splitting one gets a precise determination of the mixed condensate $\langle\bar\psi\sigma G\psi\rangle$ with the value [401]:

$$\langle\bar\psi\sigma G\psi\rangle = -(9\pm1)\times10^{-3}\ \mathrm{GeV}^5\ , \tag{53.49}$$

which combined with the value of M_0^2 given in Eq. (53.48) gives our first result for the value of $\langle\bar\psi\psi\rangle$ at the nucleon scale:

$$\langle\bar\psi\psi\rangle(M_N) = -[(225\pm9\pm9)\ \mathrm{MeV}]^3\ , \tag{53.50}$$

where the last error is our estimate of the systematics and higher-order contributions. Using the GMOR relation, one can translate the previous result into a prediction on the sum of light quark masses. The resulting value is [423]:

$$(\bar m_u+\bar m_d)(2\ \mathrm{GeV}) = (10.6\pm1.8\pm0.5)\ \mathrm{MeV}\ , \tag{53.51}$$

where we have added the second error due to the quark mass evolution. Combining this value with the ChPT mass ratio, one obtains:

$$\bar m_s(2\ \mathrm{GeV}) \simeq 129.3\pm23.2\ \mathrm{MeV}\ . \tag{53.52}$$

Alternatively, one can use the central value of the range given in Eq. (53.21) in order to deduce the estimate:

$$(\bar m_u+\bar m_d)(2\ \mathrm{GeV}) = (9.6\pm4\pm0.4)\ \mathrm{MeV} \quad\Longrightarrow\quad \bar m_s(2\ \mathrm{GeV}) \simeq (117.1\pm49.0)\ \mathrm{MeV}\ . \tag{53.53}$$

The results for m_s are shown in Table 53.3.

53.6 Final estimate of (m_u+m_d) from QSSR and consequences on m_u, m_d and m_s

One can also notice the impressive agreement of the previous results from pseudoscalar and from the other channels. As the two results in Eqs. (53.42), (53.51) and (53.53) come from completely independent analysis, we can take their geometric average and deduce *the final value from QSSR*:

$$(\bar m_u+\bar m_d)(2\ \mathrm{GeV}) = (10.1\pm1.3\pm1.3)\ \mathrm{MeV}\ , \tag{53.54}$$

where the last error is our estimate of the systematics. One can combine this result with the one for the light quark mass ratios from ChPT [57]:

$$r_2^{CA} \equiv \frac{m_u}{m_d} = 0.553 \pm 0.043 , \qquad r_3^{CA} \equiv \frac{2m_s}{(m_d + m_u)} = 24.4 \pm 1.5 . \qquad (53.55)$$

Therefore, one can deduce the running masses at 2 GeV:

$$\bar{m}_u(2) = (3.6 \pm 0.6)\ \text{MeV} , \quad \bar{m}_d(2) = (6.5 \pm 1.2)\ \text{MeV} , \quad \bar{m}_s(2) = (123.2 \pm 23.2)\ \text{MeV} . \qquad (53.56)$$

Alternatively, we can use the relation between the invariant mass $\hat{m}_q$ and running mass $\bar{m}_q(2)$ to order α_s^3 in order to get:

$$\hat{m}_q = (1.14 \pm 0.05)\, \bar{m}_q(2) , \qquad (53.57)$$

for $\Lambda_3 = (375 \pm 50)$ MeV. Therefore, one can deduce the invariant masses:

$$\hat{m}_u = (4.1 \pm 0.7)\ \text{MeV} , \qquad \hat{m}_d = (7.4 \pm 1.4)\ \text{MeV} , \qquad \hat{m}_s = (140.4 \pm 26.4)\ \text{MeV} . \qquad (53.58)$$

53.7 Light quark mass from the scalar sum rules

As can be seen from Eq. (53.24), one can also (in principle) use the isovector-scalar sum rule for extracting the quark mass-differences $(m_d - m_u)$ and $(m_s - m_u)$, and the isoscalar-scalar sum rules for extracting the sum $(m_d + m_u)$.

53.7.1 The scalar $\bar{u}d$ channel

In the isovector channel, the analysis relies heavily on the less controlled nature of the $a_0(980)$ [3,420,422,666], which has been speculated to be a four-quark state [73]. However, it appears that its $\bar{q}q$ nature is favoured by the present data [690], and further tests are needed for confirming its real $\bar{q}q$ assignment.

In the $I = 0$ channel, the situation of the π-π continuum is much more involved due to the possible gluonium nature of the low mass and wide σ meson [686,687,689,688,690], which couples strongly to π-π and then can be missed in the quenched lattice calculation of scalar gluonia states.

Assuming that these previous states are quarkonia states, bounds on the quark mass difference and sum of quark masses have been derived in [666,671,672], while an estimate of the sum of the quark masses has been recently derived in [691]. However, in view of the hadronic uncertainties, we expect that the results from the pseudoscalar channels are much more reliable than the ones obtained from the scalar channel. Instead, we think that it is more useful to use these sum rules the other way around. Using the values of the quark masses from the pseudoscalar sum rules and their ratio from ChPT, one can extract their decay constants, which are useful for testing the $\bar{q}q$ nature of the scalar resonances [3,688] (we shall come back to this point in the next section). The agreement of the values of the quark masses from the isovector scalar channel with the ones from the pseudoscalar channel can be interpreted as a strong indication for the $\bar{q}q$ nature of the $a_0(980)$. In the isoscalar channel, the value of the sum of light quark masses obtained recently in [691], although slightly lower, agrees within the errors with that from the pseudoscalar channel. This result supports the maximal quarkonium-gluonium scheme for the broad low mass σ and narrow $f_0(980)$ meson: the narrowness of the f_0 is due to a destructive interference, while the broad nature of the σ is due to a contructive interference allowing its strong coupling with π-π. These features are very important for the scalar meson phenomenology, and need to be tested further.

53.7.2 The scalar $\bar{u}s$ channel

Here, the analysis is mostly affected by the parametrization of the $K\pi$ phase shift data, which strongly affects the resulting value of the strange quark mass as can be seen from the different determinations given in Table 53.3.

53.8 Light quark mass difference from $(M_{K^+} - M_{K^0})_{\rm QCD}$

The mass difference $(m_d - m_u)$ can be related to the QCD part of the kaon mass difference $(M_{K^+} - M_{K^0})_{\rm QCD}$ from the current algebra relation [57]:

$$r_2^{CA} \equiv \frac{(m_d - m_u)}{(m_d + m_u)} = \frac{m_\pi^2}{M_K^2} \frac{\left(M_{K^0}^2 - M_{K^+}^2\right)_{\rm QCD}}{M_K^2 - m_\pi^2} \frac{m_s^2 - \hat{m}^2}{(m_u + m_d)^2} = (0.52 \pm 0.05) 10^{-3} \left(r_3^2 - 1\right) , \tag{53.59}$$

where $2\hat{m} = m_u + m_d$; the QCD part of the $K^+ - K^0$ mass difference comes from the estimate of the electromagnetic term using the Dashen theorem including next-to-leading chiral corrections [677]. Using the sum rule prediction of r_3 from the ratio of $(m_u + m_d)$ in Eq. (53.54) with the average value of m_s in Table 53.3 or the ChPT ratio given in the previous section, one can deduce to order α_s^3:

$$(\bar{m}_d - \bar{m}_u)\,(2\ \text{GeV}) = (2.8 \pm 0.6)\ \text{MeV}\ . \tag{53.60}$$

An analogous result has been obtained from the heavy-light meson mass-differences [692]. We shall come back to the values of these masses at the end of this chapter.

53.9 The strange quark mass from e^+e^- and τ decays

53.9.1 $e^+e^- \to I = 0$ hadrons data and the ϕ-meson channel

Its extraction from the vector channel has been done in [693,3] and more recently in [354], while its estimate from an improved Gell-Mann–Okubo mass formula, including the quadratic mass corrections, has been done in [32,399,3]. More recently, the vector channel has been re-analysed in [354] using a τ-like inclusive decay sum rule in a modern

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version of the Das–Mathur–Okubo (DMO) sum rule [27] discussed in a previous chapter. The analysis in this vector channel is interesting as we have complete data from e^+e^- in this channel, which is not the case of (pseudo) scalar channels where some theoretical inputs related to the realization of chiral symmetry have to be used in the parametrization of spectral function. One can combine the $e^+e^- \to I = 0,\ 1$ hadrons and the rotated recent $\Delta S = 0$ component of the τ-decay data in order to extract m_s. Unlike previous sum rules, one has the advantage to have a complete measurement of the spectral function in the region covered by the analysis. We shall work with:

$$R_{\tau,\phi} \equiv \frac{3|V_{ud}|^2}{2\pi\alpha^2} S_{EW} \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) \frac{s}{M_\tau^2} \sigma_{e^+e^- \to \phi,\phi',\dots} \ ,$$

and the $SU(3)$-breaking combinations [354]:

$$\Delta_{1\phi} \equiv R_{\tau,1} - R_{\tau,\phi}, \qquad \Delta_{10} \equiv R_{\tau,1} - 3R_{\tau,0} \ , \tag{53.61}$$

which vanish in the $SU(3)$ symmetry limit; Δ_{10} involves the difference of the isoscalar ($R_{\tau,0}$) and isovector ($R_{\tau,1}$) sum rules à la DMO. The PT series converges quite well at the optimization scale of about 1.6 GeV [354]. For example, normalized to $\bar{m}_s^2$, one has:

$$\begin{aligned}\Delta_{1\phi} \simeq &-12\frac{\bar{m}_s^2}{M_\tau^2}\left\{1 + \frac{13}{3}a_s + 30.4a_s^2 + (173.4 \pm 109.2)a_s^3\right\} \\ &+ 36\frac{\bar{m}_s^4}{M_\tau^2} - 36\alpha_s^2\frac{\langle m_s\bar{s}s - m_d\bar{d}d\rangle}{M_\tau^4} \ .\end{aligned} \tag{53.62}$$

The different combinations $\Delta_{1\phi}$ and Δ_{10} have the advantage to be free (to leading order) from flavour-blind combinations like the tachyonic gluon mass and instanton contributions. We have checked using the result in [161] that, to non-leading in m_s^2, the tachyonic gluon contribution is also negligible. It has been argued in [355] that Δ_{10} can be affected by large $SU(2)$ breakings. This claim has been tested using some other sum rules not affected by these terms [354] but has not been confirmed. The average from different combinations is given in Table 53.3. An upper bound deduced from the positivity of $R_{\tau,\phi}$ is also given in Table 53.2.

