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EVERY CHAOTIC INTERVAL MAP HAS A SCRAMBLED SET IN THE RECURRENT SET

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Let I denote a compact real interval and let $f \in C^0(I, I)$. In this note we show that if f is chaotic in the sense of Li and Yorke, then there is an uncountable perfect δ-scrambled set S for f in the recurrent set of f. Furthermore, the ω-limit set of every $x \in S$ under f contains S and contains infinitely many periodic points of f with arbitrarily large periods.

1. INTRODUCTION

Let I denote a compact interval on the real line and let f be a continuous map from I into itself. For every positive integer n, let f^n denote the nth iterate of f : $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n > 1$. Let x_0 be a point of I. x_0 is called a periodic point of f if $f^m(x_0) = x_0$ for some positive integer m and the smallest such positive integer m is called the period of x_0 (under f). x_0 is called a recurrent point of f if for every open neighbourhood V of x_0, $f^k(x_0) \in V$ for some positive integer k. The set of all recurrent points of f is called the recurrent set of f and is denoted by $R(f)$. The set of all limit points of the set $\{x_0, f(x_0), f^2(x_0), \ldots\} = \{f^n(x_0) \mid n \geqq 0\}$ is called the ω-limit set of x_0 under f and is denoted by $\omega_f(x_0)$. If f has a periodic point of period not an integral power of 2, then we say that f is chaotic in the sense of Li and Yorke ([**3, 6, 8, 11, 16**]). Note that this is slightly different from the definition used in [**14, 15**] and [**18**]. It is well-known ([**2, 4, 5, 8–15, 18, 19**]) that if f is chaotic in the sense of Li and Yorke, then there exist a positive real number δ and an uncountable perfect set S (called a δ-scrambled set of f) in I such that the following hold:

(1) For any two distinct points x and y in S,

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$
$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geqq \delta.$$

(2) For any point x in S and any periodic point p of f,

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \geqq \delta/2.$$

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In this note we extend the ideas of Auslander and Yorke [**1**], and Osikawa and Oono [**11**] to show that if f is chaotic in the sense of Li and Yorke, then the δ-scrambled set S mentioned above can be chosen in the recurrent set $R(f)$ of f. To be more precise, we prove the following result (see also [**11**]).

THEOREM. *If $f \in C^0(I, I)$ is chaotic in the sense of Li and Yorke, then there exist a positive real number δ and an uncountable perfect set S (called a δ-scrambled set of f) in $R(f)$ such that the following hold:*

(i) *For every $x \in S$, the ω-limit set of x under f contains S and contains infinitely many periodic points of f with arbitrarily large periods (see also [**17**]).*

(ii) *For any two distinct points x and y in S,*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$
$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geqq \delta.$$

(iii) *For any point x in S and any periodic point p of f,*

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \geqq \delta/2.$$

(iv) *For any positive number ε, there are infinitely many periodic points p of f such that, for every $x \in S$,*

$$\liminf_{n\to\infty} |f^n(x) - p| < \varepsilon.$$

2. PROOF OF THE THEOREM

We first recall some terminology from symbolic dynamics. Let $\Sigma_2 = \{a \mid a = (a_0, a_1, a_2, \cdots), a_k = 0 \text{ or } 1 \text{ for all } k \geqq 0\}$. A metric for Σ_2 is given by putting $d((a_0, a_1, a_2, \cdots), (b_0, b_1, b_2, \cdots)) = \Sigma |a_k - b_k|/2^k$. Let $\sigma: \Sigma_2 \to \Sigma_2$ be the shift map $\sigma((a_0, a_1, a_2, \cdots)) = (a_1, a_2, \cdots)$, that is $(\sigma(a))_k = a_{k+1}$ for all $k \geqq 0$ if $a = (a_0, a_1, a_2, \cdots)$. Then (Σ_2, d) is a compact metric space and σ is a continuous, onto, two to one map. The pair (Σ_2, σ) is called the one-sided shift map on two symbols. In the following, we omit parentheses and commas in writing the elements of Σ_2. We call $[a_0 | a_1 | \cdots | a_{k-1}] = \{\omega \in \Sigma_2 | \omega = \omega_0\omega_1\omega_2\cdots \text{ and } \omega_j = a_j \text{ for } 0 \leqq j \leqq k-1\}$ a k-cylinder in Σ_2. We note that there are uncountably many elements $\omega \in \Sigma_2$ (called transitive elements) such that for every cylinder set Q in Σ_2, $\sigma^m\omega \in Q$ for infinitely many positive integers m.

Let $g \in C^0(I, I)$. Assume that there are two disjoint closed subintervals I_0, I_1 of I such that $g(I_0) \cap g(I_1) \supseteq I_0 \cup I_1$. Let $Z = I_0 \cup I_1$. For all $a_k = 0$ or 1,

$k \geqq 0$, we define $I(a_0a_1 \ldots a_k)$ inductively by putting $I(0) = I_0$, $I(1) = I_1$, and letting $I(a_0a_1 \cdots a_{k+1})$ be a closed subinterval of $I(a_0a_1 \cdots a_k)$ such that $g(I(a_0a_1 \cdots a_{k+1})) = I(a_1a_2 \cdots a_{k+1})$. For every $a = a_0a_1a_2 \cdots \in \Sigma_2$, let $I(a) = \bigcap_k I(a_0a_1 \cdots a_k)$. Then $I(a)$ is either a compact interval or consists of one point and $g(I(a)) = I(\sigma a)$. Let Z^* be the union of $I(a)$ for all $a \in \Sigma_2$ and let B be the set of all $b \in \Sigma_2$ with $I(b)$ consisting of one point. It is easy to see that Z^* is a nonempty compact subset of Z with $g(Z^*) = Z^*$. Let $h\colon Z^* \to \Sigma_2$ be defined by putting $h(z) = a = a_0a_1a_2 \cdots$ for $z \in Z^*$, where $a_n = k$ if $g^n(z) \in I_k$, $n \geqq 0$. Note that the map h defined above is the same as the map $\tau\colon Z^* \to \Sigma_2$ defined by putting $\tau(z) = a$ for every $z \in I(a) \subset Z^*$. Now we can state the following result (see also [**1**]).

LEMMA 1. *Let g, Z^*, and h be defined as above. Then the following hold.*

(a) *h is continuous and onto.*

(b) *g is semiconjugate through h to σ on Z^*, that is $hg = \sigma h$ on Z^*.*

(c) *If $z \in Z^*$, $h(z) = a$, and $\overline{\{g^n(z) \mid n \geqq 0\}}$ contains a periodic point of g with period m, then $\overline{\{\sigma^n(a) \mid n \geqq 0\}}$ contains a periodic point of σ with period dividing m. In particular, h sends periodic points of g in Z^* into periodic points of σ.*

(d) *If $a \in \Sigma_2$ is a periodic point of σ with period m, then $I(a)$ contains a periodic point of g with period m. Furthermore, if $a \notin B$, then at least one endpoint of $I(a)$ is a periodic point of g with period m or $2m$. Consequently, for every positive integer m, g has at least as many periodic points of period m as the shift map σ in Σ_2.*

(e) *h sends recurrent points of g in Z^* into recurrent points of σ.*

(f) *If $a \in \Sigma_2$ is a recurrent point of σ, then $I(a)$ contains a recurrent point of g. Furthermore, if $a \notin B$, then at least one endpoint of $I(a)$ is a recurrent point of g.*

(g) *Assume that $b \in \Sigma_2 - B$ is not transitive for σ. Then there is an endpoint y of $I(b)$ such that if $I(\sigma^n b)$ is a nondegenerate closed interval for some positive integer n, then $g^n(y)$ is an endpoint of $I(\sigma^n b)$.*

(h) *For every transitive element $b \in \Sigma_2$ of σ, the ω-limit set of x_b under g for every $x_b \in I(b)$ contains all elements of $I(c)$ with $c \in B$ and contains at least one endpoint of $I(a)$ for every point $a \in \Sigma_2 - B$. In particular, if $b \in \Sigma_2$ is transitive for σ, then the ω-limit set $\omega_g(x_b)$ of x_b under g for every $x_b \in I(b)$ contains infinitely many periodic points of g. Consequently, there are uncountably many points in Z^* which are not asymptotically periodic.*