53.9.2 Tau decays

As in the case of e^+e^-, one can use tau decays for extracting the value of m_s. However, data from τ decays are more accurate than those from e^+e^-. A suitable combination of sum rules that are sensitive to leading order to the $SU(3)$ breaking parameter is needed. It is easy to construct such a combination which is very similar to the one for e^+e^-. One can work with the DMO-like sum rule involving the difference between the $\Delta S = 0$ and $\Delta S = -1$ processes [348–353]:

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,V+A}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} = 3S_{EW}\sum_{D\geq 2}\left\{\delta_{ud}^{kl(D)} - \delta_{us}^{kl(D)}\right\} , \tag{53.63}$$

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where the moments are defined as:

$$R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds} , \tag{53.64}$$

with $R_\tau^{00} \equiv R_\tau$ is the usual τ-hadronic width. The QCD expression reads:

$$\delta R_\tau^{kl} \simeq 24 S_{EW} \left\{ \frac{\bar{m}_s^2}{M_\tau^2} \Delta_{kl}^{(2)} - 2\pi^2 \frac{\langle m_s \bar{s}s - m_d \bar{d}d \rangle}{M_\tau^4} \Delta_{kl}^{(4)} \right\} , \tag{53.65}$$

where $\Delta_{kl}^{(D)}$ are perturbative coefficients known to order α_s^2:

$$\Delta_{kl}^{(D)} \equiv \frac{1}{4} \left\{ 3\Delta_{kl}^{(D)}\Big|_{\mathrm{L+T}} + \Delta_{kl}^{(D)}\Big|_{\mathrm{L}} \right\} , \tag{53.66}$$

where the indices T and L refer to the tranverse and longitudinal parts of the spectral functions. For $D = 2$, the L part converges quite badly while the L + T converge quite well such that the combination can still have an acceptable convergence. For the lowest moments, one has:

$$\begin{aligned} \Delta_{00}^{(2)} &= 0.973 + 0.481 + 0.372 + 0.337 + \cdots \\ \Delta_{10}^{(2)} &= 1.039 + 0.558 + 0.482 + 0.477 + \cdots \\ \Delta_{20}^{(2)} &= 1.115 + 0.643 + 0.608 + 0.647 + \cdots \end{aligned} \tag{53.67}$$

The authors advocate that although the convergence is quite bad, the behaviour of the series is typical for an asymptotic series close to their point of minimum sensistivity. Therefore, the mathematical procedure for doing a reasonable estimate of the series is to truncate the expansion where the terms reach their minimum value. However, the estimate of the errors is still arbitrary. The authors assume that the error is given by the last term of the series. The result of the analysis is given in Table 53.3. The different numbers given in the table reflects the difference of methods used to get m_s but the results are consistent with each other within the errors. As in the case of the e^+e^- DMO-like sum rule, the combination used here is not affected to leading order by flavour-blind contribution like the tachyonic gluon and instanton contribution. We have checked [161] that the contribution of the tachyonic gluon to order $m_s^2 \alpha_s \lambda^2 / M_\tau^2$ gives a tiny correction and does not affect the estimate done without the inclusion of this term.

53.9.3 Summary for the estimate of light quark masses

Here, we summarize the results from the previous analysis:

- The sum $(\bar{m}_u + \bar{m}_d)$ of the running up and down quark masses from the pion sum rules is given in Eq. (53.42), while the one of the strange quark mass from the kaon channel is given in Eq. (53.44). Their values lead to the pseudoscalar sum rules prediction for the mass ratio in Eq. (53.45) which agrees nicely with the ChPT mass ratio.

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- The sum $(\bar{m}_u + \bar{m}_d)$ of the running up and down quark masses averaged from the pseudoscalar sum rule and from a direct extraction of the chiral condensate $\langle \bar{u}u \rangle$ obtained from a global fit of the nucleon, B^*-B mass-splitting and the vector part of the $D^* \to K^* l \nu$ form factor is given in Eq. (53.51) and reads for $\Lambda_3 = (375 \pm 50)$ MeV:

$$(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (10.1 \pm 1.8)\ \text{MeV}\ , \tag{53.68}$$

  implying with the help of the ChPT mass ratio m_u/m_d, the value:

$$\bar{m}_u(2\ \text{GeV}) = (3.6 \pm 0.6)\ \text{MeV}\ , \qquad \bar{m}_d(2\ \text{GeV}) = (6.5 \pm 1.2)\ \text{MeV}\ , \tag{53.69}$$

  which leads to the invariant mass in Eq. (53.58):

$$\hat{m}_u = (4.1 \pm 0.7)\ \text{MeV}\ , \qquad \hat{m}_d = (7.4 \pm 1.4)\ \text{MeV}\ , \tag{53.70}$$

- We have combined the result in Eq. (53.54) with the sum rule prediction for $m_s/(m_u + m_d)$ in order to deduce the quark mass difference $(m_d - m_u)$ from the QCD part of the $K^0 - K^+$ mass difference. We obtain the result in Eq. (53.60):

$$(\bar{m}_d - \bar{m}_u)(2\ \text{GeV}) = (2.8 \pm 0.6)\ \text{MeV}\ . \tag{53.71}$$

  This result indeed agrees with the one taking the difference of the mass given previously. The fact that $(m_u + m_d) \neq (m_d - m_u)$ does not favour the possibility of having $m_u = 0$.
- We give in Table 53.3 the different sum rules determinations of m_s. The results from the pion SR and $\langle \bar{\psi}\psi \rangle$ come from the determination of $(m_u + m_d)$ to which we have added the ChPT contraint on $m_s/(m_u + m_d)$. One can see from this table that different determinations are in good agreement with each others. Doing an average of these different results, we obtain:

$$\bar{m}_s(2\ \text{GeV}) = (117.4 \pm 23.4)\ \text{MeV} \quad \Longrightarrow \quad \hat{m}_s = (133.8 \pm 27.3)\ \text{MeV}\ . \tag{53.72}$$

  Aware on the possible correlations between these estimates, we have estimated the error as an arithmetic average which is about 10% as generally expected for the systematics of the SR approach.

It is informative to compare the above results with the average of different quenched and unquenched lattice values [694]:

$$\begin{aligned} &\bar{m}_{ud}(2\ \text{GeV}) \approx \frac{1}{2}(\bar{m}_u + \bar{m}_d)(2\ \text{GeV}) = (4.5 \pm 0.6 \pm 0.8)\ \text{MeV}\ , \\ &\bar{m}_s(2\ \text{GeV}) = (110 \pm 15 \pm 20)\ \text{MeV}\ , \end{aligned} \tag{53.73}$$

where the last error is an estimate of the quenching error. We show in Table 53.4 a compilation of the lattice unquenched results including comments on the lattice characterisitcs (action, lattice spacing a, β). Also shown is the ratio over m_s/m_{ud} and quenched (quen) over unquenched (unq) results.

53.10 Decay constants of light (pseudo)scalar mesons

53.10.1 Pseudoscalar mesons

Due to the Goldstone nature of the pion and kaon, we have seen that their radial excitations play an essential rôle in the sum rule. This unusual property allows a determination of

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Table 53.4. *Simulation details and physical results of unquenched lattice calculations of light-quark masses from [694], where original references are quoted*

	Action	a^{-1}[GeV]	$\#_{(\beta,K_{sea})}$	Z_m	$\bar{m}_s(2)$		$\frac{m_s}{m_{ud}}$	$\frac{\bar{m}_s^{\text{quen}}}{\bar{m}_s^{\text{unq}}}$
SESAM 98	Wilson	2.3	4	PT	151(30)	$(m_{K,\phi})$	55(12)	1.10(24)
MILC 99	Fatlink	1.9	1	PT	113(11) 125(9)	(m_K) (m_ϕ)	22(4)	1.08(13)
APE 00	Wilson	2.6	2	NP-RI	112(15) 108(26)	(m_K) (m_ϕ)	26(2)	1.09(20)
CP-PACS 00	MF-Clover	$a \to 0$	12	PT	88^{+4}_{-6} 90^{+5}_{-11}	(m_K) (m_ϕ)	26(2)	1.25(7)
JLQCD 00	NP-Clover	2.0	5	PT	$94(2)^\dagger$ $88(3)^\ddagger$ $109(4)^\dagger$ $102(6)^\ddagger$	(m_K) (m_ϕ)	–	–
QCDSF + UKQCD 00	NP-Clover	2.0	6	PT	90(5)	(m_K)	26(2)	–

† From vector WI; ‡ from axial WI. The errors on the ratios m_s/m_{ud} and $\bar{m}_s^{\text{quen}}/\bar{m}_s^{\text{unq}}$ are estimates based on the original data.

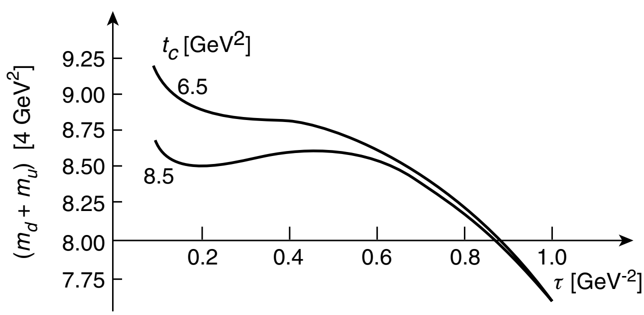

Fig. 53.4. LSR analysis of the ratio $r_\pi \equiv M^4_{\pi'} f^2_{\pi'}/M^4_\pi f^2_\pi$. For a given value $r_\pi = 9.5$, we show the value of $(\bar{m}_d + \bar{m}_u)(2)$ for two values of the QCD continuum t_c.

the radial excitation parameters. In the strange quark channels, an update of the results in [354,422,420,3,675] gives:

$$r_K \equiv M^4_{K'} f^2_{K'}/M^4_K f^2_K \simeq 9.5 \pm 2.5 \simeq r_\pi \ , \tag{53.74}$$

where r_π has been defined previously. The optimal value has been obtained at the LSR scale $\tau \approx 0.5$ GeV^{-2} and $t_c \simeq 4.5 - 6.5$ GeV^2 as shown in Fig. 53.4. This result implies for $\pi'(1.3)$ and $K'(1.46)$:

$$f_{\pi'} \simeq (3.3 \pm 0.6) \text{ MeV} \ , \qquad f_{K'} \simeq (39.8 \pm 7.0) \text{ MeV} \ . \tag{53.75}$$

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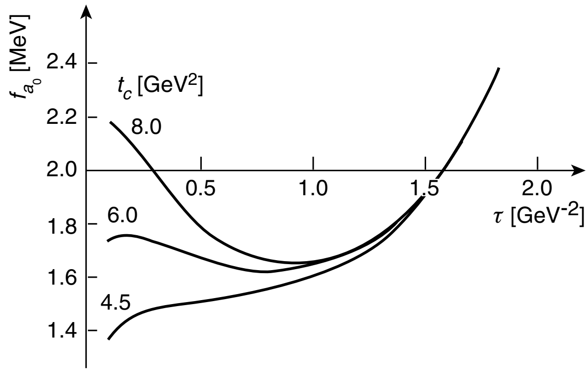

Fig. 53.5. LSR analysis of the decay constant f_{a_0} of the $a_0(.98)$ meson normalized as $f_\pi = 92.4$ MeV. We use $(\bar{m}_d - \bar{m}_u)(2) = 2.8$ MeV.