PROOF: We give a proof of (h) only. The other proofs are easy and are omitted. Let a be a transitive element of σ in B with $I(a) = \{x_a\}$. Let $c \in B$ and $I(c) = \{x_c\}$. Then, for $\varepsilon > 0$, there is a positive integer m such that $I(c_0c_1 \cdots c_k) \subseteq (x_c - \varepsilon, x_c + \varepsilon)$ for all $k \geqq m$. Since a is transitive, $\sigma^n a \in [c_0 \mid c_1 \mid \cdots \mid c_m]$ for infinitely many positive integers n. Hence, $g^n(x_a) \in g^n(I(a)) = I(\sigma^n a) \subset I(c_0c_1 \cdots c_m) \subseteq (x_c - \varepsilon, x_c + \varepsilon)$ for infinitely many positive integers n. This shows that $x_c \in \omega_g(x_a)$.

Now let $c \notin B$ and $I(c) = [y, z]$ with $y < z$. Since $I(a)$ consists only of the point x_a and $I(c) = [y, z]$ is a nondegenerate interval, we see that $I(\sigma^n a) = g^n(I(a)) \neq I(c)$ for all $n \geqq 0$. Hence, $\sigma^n a \neq c$ for all $n \geqq 0$. By a similar argument to that above, we see that, for any $\varepsilon > 0$, $g^n(x_a) \in g^n(I(a)) = I(\sigma^n a) \subseteq (y-\varepsilon, z+\varepsilon) - [y, z] = (y-\varepsilon, y) \cup (z, z+\varepsilon)$ for infinitely many positive integers n. Consequently, $y \in \omega_g(x_a)$ or $z \in \omega_g(x_a)$.

In the following, we use a method introduced in [**11**]. We first let the map ϕ from Σ_2 into Z^* be defined as in [**11**] by the following five steps:

Step 1 For a such that $I(a)$ consists of a point x_a, let $\phi(a) = x_a$.

Step 2 If $\phi(a)$ is defined for a, let $\phi(\sigma(a)) = g(\phi(a))$.

Step 3 If $\phi(\sigma(a))$ is defined for a, let $\phi(a) \in I(a)$ be defined such that $g(\phi(a)) = \phi(\sigma(a))$.

Step 4 If ϕ is not yet defined on $\sigma^k(a)$ for all integers $k \geqq 0$ and no $\sigma^k(a)$ is a periodic point of σ, let $\phi(a)$ be chosen as an arbitrary point in $I(a)$.

Step 5 If ϕ is not yet defined on $\sigma^k(a)$ for all integers $k \geqq 0$ and a is a periodic point of σ with period $n \geqq 1$, let $\phi(a)$ be chosen as an arbitrary point in $I(a)$ such that $g^n(\phi(a)) = \phi(a)$ (such $\phi(a)$ exists since $g^n(I(a)) = I(\sigma^n(a)) = I(a)$).

Since, for every $a \in \Sigma_2$, we have $\phi(a) \in I(a)$, it is clear that ϕ is one-to-one, and from the construction of ϕ, $g\phi = \phi\sigma$ on Σ_2. Note that if $a \in \Sigma_2$ is a periodic point of σ, then $\phi(a)$ is a periodic point of g, and vice versa. From now on, we let $a = a_0a_1a_2\cdots \in \Sigma_2$ be any transitive point such that $I(a)$ consists of one point. For any point $b = b_0b_1b_2\cdots \in \Sigma_2$, define $\omega_b \in \Sigma_2$ by putting $\omega_b = a_0b_0a_0a_1b_0b_1a_0a_1a_2b_0b_1b_2\cdots a_0a_1a_2\cdots a_kb_0b_1b_2\cdots b_k\cdots$ and let $T = \{\phi(\omega_b) \mid b \in \Sigma_2$ and $I(\omega_b)$ consists of one point $\}$. Then T is an uncountable Borel set. We now show that g behaves chaotically on T.