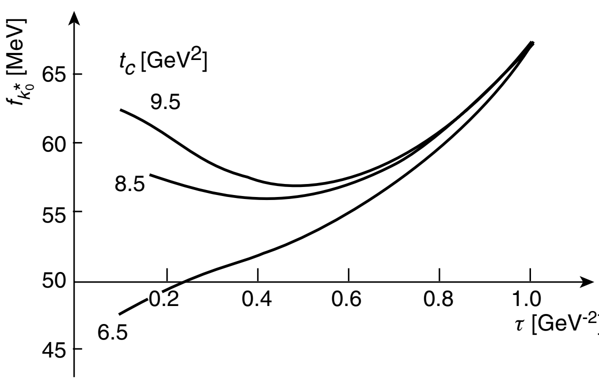

Fig. 53.6. LSR analysis of the decay constant f_{K*_0} of the $K_0^*(1.43)$ meson normalized as $f_\pi = 92.4$ MeV. We use $\bar{m}_s(2) = 117.4$ MeV.

It is easy to see that the result satisfies the relation:

$$\frac{f_{K'}}{f_{\pi'}} \approx \frac{M_K^2}{m_\pi^2} \approx \frac{m_s}{m_d} \,, \tag{53.76}$$

as expected from chiral symmetry arguments.

53.10.2 Scalar mesons

We expect that the scalar channel is more useful for giving the decay constants of the mesons which are not well known rather than predicting the value of the quark masses. Such a programme has been initiated in [420,422,3]. Since then, the estimate of the decay constants has not mainly changed. The analysis is shown in Figs. 53.5 and 53.6.

A recent estimate gives [354]:

$$f_{a_0} = (1.6 \pm 0.15 \pm 0.35 \pm 0.25)\,\text{MeV}\,, \qquad f_{K_0^*} \simeq (46.3 \pm 2.5 \pm 5 \pm 5)\,\text{MeV}\,, \quad (53.77)$$

where the errors are due respectively to the choice of t_c from 4.5 to 8 GeV^2, the value of the quark mass difference obtained previously and the one of Λ_3. The decay constants are normalized as:

$$\langle 0|\partial_\mu V^\mu(x)|a_0\rangle = \sqrt{2} f_a M_a^2\,, \qquad (53.78)$$

corresponding to $f_\pi = 92.4$ MeV. We have used the experimental masses 0.98 and 1.43 GeV in our analysis.[5] It is also interesting to notice that the ratio of the decay constants are:

$$\frac{f_{K_0^*}}{f_{a_0}} \simeq 29 \approx \frac{m_s - m_u}{m_d - m_u} \simeq 40\,, \qquad (53.79)$$

as naïvely expected. We are aware that the values of these decay constants might have been overestimated due to the eventual proliferations of nearby radial excitations. Therefore, it will be interesting to have a direct measurement of these decay constants for testing these predictions. The values of these decay constants will be given like other meson decay constants in Table 54.1 in the next chapter.

53.11 Flavour breaking of the quark condensates

53.11.1 $SU(3)$ *corrections to kaon PCAC*

Let us remind ourselves that the (pseudo)scalar two-point function obeys the twice-subtracted dispersion relation:

$$\Psi_{(5)}(q^2) = \Psi_{(5)}(0) + q^2\Psi'_{(5)}(0) + q^4\int_0^\infty \frac{dt}{t^2(t - q^2 - i\epsilon)}\text{Im}\Psi_{(5)}(t)\,. \qquad (53.80)$$

The deviation from kaon PCAC was first studied in [400] using the once-subtracted pseudoscalar sum rule of the quantity:

$$\frac{\Psi_{(5)}(q^2) - \Psi_{(5)}(0)}{q^2} \qquad (53.81)$$

sensitive to the value of the value of the correlator at $q^2 = 0$.[6] The Ward identity obeyed by the (pseudo)scalar two-point function leads to the low-energy theorem:

$$\Psi_{(5)}(0) = -(m_i \pm m_j)\langle \bar\psi_i\psi_i \pm \bar\psi_j\psi_j\rangle\,, \qquad (53.82)$$

in terms of the *normal ordered condensates.* However, as emphasized in different papers [167,399,444,442], $\Psi_{(5)}(0)$ contains a perturbative piece which cancels the mass singularities appearing in the OPE evaluation of $\Psi_{(5)}(q^2)$. This leads to the fact that the quark

[5] The masses of the a_0 and K_0^* are also nicely reproduced by the ratio of moments [357,3].

[6] This sum rule has also been used in [260,261,265] for estimating the $U(1)_A$ topological suceptibility and its slope. The result has been confirmed on the lattice [266].

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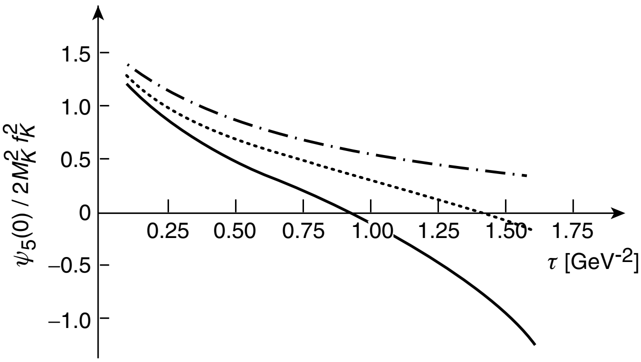

Fig. 53.7. LSR analysis of the subtraction constant $\Psi_5(0)$. We use $\bar{m}_s(2) = 117.4$ MeV, $r_K = 9.5$ and $t_c = 6\,\text{GeV}^2$. The curves correspond to different truncations of the PT series: to $\mathcal{O}(\alpha_s)$: dotted-dashed; to $\mathcal{O}(\alpha_s^2)$: dashed; to $\mathcal{O}(\alpha_s^3)$: continuous.

condensate entering in Eq. 53.82 are defined as a *non-normal ordered condensate*, which has a slight dependence on the scale and renormalization scheme. This mass correction effect is only quantitatively relevant for the $\bar{u}s$ channel but not for the $\bar{u}d$ one. To order α_s^3 for the perturbative term and to leading order for the condensates, the (pseudo)scalar sum rule for the $\bar{u}s$ channel reads, by neglecting the up quark mass:

$$\begin{aligned}
&\int_0^{t_c} \frac{dt}{t} \exp(-t\tau) \frac{1}{\pi} \text{Im}\Psi_{(5)}(t) \simeq \Psi_{(5)}(0) \\
&+ (\bar{m}_u \pm \bar{m}_s)^2 \frac{3}{8\pi^2} \tau^{-1} \left\{ (1-\rho_0) \left[1 + 6.82\left(\frac{\bar{\alpha}_s}{\pi}\right) + 58.55\left(\frac{\bar{\alpha}_s}{\pi}\right)^2 + 537.6\left(\frac{\bar{\alpha}_s}{\pi}\right)^3 \right] \right. \\
&+ 3.15\bar{m}_s^2\tau \left[1 + 3.32\left(\frac{\bar{\alpha}_s}{\pi}\right) \right] \\
&- \left[\frac{\pi}{3}\langle\alpha_s G^2\rangle - \frac{8\pi^2}{3} \left[\left(\bar{m}_s - \frac{\bar{m}_u}{2} \right) \langle\bar{u}u\rangle \pm (u \leftrightarrow s) \right] \right] \tau^2 \\
&\left. + \frac{1}{2}(2 \mp 9)\left(\frac{128}{81}\right) \pi^3 \rho \alpha_s \langle\bar{u}u\rangle^2 \tau^3 \right\} ,
\end{aligned} \tag{53.83}$$

where we have neglected the $SU(3)$ breaking for the four-quark condensates. This assumption does not, however, affect the analysis due to the small contribution of this operator at the optimization scale. The analysis is shown in Fig. 53.7. Examining the different curves, on can notice that they deviate notably from the kaon PCAC prediction:

$$\Psi_5(0) \simeq 2M_K^2 f_K^2 , \tag{53.84}$$

therefore confirming the early findings in [400]. The LSR indicates a slight stability point at $\tau \approx (0.50 \sim 0.75)$ GeV^{-2}, where:

$$\Psi_5(0) \simeq (0.5 \pm 0.2) 2M_K^2 f_K^2 . \tag{53.85}$$

However, at this scale, PT series has a bad convergence:

$$\begin{aligned}\text{Pert} = \text{Parton} \times \left\{1 + 2.17\alpha_s + 5.93\alpha_s^2 + 17.34\alpha_s^3\right\} \\ \simeq \text{Parton} \times \{1 + 0.86 + 0.92 + 1.06\} ,\end{aligned} \quad (53.86)$$

which might not be of concern if one considers that an asymptotic series close to its point of *minimum sensitivity* can be truncated when its reaches the extremum value and the last term added as a truncation error.[7] This convergence might *a priori* be improved if one works with the combination of sum rules which is less sensitive to the high-energy behaviour of the spectral function (and then to the perturbative contribution) than the former [675,680,420,3,354]. The modified sum rule reads [3].[8]

$$\begin{aligned}&\int_0^\infty \frac{dt}{t}\exp(-t\tau)(1-t\tau)\frac{1}{\pi}\text{Im}\Psi_{(5)}(t) \simeq \Psi_{(5)}(0) + (\bar{m}_u \pm \bar{m}_s)^2\frac{3}{8\pi^2}\tau^{-1} \\ &\times\left\{2\left(\frac{\bar{\alpha}_s}{\pi}\right)\left[1 + 18.3\left(\frac{\bar{\alpha}_s}{\pi}\right) + 242.2\left(\frac{\bar{\alpha}_s}{\pi}\right)^2\right] + 5.15\bar{m}_s^2\tau\left[1 + 5.0\left(\frac{\bar{\alpha}_s}{\pi}\right)\right]\right. \\ &\left. + 2\left[\frac{\pi}{3}\langle\alpha_s G^2\rangle - \frac{8\pi^2}{3}\bar{m}_s\left[\langle\bar{u}u\rangle \mp \frac{1}{2}\langle\bar{s}s\rangle\right]\right]\tau^2 + \frac{3}{2}(2\mp 9)\left(\frac{128}{81}\right)\pi^3\rho\alpha_s\langle\bar{u}u\rangle^2\tau^3\right\} .\end{aligned} \quad (53.87)$$

The analysis also leads to a similar result. The LSR has been also studied recently in [695], by including threshold effects and higher mass resonances, which enlarge the region of stability in the LSR variable. Within the previous hadronic parametrization, one obtains:

$$\Psi_5(0) \simeq (0.56 \pm 0.04 \pm 0.15)2M_K^2 f_K^2 , \quad (53.88)$$

where we have added the error due to our estimate of the truncation of the QCD PT series as deduced from Fig. 53.7. An alternative estimate is obtained with the use of FESR [679]. Parametrizing the subtraction constant as:

$$\Psi_5(0)_s^u = 2M_K^2 f_K^2(1-\delta_K) , \quad (53.89)$$

one has the sum rule [679]:

$$\delta_K \simeq \frac{3}{16\pi^2}\frac{\bar{m}_s^2 t_c}{f_K^2 M_K^2}\left\{1 + \frac{23}{3}a_s + \mathcal{O}\left(a_s^2\right)\right\} - r_K\left(\frac{M_K}{M_{K'}}\right)^2 , \quad (53.90)$$

which gives, after using the *correlated values* of the input parameters [420,3,354]:

$$\delta_K = 0.34^{+0.23}_{-0.17} , \quad (53.91)$$

[7] A similar argument has been used for the extraction of the strange quark mass from τ-decay data discussed in the previous section, where the QCD series has also quite bad behaviour.

[8] Notice that we have not yet introduced the QCD continuum into the LHS of the sum rule.

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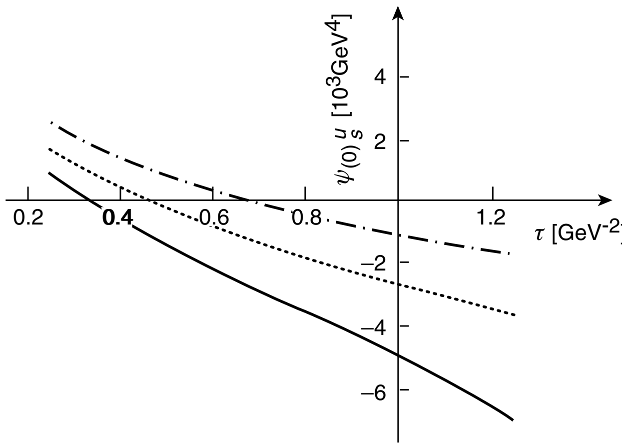

Fig. 53.8. LSR analysis of the subtraction constant $\Psi(0)$. We use $\bar{m}_s(2) = 117.4$ MeV, $f_{K*_0} = 46$ MeV and $t_c = 6.5$ GeV^2. The curves correspond to different truncations of the PT series: to $\mathcal{O}(\alpha_s)$: dotted-dashed; to $\mathcal{O}(\alpha_s^2)$: dashed; to $\mathcal{O}(\alpha_s^3)$: continuous.

leading to:

$$\Psi_5(0) \simeq (0.66 \pm 0.20) 2 M_K^2 f_K^2 \ , \tag{53.92}$$

confirming the large violation of kaon PCAC obtained from LSR.

53.11.2 Subtraction constant from the scalar sum rule

One can do a similar analysis for the scalar channel. The analysis from LSR is shown in Fig. 53.8. One can also see that there is a slight stability for $\tau \approx (0.50 \sim 0.75)$ GeV^{-2}, which gives:

$$\Psi(0) \approx -10^{-3} \text{ GeV}^4 \ , \tag{53.93}$$

in agreement with previous results [3,420,422,675]. In [695], using LSR, a similar result but from a larger range of LSR stability, has been obtained within an Omnés representation for relating the scalar form factor to the $K\pi$ phase shift data:

$$\Psi(0) \simeq -(1.06 \pm 0.21 \pm 0.20) 10^{-3} \text{ GeV}^4 \ , \tag{53.94}$$

where the last term is our estimate of the error due to the truncation of the QCD series. We show the analysis in Fig. 53.9.

One can use an alternative approach by working with FESR:

$$\Psi(0)_s^u = 2M_{K_0^*}^2 f_{K_0^*}^2 - \frac{3}{16\pi^2}\bar{m}_s^2 t_c \left\{1 + \frac{23}{3}a_s + \mathcal{O}\big(a_s^2\big)\right\} , \tag{53.95}$$

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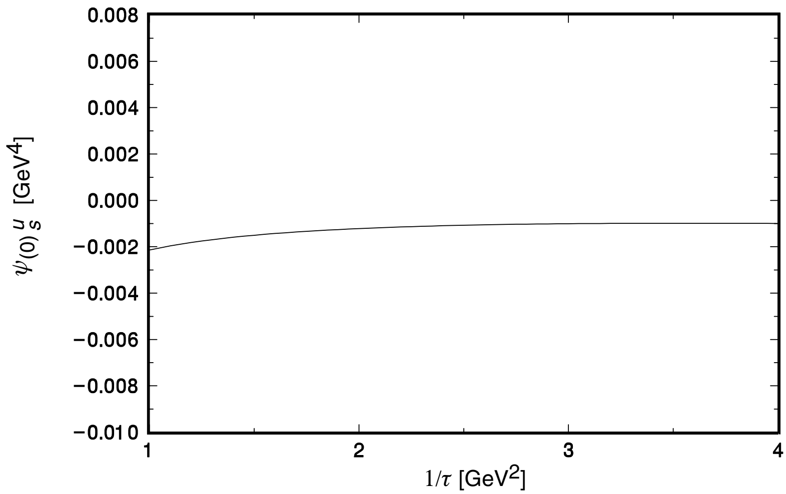

Fig. 53.9. LSR analysis of the subtraction constant $\Psi(0)$ versus the sum rule scale using $K\pi$ phase shift data, from [695].

which gives [354]:

$$\Psi(0)_s^u = -\left(7.8^{+5.5}_{-2.7}\right) 10^{-4}\ \mathrm{GeV}^4. \tag{53.96}$$

53.11.3 $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$ from the (pseudo)scalar sum rules

We take the arithmetic average of the previous determinations for our final estimate:

$$\Psi_5(0) \simeq (0.57 \pm 0.19) 2M_K^2 f_K^2\ , \qquad \Psi(0) \simeq -(0.92 \pm 0.35)10^{-3}\ \mathrm{GeV}^4\ , \tag{53.97}$$

Taking the ratio of the scalar over the pseudoscalar subtraction constants expressed in terms of the *normal-ordered* condensates, one can deduce:

$$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle = 0.57 \pm 0.12\ . \tag{53.98}$$

53.11.4 $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$ from the B_s meson

One can also extract the flavour breakings of the condensates from a sum rule analysis of the B_s and B_s^* masses, which are sensitive to the chiral condensate as it enters like $m_b\langle\bar{s}s\rangle$ in the OPE of the heavy-light meson (see next section). The masses of the mesons are found to decrease linearly with the value of the chiral condensate. Using the observed value of the B_s meson mass $M_{B_s} = 5.375$ GeV, one can deduce from Fig. 3 of [401]:

$$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle \simeq 0.75 \pm 0.08\ , \tag{53.99}$$

where the error is the expected typical sum rule estimate. The effect of the strange quark mass is less important than the one here, such that the result given in [401] remains valid although obtained with slightly different values of m_s and Λ_5. This estimate is expected to be more reliable than the one from the (pseudo)scalar light mesons, which are affected by

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the bad convergence of the PT QCD series. Using this value of ratio of the condensates in the (old) curve of the B_s^* mass, leads to a value higher than the measured one, which needs to be clarified.

53.11.5 Final sum rule estimate of $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$

Using the previous results, one can deduce that the sum rules from the light (pseudo)scalar and from the B_s meson predict for the *normal ordered* condensate ratio:

$$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle \simeq 0.66 \pm 0.10 \,, \tag{53.100}$$

confirming earlier findings [675,3,420,354] on the large flavour breaking of the chiral condensate. This number comes from the arithmetic average of the two values in Eqs. (53.98) and (53.99). If one instead works with the *non-normal ordered* condensate, one should add to the expression in Eq. (53.82) a small perturbative part first obtained by Becchi *et al.* [167] (see also [3,399,441]):

$$\langle\bar{s}s\rangle_{\overline{MS}} = \langle\bar{s}s\rangle - \frac{3}{2\pi^2}\frac{2}{7}\left(\frac{1}{a_s} - \frac{53}{24}\right)\bar{m}_s^3. \tag{53.101}$$

This leads to the ratio of the *non-normal ordered* condensates:

$$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle|_{\overline{MS}} = 0.75 \pm 0.12. \tag{53.102}$$

The previous estimates are in good agreement with those from chiral perturbation theory [57] (see also [696]). They are also in fair agreement with the one from the baryonic sum rules [424–430], although we expect that the result from the latter is less accurate due to the complexity of the analysis in this channel (choice of the interpolating operators, eventual large effects of the continuum due to the nearby Roper resonances, . . .).

53.11.6 $SU(2)$ breaking of the quark condensate

The $SU(2)$ breaking of the quark condensate has been studied for the first time in [680] and in [679,3]. Using similar approaches, the estimate is [3,420]:

$$\langle\bar{d}d\rangle/\langle\bar{u}u\rangle \simeq 1 - 9\times 10^{-3} \,. \tag{53.103}$$

The previous estimate is in good agreement with the one from FESR [679].

53.12 Heavy quark masses

In the previous part of this book, we have already discussed the different definitions of the heavy quark masses and given their values. Contrary to the light quark masses, the definition of pole quark masses $p^2 = M_H^2$ can (in principle) be introduced perturbatively for heavy quarks [147,133,148], similarly to that of the electron, as here the quark mass is much heavier than the QCD scale Λ such that the perturbative approach makes sense. However, a complication arises due to the resummation of the QCD series [154] such that the pole mass definition has an intrinsic ambiguity, which can be an obstacle to its improved accuracy determination, although the effect is relatively small. Alternative definitions that are free from such ambiguities have been proposed in the literature [157,159]. In this section, we shall discuss the determinations of the perturbative running quark masses which do not have such problems.