Let $b = b_0b_1b_2\cdots \in \Sigma_2$ and $c = c_0c_1c_2\cdots \in \Sigma_2$ with $b \neq c$. Then there exists a least integer $k \geqq 0$ such that $b_k \neq c_k$. Since a is transitive, $g^m(\phi(\omega_b)) \in I(b_k)$ and $g^m(\phi(\omega_c)) \in I(c_k)$ for infinitely many m. Therefore, $\limsup_{n\to\infty} |g^n(\phi(\omega_b)) - g^n(\phi(\omega_c))| \geqq \operatorname{dist}(I_0, I_1)$.

On the other hand, $\{\sigma^{n(n+1)}(\omega_b), \sigma^{n(n+1)}(\omega_c)\} \subseteq [a_0 \mid a_1 \mid \cdots \mid a_n]$ for all $n \geqq 1$ and $\phi([a_0 \mid a_1 \mid \cdots \mid a_n]) \subseteq I(a_0a_1\cdots a_n)$ converges to the set $I(a)$ which consists of one point. So, $\liminf_{n\to\infty} |g^n(\phi(\omega_b)) - g^n(\phi(\omega_c))| = 0$.

Now let p be any periodic point of g with period k. Since a is transitive, there are infinitely many integers m and n such that $a_m = a_{m+1} = \cdots = a_{m+k+1} = 0$ and $a_n = a_{n+1} = \cdots = a_{n+k+1} = 1$. Consequently, $\limsup_{n\to\infty} |g^n(\phi(\omega_b)) - g^n(p)| \geqq \operatorname{dist}(I_0, I_1)/2$.

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Finally, for any $\varepsilon > 0$, since $I(a) = \{\phi(a)\}$, there is a cylinder Q such that the length of the smallest interval V containing $\phi(Q)$ is less than ε. Since Q contains infinitely many periodic points of σ, $\phi(Q) \subseteq V$ contains infinitely many periodic points p of g. Since a is transitive, we see that, for every $x \in T$, $g^n(x) \in V$ for infinitely many $n \geqq 1$. Consequently, we have, for every $x \in T$,

$$\liminf_{n\to\infty} |g^n(x) - p| < \varepsilon.$$

∎

It is well-known [7] that every uncountable Borel set contains a perfect set. Let S denote any such perfect set in T. From what we have proved above, together with Lemma 1, we obtain immediately the following result.

LEMMA 2. *Let $g \in C^0(I, I)$. Assume that there exist two disjoint closed subintervals I_0, I_1 of I such that $g(I_0) \cap g(I_1) \supseteq I_0 \cup I_1$. Let $\delta =$ dist(I_0, I_1). Then there exists an uncountable perfect set S (called a δ-scrambled set of g) in $R(g)$ such that the following hold.*

(i) *For every $x \in S$, the ω-limit set of x under g contains S and contains infinitely many periodic points of g with arbitrarily large periods.*

(ii) *For any two distinct points x and y in S,*

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0,$$
$$\limsup_{n\to\infty} |g^n(x) - g^n(y)| \geqq \delta.$$

(iii) *For any point x in S and any periodic point p of g,*

$$\limsup_{n\to\infty} |g^n(x) - g^n(p)| \geqq \delta/2.$$

(iv) *For any positive number ε, there are infinitely many periodic points p of g such that, for every $x \in S$,*

$$\liminf_{n\to\infty} |g^n(x) - p| < \varepsilon.$$

Now we can prove the theorem. Assume that f has a periodic point of period $2^m(2n+1)$. Then f^{2^m} has a periodic point of period $2n+1$. By Sharkovskii's theorem [16], the map $g = f^{2^{m+1}}$ has a periodic point of period 3. Without loss of generality, we may assume that $x < y < z$, $g(x) = y$, $g(y) = z$ and $g(z) = x$. So, there is a point $w \in (y, z)$ such that $g(w) = y$. Let $I_0 = [x, y]$ and $I_1 = [w, z]$. Then it is easy to see that $g^3(I_0) \cap g^3(I_1) \supseteq I_0 \cup I_1$. The theorem now follows easily from Lemma 2 since $R(f) = R(g)$. This completes the proof of the theorem.

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