53.12.1 The quarkonia channel

Charmonium and bottomium are the standard channels for extracting the charm and bottom quark masses. Most of the sum rule analysis are based on the $Q^2 = 0$ moments (MOM) originally introduced by SVZ for the study of the charmonium systems:

$$\mathcal{M}_n \equiv \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n \Pi\bigg|_{Q^2=0} = \int_{4m^2}^{\infty} \frac{dt}{t^{n+1}} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,, \tag{53.104}$$

but convenient for the bottomium systems due to a much better convergence of the OPE. In [357], the $Q^2 \neq 0$ moments have been introduced for improving the convergence of the QCD series:

$$\mathcal{M}_n(Q_0^2) \equiv \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n \Pi\bigg|_{Q^2=Q_0^2} = \int_{4m^2}^{\infty} \frac{dt}{\left(t+Q_0^2\right)^{n+1}} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,, \tag{53.105}$$

The spectral function can be related to the $e^+e^- \to Q\bar{Q}$ total cross-section via the optical theorem:

$$\mathrm{Im}\Pi(t+i\epsilon) = \frac{1}{12\pi Q_Q^2} \frac{\sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,. \tag{53.106}$$

Q_Q is the heavy quark charge in units of e. The contribution to the spectral function is as usual saturated by the lowest few resonances plus the QCD continuum above the threshold t_c:

$$\mathrm{Im}\Pi_Q(t) = \frac{3}{4\alpha^2}\frac{1}{Q_Q^2}\sum_i \Gamma_i M_i \delta\left(t - M_i^2\right) + \theta(t-t_c)\mathrm{Im}\Pi_Q^{\mathrm{QCD}}(t), \tag{53.107}$$

where Γ_i is the electronic width of the resonances with the value given in PDG [16]. Retaining the observed resonances, the value of $\sqrt{t_c}$ fixed from stability analysis is about $(11 \sim 12)$ GeV for the Υ-and about 5 GeV for the J/Ψ-families. However, the result will be practically independent from this choice of t_c due to the almost complete dominance of the lowest ground state to the spectral function at the stability point. An alternative approach that is used in [148,149] is the LSR:

$$\mathcal{L}(\tau) = \int_{4m^2}^{\infty} dt\ \exp^{-t\tau} \frac{1}{\pi}\mathrm{Im}\Pi(t)\,. \tag{53.108}$$

This sum rule is particularly convenient for the analysis of the charmonium systems as the corresponding OPE converges faster than the moment sum rules. It has been noted in [149] that the ratios of sum rules (and their finite energy sum rule (FESR) variants) are

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more appropriate for the estimate of the quark mass as these ratios equate *directly* the mass squared of ground state to that of the quark:

$$\mathcal{R}_n \equiv \frac{\mathcal{M}^{(n)}}{\mathcal{M}^{(n+1)}} \quad \text{and} \quad \mathcal{R}_\tau \equiv -\frac{d}{d\tau} \log \mathcal{L} \,, \tag{53.109}$$

They also eliminate, to leading order, some artefact dependence due to the sum rules (exponential weight factor or number of derivatives) and some other systematic errors appearing in each of the individual moments. For the perturbative part, we shall use (without expanding in 1/M) the Schwinger interpolating formula to two loops:

$$\mathrm{Im}\Pi_Q^{pert}(t) \simeq \frac{3}{12\pi} v_Q \left(\frac{3 - v_Q^2}{2}\right) \left\{1 + \frac{4}{3}\alpha_s f(v_Q)\right\}, \tag{53.110}$$

where:

$$v_Q = \sqrt{1 - 4M_Q^2/t}\,, \qquad f(v_Q) = \frac{\pi}{2v_Q} - \frac{(3 + v_Q)}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right) \tag{53.111}$$

are respectively the quark velocity and the Schwinger function [319]. We express this spectral function in terms of the running mass by using the two-loops relation given in a previous chapter and including the $\alpha_s \log(t/M_Q^2)$-term appearing for off-shell quark. We shall add to this perturbative expression the lowest dimension $\langle \alpha_s G^2 \rangle$ non-perturbative effect (it has been explained in a previous part of this book that, for a heavy-heavy quark correlator, the heavy-quark condensate contribution is already absorbed into the gluon one), which among the available higher-dimension condensate terms can only give a non-negligible contribution. The gluon condensate contribution to the moments $\mathcal{M}^{(n)}$ and so to $\mathcal{R}_n$ can be copied from the original work of SVZ [1] and reads:

$$\mathcal{M}_G^{(n)} = -\mathcal{M}_{pert}^{(n)} \frac{(n+3)!}{(n-1)!(2n+5)} \frac{4\pi}{9} \frac{\langle \alpha_s G^2 \rangle}{\left(4M_Q^2\right)^2} \,, \tag{53.112}$$

where $\mathcal{M}_{pert}^{(n)}$ is the lowest perturbative expression of the moments. The one to the Laplace ratio $\mathcal{R}_\tau$ can be also copied from the original work of Bertlmann [93], which has been expanded recently in $1/M_Q$ by [697]. It reads:

$$\mathcal{R}_\tau^G \simeq \left(4M_Q^2\right) \frac{2\pi}{3} \langle \alpha_s G^2 \rangle \tau^2 \left(1 + \frac{4}{3\omega} - \frac{5}{12\omega^2}\right), \tag{53.113}$$

where $\omega \equiv 4M_Q^2 \tau$. The results of the analysis from the ratios of moments and Laplace sum rules give the values of the running masses to order α_s:[9]

$$\bar{m}_c(\bar{m}_c) = (1.23 \pm 0.03 \pm 0.03)\ \mathrm{GeV}\,, \qquad \bar{m}_b(\bar{m}_b) = (4.23 \pm 0.04 \pm 0.02)\ \mathrm{GeV}\,, \tag{53.114}$$

where the errors are respectively due to $\alpha_s(M_Z) = 0.118 \pm 0.006$ and $\langle \alpha_s G^2 \rangle = (0.06 \pm 0.03)\ \mathrm{GeV}^4$ used in the original work. These running masses can be converted into the pole

[9] The inclusion of the α_s^2 correction is under study.

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Table 53.5. *QSSR direct determinations of $\bar{m}_c(\bar{m}_c)$ in $\overline{MS}$ scheme and of the pole mass M_c from J/Ψ-family, e^+e- data and D-meson and comparisons with lattice results. Determinations from some other sources are quoted in PDG [16]. The results are given in units of GeV. The estimated error in the SR average comes from an arithmetic average of the different errors. The average for the pole masses is given at NLO. The one of the running masses is almost unchanged from NLO to NNLO determinations. $\Longleftarrow$ means that the perturbative relations between the different mass definitions have been used to get the quoted values*

Sources	$\bar{m}_c(\bar{m}_c)$	M_c	Comments	Authors
J/Ψ**-family**				
MOM and LSR at NLO	$(1.27 \pm 0.02) \Longleftarrow$	(1.45 ± 0.05)	$\Longleftarrow m\left(-m_c^2\right) = (1.26 \pm 0.02)$	SN89 [148]
Ratio of LSR at NLO	$(1.23 \pm 0.04) \Longrightarrow$	(1.42 ± 0.03)		SN94 [149]
NRSR at NLO	$(1.23 \pm 0.04) \Longleftarrow$	(1.45 ± 0.04)		SN94 [149]
SR at NLO	$(1.22 \pm 0.06) \Longleftarrow$	(1.46 ± 0.04)		DGP94 [697]
NRSR at NNLO	(1.23 ± 0.09)	$(1.70 \pm 0.13)^*$		EJ01 [699]
e^+e^- **data**				
FESR at NLO	(1.37 ± 0.09)			PS01 [700]
MOM at NNLO	(1.30 ± 0.03)			KS01 [701]
NLO	$(1.04 \pm 0.04) \Longleftarrow$	$1.33 \sim 1.4$		M01 [702]
D **meson**				
Ratio of LSR at NNLO	(1.1 ± 0.04)	(1.47 ± 0.04)		SN01 [150]
SR average	(1.23 ± 0.05)	(1.43 ± 0.04)		
Quenched lattice				
	(1.33 ± 0.08)			FNAL98 [703]
	(1.20 ± 0.23)			NRQCD99 [704]
	(1.26 ± 0.13)			APE01 [705]

* Not included in the average.

masses at this order. Non-relativistic versions of these sum rules (NRSR) introduced by [155] have also been used in [148,149] for determining the b quark mass. These NRSR approaches have been improved by the inclusion of higher-order QCD corrections and resummation of the Coulomb corrections from ladder gluonic exchanges. Some recent different determinations are given in Tables 53.5 and 53.6.

53.12.2 The heavy-light D and B meson channels

Heavy quark masses can also be extracted from the heavy-light quark channels because the corresponding correlators are sensitive to leading order to the values of these masses

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Table 53.6. *The same as in Table 53.5 but for the b-quark*

Sources	$\bar{m}_b(\bar{m}_b)$	M_b	Comments	Authors
Υ-family				
MOM and LSR at NLO	$(4.24 \pm 0.05) \Longleftarrow$	(4.67 ± 0.10)	$\Longleftarrow m_b\left(-m_b^2\right) = (4.23 \pm 0.05)$	SN89 [148]
Ratio of LSR at NLO	$(4.23 \pm 0.04) \Longrightarrow$	(4.62 ± 0.02)		SN94 [149]
NRSR at NLO	$(4.29 \pm 0.04) \Longleftarrow$	(4.69 ± 0.03)		SN94 [149]
FESR at NLO	$(4.22 \pm 0.05) \Longleftarrow$	(4.67 ± 0.05)		SN95 [149]
	$(4.14 \pm 0.04) \Longleftarrow$	(4.75 ± 0.04)		KPP98 [706]
NRSR at NNLO	(4.20 ± 0.10)			PP99, MY99 [707]
MOM at NNLO	(4.19 ± 0.06)			JP99 [708]
NR at NNNLO	(4.45 ± 0.04)			PY00, LS00 [94,709]
NR at NNNLO	(4.21 ± 0.09)			P01 [602]
NR at NNLO	(4.25 ± 0.08)		$\Longleftarrow$ Residual mass	BS99 [710]
NR at NNLO	(4.20 ± 0.06)		$\Longleftarrow$ $1S$ mass	H00 [711]
MOM at NNNLO	(4.21 ± 0.05)			KS01 [701]
***B* and *B** mesons**				
Ratio of LSR at NLO	$(4.24 \pm 0.07) \Longleftarrow$	(4.63 ± 0.08)		SN94 [149]
Ratio of LSR at NNLO	(4.05 ± 0.06)	(4.69 ± 0.06)	B-meson only	SN01 [150]
SR average	(4.24 ± 0.06)	(4.66 ± 0.06)	$\Longrightarrow \bar{m}_b(M_Z) = (2.83 \pm 0.04)$	
Average LEP				
Three-jets at M_Z	(4.23 ± 0.94)		$\Longleftarrow \bar{m}_b(M_Z) = (2.82 \pm 0.63)$	LEP [712]
Unquenched lattice				
	(4.23 ± 0.09)			APE00 [713]

[3,401,149,698,150]. Again, we shall be concerned here with the LSR $\mathcal{L}(\tau)$ and the ratio $\mathcal{R}(\tau)$. The latter sum rule, or its slight modification, is useful, as it is equal to the resonance mass squared, in the simple duality ansatz parametrization of the spectral function:

$$\frac{1}{\pi}\,\mathrm{Im}\psi_5(t) \simeq f_D^2 M_D^4 \delta\left(t - M_D^2\right) + \text{“QCD continuum”}\theta(t - t_c)\,, \tag{53.115}$$

where f_D is the decay constant analogue to[10] $f_\pi = 130.56$ MeV. The QCD side of the sum rule reads:

$$\mathcal{L}_{\rm QCD}(\tau) = M_Q^2 \left\{ \int_{M_Q^2}^{\infty} dt\ \mathrm{e}^{-t\tau}\ \frac{1}{8\pi^2}\left[3t(1-x)^2\left(1+\frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)f(x)\right) + \left(\frac{\alpha_s}{\pi}\right)^2 R2s\right] \right.$$
$$\left. + \left[C_4\langle O_4\rangle + C_6\langle O_6\rangle\tau\right]\mathrm{e}^{-M_Q^2\tau} \right\}\,, \tag{53.116}$$

[10] Notice that we have adopted here the lattice normalization for avoiding confusion. We shall discuss its determination in the next chapter.

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where $R2s$ is the new α_s^2-term obtained semi-analytically in [448] and is available as a Mathematica package program Rvs.m. Neglecting m_d, the other terms are:

$$\begin{aligned}
x &\equiv M_Q^2/t\ ,\\
f(x) &= \frac{9}{4} + 2\mathrm{Li}_2(\mathrm{x}) + \log\mathrm{x}\log(1-\mathrm{x}) - \frac{3}{2}\log(1/\mathrm{x}-1)\\
&\quad - \log(1-x) + x\log(1/x-1) - (x/(1-x))\log x\,,\\
C_4\langle O_4\rangle &= -M_Q\langle\bar{d}d\rangle + \langle\alpha_s G^2\rangle/12\pi\\
C_6\langle O_6\rangle &= \frac{M_Q^3\tau}{2}\left(1-\frac{1}{2}M_Q^2\tau\right) g\left\langle\bar{d}\sigma_{\mu\nu}\frac{\lambda_a}{2}G_a^{\mu\nu}d\right\rangle\\
&\quad - \left(\frac{8\pi}{27}\right)\left(2-\frac{M_Q^2\tau}{2}-\frac{M_Q^4\tau^2}{6}\right)\rho\alpha_s\langle\bar{\psi}\psi\rangle^2\ .
\end{aligned} \tag{53.117}$$

The previous sum rules can be expressed in terms of the running mass $\bar{m}_Q(\nu)$.[11] From this expression, one can easily deduce the expression of the ratio $\mathcal{R}(\tau)$, where the unknown decay constant disappears, and from which we obtain the running quark masses:

$$\bar{m}_c(m_c) = (1.10 \pm 0.04)\ \mathrm{GeV}\ . \tag{53.118}$$

The analysis is shown in Fig. 53.10, where a simultaneous fit of the decay constant from $\mathcal{L}$ and of $\bar{m}_c(\bar{m}_c)$ from $\mathcal{R}$ is shown.[12]

Our optimal results correspond to the case where both stability in τ and in t_c are reached. However, for a more conservative estimate of the errors we allow deviations from the stability points, and we take:

$$t_c \simeq (6 \sim 9.5)\ \mathrm{GeV}^2\ , \qquad \tau \simeq (1.2 \pm 0.2)\ \mathrm{GeV}^{-2}\ , \tag{53.119}$$

and where the lowest value of t_c corresponds to the beginning of the τ-stability region. Values outside the above ranges are not consistent with the stability criteria. One can check that the dominant non-perturbative contribution is due to the dimension-four $M_c\langle\bar{d}d\rangle$ light quark condensate, and test that the OPE is not broken by high-dimension condensates at the optimization scale. However, the perturbative radiative corrections converge slowly, as the value of f_D increases by 12% after the inclusion of the α_s correction and the sum of the lowest order plus α_s-correction increases by 21% after the inclusion of the α_s^2 term, indicating that the total amount of corrections of 21% is still a reasonnable correction despite the slow convergence of the perturbative series, which might be improved using a resummed series. However, as the radiative corrections are both positive, we expect that this slow convergence will not affect the final estimate in a significant way. A similar analysis is done for the pole mass. The discussions presented previously also apply here, including the one of the radiative corrections. We quote the final result:

$$M_c = (1.46 \pm 0.04)\ \mathrm{GeV}\ , \tag{53.120}$$

[11] It is clear that, for the non-perturbative terms which are known to leading order of perturbation theory, one can use either the running or the pole mass. However, we shall see that this distinction does not notably affect the present result.
[12] We shall discuss the decay constant in the next section.

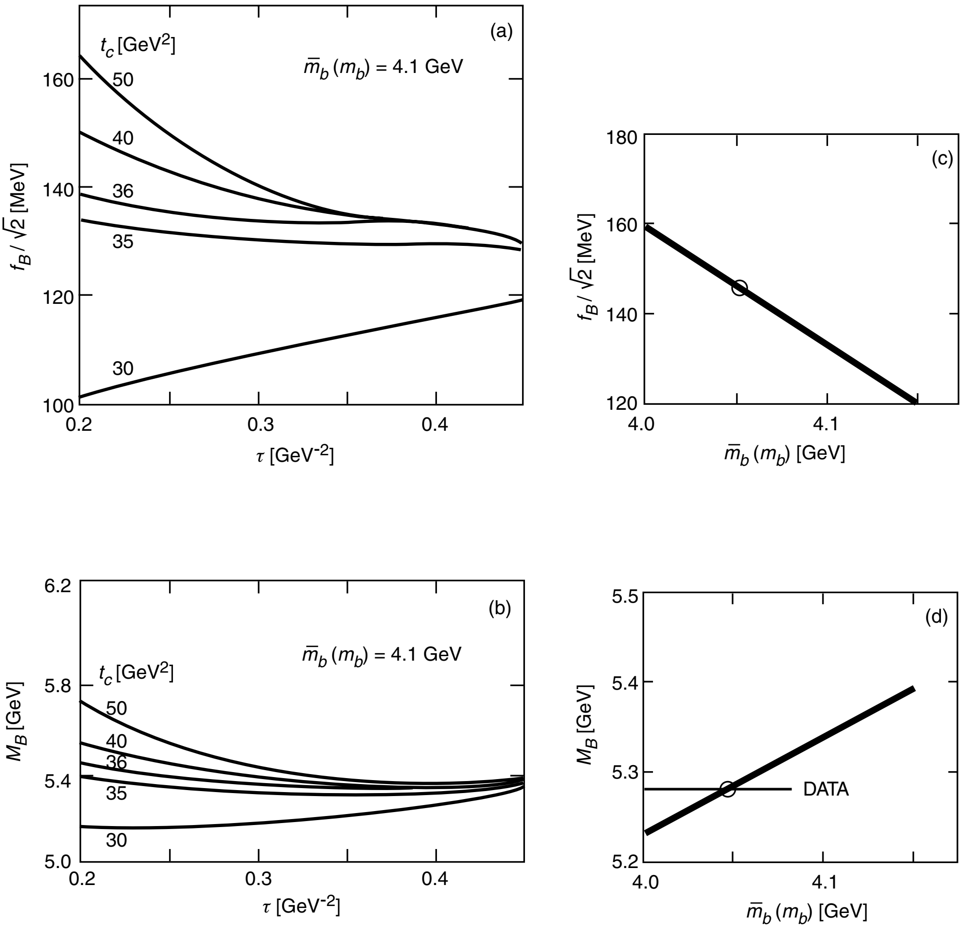

Fig. 53.10. Laplace sum rule analysis of f_D and $\bar{m}_c(\bar{m}_c)$.

where the error is slightly smaller here due to the absence of the subtraction scale uncertainties. One can cross-check that the two values of $\bar{m}_c(m_c)$ and M_c give the ratio:

$$M_c/\bar{m}_c(m_c) \simeq 1.33 \,, \tag{53.121}$$

which satisfies quite well the three-loop perturbative relation $M_c/\bar{m}_c(m_c) = 1.33$. This could be a non-trivial result if one has in mind that the quark pole mass definition can be affected by non-perturbative corrections not present in the standard SVZ-OPE. In particular, it may signal that $1/q^2$ correction of the type discussed in [162,161,394], if present, will only affect weakly the standard SVZ-phenomenology as observed explicitly in the light quark, gluonia and hybrid channels [161]. Using analogous analysis for the B meson, we obtain at the optimization scale $\tau = 0.375$ GeV^{-2} and $t_c = 38$ GeV^2:

$$\bar{m}_b(m_b) = (4.05 \pm 0.06)\ \text{GeV} \,, \tag{53.122}$$

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while using the pole mass as a free parameter, we get:

$$M_b = (4.69 \pm 0.06)\ \text{GeV} . \tag{53.123}$$

One can again cross-check that the two values of $\bar{m}_b(m_b)$ and M_b lead to:

$$M_b/\bar{m}_b(m_b) = 1.16 , \tag{53.124}$$

to be compared with 1.15 from the three-loop perturbative relation between M_b and $\bar{m}_b$, and might indirectly indicate the smallness of the $1/q^2$ correction if any. One can immediately notice the agreement of the results from quarkonia and heavy-light quark channels. Comparisons with other determinations are given in Tables 53.5 and 53.6.

Summary for the heavy quark masses and consequences

From Tables 53.5 and 53.6, we conclude that the running c and b quark masses to order α_s^2 from the different sum rules analysis are likely to be:

$$\bar{m}_c(\bar{m}_c) = (1.23 \pm 0.05)\ \text{GeV} , \qquad \bar{m}_b(\bar{m}_b) = (4.24 \pm 0.06)\ \text{GeV} , \tag{53.125}$$

where the estimated errors come from the arithmetical average of different errors. We have not tried to minimize the errors from weighted average as the correlations between these different determinations are not clear at all. However, as one can see in the tables, the quoted errors are typical for each individual determination. These results are consistent with other determinations given in [16] and in particular with the LEP average from three-jet events and lattice values reported in the tables. Using the previous relation between the short distance perturbative pole and running masses, one obtains, to order α_s:

$$M_c^{PT2} = (1.41 \pm 0.06)\ \text{GeV} , \qquad M_b^{PT2} = (4.63 \pm 0.07)\ \text{GeV} , \tag{53.126}$$

and to order α_s^2:

$$M_c^{PT3} = (1.64 \pm 0.07)\ \text{GeV} , \qquad M_b^{PT3} = (4.88 \pm 0.07)\ \text{GeV} , \tag{53.127}$$

which are consistent with the average values to order α_s quoted in the tables and in [149]. However, one should notice the large effects due to radiative corrections which can reflect the uncertainties in the pole mass definition. From the previous values of the running masses, one can also deduce the values of the RG invariant masses to order α_s^2:

$$\hat{m}_c = (1.21 \pm 0.07)\ \text{GeV} , \qquad \hat{m}_b = (6.9 \pm 0.2)\ \text{GeV} . \tag{53.128}$$

We have used $\Lambda_4 = 325 \pm 40$ MeV and $\Lambda_5 = 225 \pm 30$ MeV. Taking into account threshold effects and using matching conditions, we can also evaluate the running masses at the scale 2 GeV and obtain:

$$\bar{m}_c(2) = (1.23 \pm 0.05)\ \text{GeV} , \qquad \bar{m}_b(2) = (5.78 \pm 0.26)\ \text{GeV} . \tag{53.129}$$

Combining the values of m_b and m_s obtained in the previous section, one can deduce the scale-independent mass ratio:

$$\frac{m_b}{m_s} = 48.8 \pm 9.8 \,, \tag{53.130}$$

which is useful for model building.

One can also run the value of m_b at the Z-mass, and obtains the value of $\bar{m}_b(M_Z)$ quoted in the table:

$$\bar{m}_b(M_Z) = (2.83 \pm 0.04)\ \text{GeV}. \tag{53.131}$$

This value compares quite well with the ones measured at M_Z from three-jet heavy quark production at LEP where the average (2.83 ± 0.04) GeV of different measurements [712] is also given in the table. This is a *first indication* for the running of m_b in favour of the QCD predictions based on the renormalization group equation.

53.13 The weak leptonic decay constants $f_{D_{(s)}}$ and $f_{B_{(s)}}$

In this section,[13] we summarize the different results obtained from the QCD spectral sum rules (QSSR) on the leptonic decay constants of the B and D mesons which are useful in the analysis of the leptonic decay and on the B-$\bar{B}$ mixings. Intensive studies have been carried out on this subject during the last few years using QSSR and lattice calculations.

The leptonic constant of the pseudoscalar $P \equiv D,\ B$ meson is defined as:

$$\langle 0|\partial_\mu A^\mu|P\rangle = f_P M_P^2 \vec{P} \,, \tag{53.132}$$

where $\vec{P}$ is the pseudoscalar meson field and f_P is the pseudoscalar decay constant which controls the $P \to l\nu$ leptonic decay width, normalized as $f_\pi = 130.56$ MeV.[14] The current:

$$\partial_\mu A^\mu(x)^i_j = (m_i + M_j)\bar{\psi}_i(i\gamma_5)\psi_j \quad (i \equiv u,\ d,\ s;\ j \equiv c, b) \,, \tag{53.133}$$

is the divergence of the axial current. In the sum rule analysis, we shall be concerned with the pseudoscalar two-point correlator:

$$\Psi_5(q^2) = i \int d^4x e^{iqx} \langle 0|\mathbf{T}\partial_\mu A^\mu(x)^i_j \left(\partial_\mu A^\mu(0)^i_j\right)^\dagger |0\rangle \,. \tag{53.134}$$

In the case of the $B(\bar{u}b)$ meson, the decay width into $\tau\nu_\tau$ reads:

$$\Gamma(B \to \tau\nu_\tau \ + \ B \to \tau\nu_\tau\gamma) = \frac{G_F^2|V_{ub}|^2}{4\pi} M_B \left(1 - \frac{M_\tau^2}{M_B^2}\right)^2 M_\tau^2 f_B^2 \,, \tag{53.135}$$

where M_τ expresses the helicity suppression of the decay rate into light leptons e and μ.

[13] This is an extension and an update of the some parts of the reviews given in [364].

[14] In this chapter, we adopt this normalization used by the lattice and experimental groups. In the previous sections, we have used $f_\pi \equiv f_\pi/\sqrt{2} = 92.4$ MeV.

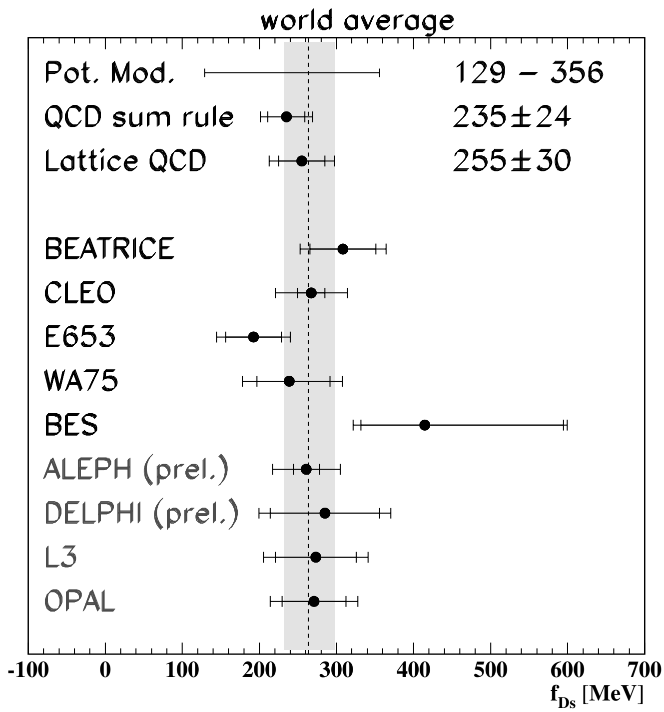

Fig. 53.11. Different measurements of f_{D_s} compared with theoretical predictions from [714].

This expression shows that a good determination of f_B will allow a precise extraction of the CKM mixing angle V_{ub}. One the other hand, f_B and the so-called bag parameter B_B also control the matrix element of the $\Delta B = 2$ B^0-$\bar{B}^0$ mixing matrix, which is of a non-perturbative origin, as we shall discuss in another chapter.

However, contrary to the case of the π and K mesons, the leptonic width of the heavy meson is small as the corresponding decay constant vanishes as $1/\sqrt{M_Q}$, while the presence of the neutrino in the final state renders the reconstruction of the signal and the rejection of background difficult. Moreover, the B leptonic rate is Cabibbo suppressed, which makes it unreachable with present measurements. ($\sim |V_{ub}|^2$), while the D_s leptonic rate is Cabbibbo favoured ($\sim |V_{cs}|^2$). Recent measurements of f_{D_s} are given in Fig. 53.11, where the quoted average is [714]:

$$f_{D_s} \simeq (264 \pm 37) \text{ MeV} . \tag{53.136}$$

53.13.1 Upper bound on the value of f_D

Within the QSSR framework, the decay constants of the B and D mesons have been firstly estimated in [414], while the first upper bounds on their values have been derived in [29] and updated in the recent review [364]. Indeed, a *rigorous* upper bound on these couplings

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can be derived from the second-lowest superconvergent moment:

$$\mathcal{M}^{(2)} \equiv \frac{1}{2!} \frac{\partial^2 \Psi_5(q^2)}{(\partial q^2)^2}\bigg|_{q^2=0}, \tag{53.137}$$

where for this low moment, the OPE behaves well. Using the positivity of the higher-state contributions to the spectral function, one can deduce [29,399]:

$$f_P \leq \frac{M_P}{4\pi} \left\{ 1 + 3\frac{m_q}{M_Q} + 0.751\bar{\alpha}_s + \cdots \right\}, \tag{53.138}$$

where one should not misinterpret the mass dependence in this expression compared with that expected from heavy-quark symmetry. Applying this result to the D meson, one obtains:

$$f_D \leq 2.14 f_\pi , \tag{53.139}$$

which is not dependent to leading order on the value of the charm quark mass. Although presumably quite weak, this bound, when combined with the recent determination of the $SU(3)_F$ breaking effects to two loops on the ratio of decay constants [716]:

$$\frac{f_{D_s}}{f_D} \simeq (1.15 \pm 0.04) , \tag{53.140}$$

implies

$$f_{D_s} \leq (2.46 \pm 0.09) f_\pi \simeq (321.2 \pm 11.8)\ \text{MeV} , \tag{53.141}$$

which is useful for a comparison with the recent measurement of f_{D_s}, with the average value given previously. One cannot push, however, the uses of the moments to higher n values in this D channel, in order to minimize the continuum contribution to the sum rule with the aim of derive an estimate of the decay constant from this method, and to derive its 'correct' mass dependence, because the QCD series will not converge at higher n values.

53.13.2 Estimate of the D decay constant f_D

The decay constant f_D can be extracted from the pseudoscalar Laplace sum rules given in Eq. (53.116).[15] Prior to 1987, the different sum rules estimate of the decay constant f_P were inconsistent among each other. To our knowledge, the first attempt to understand such discrepancies was reported in [717,718] (see also [719]), where it was shown, *for the first time* and a long time before the lattice results, that:

$$f_D \approx f_B \approx (1.2 \sim 1.5) f_\pi , \tag{53.142}$$

which differs from that expected from the heavy-quark symmetry scaling law [720]:

$$f_B \approx \sqrt{\frac{M_D}{M_B}} f_D \left(\frac{\alpha_s(M_c)}{\alpha_s(M_b)} \right)^{-1/\beta_1} , \tag{53.143}$$

[15] For reviews, see for example [715,360].

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valid in the extreme case where the heavy-quark mass is infinite.[16] It has also been understood that the apparent disagreement among different existing QSSR numerical results in the literature is *not only* due to the choice of the continuum threshold t_c [its effect is $(7 \sim 10)\%$ of the result when one moves t_c from the one at the beginning of sum rule variable to the one where the t_c stability is reached.][17] as misleadingly claimed in the literature. Indeed, the main effect is *also* due to the different values of the quark masses used,[18] which is shown explicitly in the table of [716].

In the D channel, the most appropriate sum rule is the (relativistic) Laplace sum rule, as the OPE of the $q^2 = 0$ moments does not converge for larger values of the number of derivatives n, at which the D meson contribution to the spectral integral is optimized. The results from different groups are consistent with each others for a given value of the c-quark mass. For the D meson, the optimal result is obtained for:

$$6 \leq t_c \leq 9.5 \text{ GeV}^2 , \qquad \tau \simeq (1.2 \pm 0.2) \text{ GeV}^{-2} . \tag{53.144}$$

where the QCD corrections are still reasonnably small. The most recent estimate including α_s^2 corrections from a simultaneous fit of the set either $(f_D, \bar{m}_c(m_c))$ or (f_D, M_c^{pole}) is given in Fig. 53.10. The obtained values of the quark masses have been quoted in Table 53.5. The resulting value of f_D is [150]:

$$f_D \simeq (203 \pm 23) \text{ MeV} , \tag{53.145}$$

in agreement with the recent evaluation (195 ± 20) MeV at order α_s^2 but using the pole mass as input [722].

53.13.3 Ratio of the decay constants f_{D_s}/f_D and f_{B_s}/f_B

The $SU(3)$ breaking ratios f_{D_s}/f_D and f_{B_s}/f_B have been obtained semi-analytically in [716]. In order to have a qualitative understanding of the size of these corrections, we start from the global hadron-quark duality sum rule:

$$\int_0^{\omega_c} d\omega \, \text{Im}\Psi_5^{res}(\omega) \simeq \int_0^{\omega_c} d\omega \, \text{Im}\Psi_5^{\bar{q}Q}(\omega) , \tag{53.146}$$

where ω_c is the continuum energy defined as:

$$t = (E + M_Q)^2 \equiv M_Q^2 + \omega M_Q . \tag{53.147}$$

Keeping the leading order terms in α_s and in $1/M_Q$, it leads to:

$$R_P \simeq \rho_P \left\{ 1 + 3\left(\frac{m_s}{\omega_c}\right)\left(1 - \frac{m_s}{M_Q}\right) - 6\left(\frac{m_s}{\omega_c}^2\right) - \left(\frac{m_s}{M_Q}\right)\left(1 - \frac{m_s}{M_Q}\right) \right\} , \tag{53.148}$$

[16] Finite mass corrections to this formula will be discussed later on.
[17] In some papers in the literature, the value of t_c is taken smaller than the previous range. In this case, the t_c effect is larger than the one given here.
[18] A critical review on the discrepancy between different existing estimates is given in [721].

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Table 53.7. *Estimate of $f_{B_{(s)}}$ to order α_s^2 and f_{B_s}/f_B to order α_s from QSSR and comparison with the lattice*

Sources	f_B (MeV)	$f_{B_{(s)}}/f_B$	f_{B_s} (MeV)	Comments	Authors
QSSR					
LSR	203 ± 23	$1.16 \pm 0.04 \Longrightarrow$	236 ± 30	M_{pole}, $\bar{m}_b$: output	SN94,01 [716,150]
	210 ± 19		244 ± 21	$\bar{m}_b$: input	JL01 [724]
HQETSR	206 ± 20			M_{pole}: input	PS01 [722]
SR average	$207 \pm 21 \Longrightarrow$		240 ± 24		
Unq. lattice					
	200 ± 30	$1.16 \pm 0.04 \Longrightarrow$	232 ± 35	average	LAT01 [725]

where:

$$\rho_P \equiv \left(\frac{M_P}{M_{P_s}}\right)^2 \left(1 + \frac{m_s}{M_Q}\right) . \tag{53.149}$$

The value of ω is fixed from stability criteria to be [717,721,634,165]:

$$\omega_c \simeq (3.1 \pm 0.1) \text{ GeV}. \tag{53.150}$$

The sum rule indicates that the $SU(3)$ breaking corrections are of two types, the one m_s/M_Q and the other m_s/ω_c. More quantitatively, we work with the Laplace sum rule:

$$\mathcal{L} = \int_0^{\omega_c} d\omega\, \mathrm{e}^{-\omega\tau}\, \mathrm{Im}\Psi_5^{res}(\omega). \tag{53.151}$$

Analogously the Laplace sum rule gives:

$$R_P^2 \simeq \rho_P^2 \left\{ 1 + 2(2.2 \pm 0.2)\left(\frac{m_s}{\omega_c}\right)\left(1 - \frac{m_s}{M_Q}\right) - 2\,(8.2 \pm 1.6)\left(\frac{m_s}{\omega_c}^2\right) \right\} , \tag{53.152}$$

where the numerical integration includes a slight M_Q dependence. Including $m_s\alpha_s$ and $m_s^2\alpha_s$-corrections, the resulting values of the ratio are:

$$R_D \equiv \frac{f_{D_s}}{f_D} = 1.15 \pm 0.04 , \qquad R_B \equiv \frac{f_{B_s}}{f_B} = 1.16 \pm 0.05 . \tag{53.153}$$

This result implies:

$$f_{D_s} \simeq (235 \pm 24) \text{ MeV} , \tag{53.154}$$

which agrees within the errors with the data [714] and lattice [723] averages both quoted in Fig. 53.11. This feature increases confidence in the use of the QSSR method for predicting the not yet measured decay constant of the B meson.

53.13.4 Estimate of the B decay constant f_B

For the estimate of f_B, one can either work with the Laplace, the moments or their non-relativistic variants because the b-quark mass is relatively heavy. The optimal result which we shall give here comes from the Laplace relativistic sum rules. They corresponds to the *conservative range* of parameters:

$$0.6 \le E_c^{(b)} \equiv \sqrt{t_c} - M_B \le 1.8 \text{ GeV} , \qquad \tau \simeq 0.38 \text{ GeV}^{-2} , \tag{53.155}$$

which have been used in the previous section for getting the b-quark mass. As shown in the figure given in [726,727], the dominant corrections come from the $\langle \bar{u}u \rangle$ quark condensate with the strengh (30∼40)% of the lowest order term in f_B, while the higher condensate effects are smaller, which are respectively $-(20\sim30)\%$, $+(5\sim8)\%$ for the $d = 5$ and 6 condensates. This shows, despite the large value of the quark condensate contribution, that the OPE is convergent. It has been noticed in [726,727], that the convergence of the OPE is faster for the relativistic LSR than for the moments, such that the most precise result should come from the LSR. In both cases the perturbative corrections are small. One obtains from the relativistic LSR , the results to order α_s^2 [150]:

$$f_B \simeq (203 \pm 23) \text{ MeV} \simeq (1.55 \pm 0.18) f_\pi , \tag{53.156}$$

and to order α_s (see previous discussion) [716]:

$$\frac{f_{B_s}}{f_B} \simeq 1.16 \pm 0.04 . \tag{53.157}$$

These values of f_B and f_D agree quite well with the previous QSSR findings in [716], [3] and [719]. They also agree with the non-relativistic sum rules estimate in the full theory [717], in HQET [633,722] and in [634,165]. However, unlike the relativistic sum rules, the HQET sum rule is strongly affected by the huge perturbative radiative corrections of about 100%, which is important at the sum rule scale of about 1 GeV at which the HQET sum rule optimizes. These results are also in good agreement with the lattice average estimate given in Table 53.7.

53.13.5 Static limit and $1/M_b$-corrections to f_B

As noticed previously, the *first* result $f_B \simeq f_D$ in [716], which has been confirmed by recent estimates from different approaches, shows a large violation of the scaling law expected from heavy-quark symmetry. This result suggests that finite quark mass corrections are still huge at the D and B meson masses. The first attempt to understand this problem analytically is in [718] in terms of large corrections of the type E_c/M_b if one expresses in this paper the continuum threshold t_c in terms of the threshold energy E_c:

$$t_c \equiv (E_c + M_b)^2. \tag{53.158}$$

More recently different approaches have been investigated for the estimate of the size of these corrections.

In the lattice approach, these mass corrections have been estimated from a fit of the obtained value of the meson decay constant at finite and infinite (static limit) values of the heavy quark mass and by assuming that these corrections are polynomial in $1/M_Q$ up to log. corrections. A similar analysis has been done with the sum rule in the full theory [726,727], by studying numerically the quark mass dependence of the decay constant up to the quark mass value ($M_Q \leq 15$ GeV) until which one may expect that the sum rule analysis is still valid. In so doing, we use the parametrization:

$$f_B\sqrt{M_B} \simeq \tilde{f}_B \alpha_s^{1/\beta_1} \left\{ 1 - \frac{2}{3}\frac{\alpha_s}{\pi} - \frac{A}{M_b} + \frac{B}{M_b^2} \right\} , \tag{53.159}$$

by including the quadratic mass corrections. The analysis gives:[19]

$$\tilde{f}_B \equiv (f_B\sqrt{M_B})_\infty \simeq (0.65 \pm 0.06)\ \mathrm{GeV}^{3/2} , \tag{53.160}$$

which one can compare with the results from the HQET Laplace sum rule [633] and [165,728]:

$$\tilde{f}_B \simeq (0.35 \pm 0.10)\ \mathrm{GeV}^{3/2} , \tag{53.161}$$

and from FESR [165]:

$$\tilde{f}_B \simeq (0.57 \pm 0.10)\ \mathrm{GeV}^{3/2} , \tag{53.162}$$

Taking the average of these three (independent) determinations, one can deduce:

$$\tilde{f}_B \simeq (0.58 \pm 0.09)\ \mathrm{GeV}^{3/2} , \tag{53.163}$$

where we have evaluated an arithmetic average of the errors. This result is in good agreement with the lattice value given in [723,729] using non-perturbative clover fermions. One can translate this result into the value of f_B in the static limit approximation:

$$f_B^{\mathrm{stat}} \simeq (1.9 \pm 0.3) f_\pi . \tag{53.164}$$

We can also use the previous value of f_B^{stat} together with the previous values of f_B and f_D at the 'physical' quark masses in order to determine numerically the coefficients A and B of the $1/M_b$ and $1/M_b^2$ corrections. In so doing, we use the values of the quark 'pole' masses given in Tables 53.5 and 53.6. Then, we obtain from a quadratic fit:

$$A \approx 0.98\ \mathrm{GeV} \quad \text{and} \quad B \approx 0.35\ \mathrm{GeV}^2 , \tag{53.165}$$

while a linear fit gives a large uncertainty:

$$A \approx (0.74 \sim 0.91)\ \mathrm{GeV} . \tag{53.166}$$

One can notice that the fit of these coefficients depends strongly on the input values of f_D and f_B. Indeed, using some other set of values as in [726,727,634], one obtains values about two-times smaller. Therefore, we consider as a conservative value of these coefficients an

[19] The numbers given in [726,727] correspond to the quark mass $M_b = 4.6$ GeV and should be rescaled until the meson mass M_B. In the following, we shall also work with f_B normalized to be $\sqrt{2}$ bigger than in the original papers.

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uncertainty of about 50%. The value of A obtained is comparable with the one from HQET sum rules [166] and [726,634] of about $0.7 \sim 1.2$ GeV. A similar value of A has been also obtained from the recent NR lattice calculations with dynamic fermions and using a linear fit [729]:

$$A \approx 0.7\ \text{GeV}\ . \tag{53.167}$$

One can *qualitatively* compare this result with the one obtained from the analytic expression of the moment in the full theory [3,718,728]:

$$f_B^2 \approx \frac{1}{\pi^2}\frac{E_c^3}{M_b}\left[1 - \frac{2}{3}\frac{\alpha_s}{\pi} - \frac{3(n+2)E_c}{M_b} + \cdots - \frac{\pi^2}{2}\frac{\langle\bar{u}u\rangle}{E_c^3} + \cdots\right]. \tag{53.168}$$

Here, one can notice that the size of the $1/M_b$ corrections depends on the number of moments, such that their estimate using literally the expression of the moments can be inconclusive. A qualitative estimate of these corrections can be done from the semi-local duality sum rule, which has more intuitive physical meaning due to its direct connection with the leptonic width and total cross-section through the optical theorem. It corresponds to $n = -2$, and leads to the *interpolating formula* [728]:

$$\sqrt{2} f_B \sqrt{M_B} \approx \frac{E_c^{3/2}}{\pi}\alpha_s^{1/\beta_1}\left(\frac{M_b}{M_B}\right)^{3/2}\left\{1 - \frac{2}{3}\frac{\alpha_s}{\pi} + \frac{3}{88}\frac{E_c^2}{M_b^2} - \frac{\pi^2}{2}\frac{\langle\bar{u}u\rangle}{E_c^3} + \cdots\right\}, \tag{53.169}$$

from which, one obtains:

$$\begin{aligned} A &\approx \frac{3}{2}(M_B - M_b) \simeq 1\ \text{GeV}, \\ B &\approx \frac{3}{88}E_c^2 + \frac{27}{32}(M_B - M_b)^2 \simeq 0.45\ \text{GeV}^2\ , \end{aligned} \tag{53.170}$$

which is in good agreement with the previous numerical estimate.

53.14 Conclusions

We have reviewed the determinations of the quark masses and leptonic decay constants of (pseudo)scalar mesons which are useful in particle physics phenomenology. The impressive agreements of the QSSR results with the data when they are measured and/or with lattice calculations in different channels indicate the self-consistency of the QSSR approach, making it one of the most powerful semi-analytical QCD non-perturbative approaches available today. Applications of these results for studying the B-$\bar{B}$ mixings and CP-violation will be discussed in the next chapter.