

1.1 Basic Properties

1.1.1 Infinite Conductivity

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes [1] in Leiden. Kamerlingh Onnes was a pioneer in reaching experimentally low temperatures and exploring physics in this by then new realm. He was the first in 1908 to liquify helium gas (he got the Nobel Prize for this in 1913), which provided him a powerful cooling agent at these temperatures.

Kamerlingh Onnes was interested in the variation of the conductivity of metals at low temperature, which was at the time a controversial matter. He chose to work on mercury because it is much easier to purify by distillation than other metals, since it is liquid at room temperature. In this way, he could get rid of impurities, which contribute to the electrical resistivity of metals, and study the intrinsic low-temperature behavior. Upon cooling his solid mercury wire, he observed a slow decrease of its resistance, corresponding to his expectations, and then around the temperature of 4.2 K, a sudden drop to a value so low that he could not actually measure it, as seen in Fig. (1.1). This experimental disappearance of the resistance implies a vanishing resistivity ρ for mercury below the “critical temperature” of $T_c = 4.2$ K,

$$\rho = 0 \quad (1.1)$$

Kamerlingh Onnes called “superconductivity” this property of a metal to have in effect an infinite conductivity $\sigma = 1/\rho$. The year after his discovery, he found that tin and lead were superconductors with critical temperatures of 3.7 K and 7.2 K, respectively.

Naturally it is meaningless to claim that Eq. (1.1) is an experimental result; it is only a logical extrapolation of the experimental result. An experiment always has a limited accuracy and can only claim that the resistivity is extremely small. Nevertheless experimentalists have pushed as far as they could to determine how small the resistivity of a superconductor is. A clever and striking way to do this indirectly is to observe persistent currents. Indeed one can generate currents by induction in a metal having the shape of a ring. For a standard metal with a nonzero resistivity, these currents decay rapidly by dissipation due to Joule heating. However, for a superconductor, Eq. (1.1) holds and no dissipation occurs, so the induced currents can persist indefinitely. These currents can be observed by the magnetic field they generate. Kamerlingh Onnes performed first such an

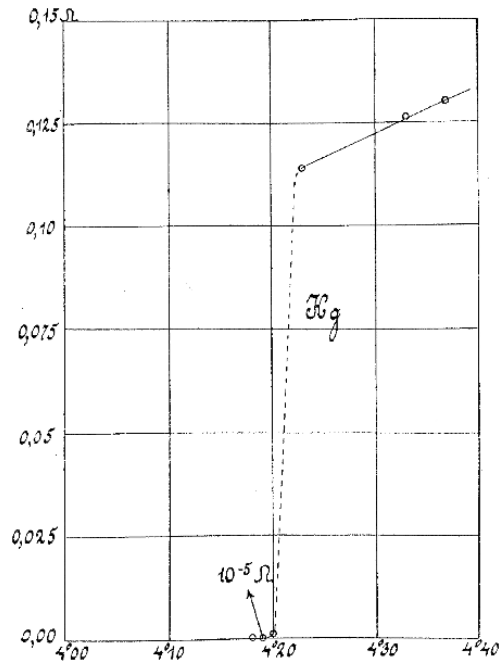


Fig. 1.1

The resistance of mercury as a function of temperature as measured by H. Kamerlingh Onnes [1].

experiment in 1914, and observed persistent currents for hours. This kind of experiment has been repeated; persistent currents have been observed for several years, and the decay time for the persistent current has been evaluated to $\sim 10^5$ years. This comes quite close to an experimental proof of Eq. (1.1).

1.1.2 Critical Temperature

Kammerlingh Onnes realized naturally that in principle, superconductivity could allow one to generate very large electric currents and, as a result, very large magnetic fields. Obviously the very low temperature at which the phenomenon occurs makes it in practice quite inconvenient to set up such a device. Hence the value of the critical temperature is not only an important quantity to characterize superconductivity, but it is also of utmost practical interest. This has led to the exploration of a number of materials for their possible superconducting properties. Among the elements of the periodic table, 33 are superconductors at atmospheric pressure (with an additional 24 which become superconductors under pressure). The one with the highest critical temperature is niobium, with $T_c = 9.26$ K.

More generally, it has been progressively realized that, far from being an exceptional phenomenon, superconductivity appears quite frequently at low temperature. Its absence may be due to the competing appearance of another kind of transition, for example toward magnetic order. It should be noted that superconducting transitions may be found at very low temperature, which while not being practically useful may be of fundamental interest. There is no lower bound for T_c . In particular copper, the best standard conductor,

does not display a superconducting transition at the lowest temperatures presently reached. Similarly gold and silver are not superconductors.

Coming back to the more practically interesting purpose of finding high T_c , a number of metallic alloys have also been explored. In this search, the one with highest critical temperature has been Nb_3Ge with $T_c = 23.2$ K. This is already above the boiling point of liquid hydrogen, which is at 20 K under atmospheric pressure, but not enough to be of practical interest. Hence, although it has a slightly lower $T_c = 18.3$ K, Nb_3Sn is rather used industrially because it can withstand high currents and magnetic fields. Nevertheless it is NbTi that is presently the industrially preferred compound for practical reasons, although its T_c is only 10 K. It is this alloy that is mostly used for the production of the high magnetic fields required in magnetic resonance imaging (MRI) in standard medical devices. Similarly this is the alloy used in large high-energy particle accelerators, although the need for higher fields induces a switch to Nb_3Sn . In all cases, the required low temperature is obtained through liquid Helium cryogenics, which has seen much development to large scale for this purpose.

Despite many efforts, progress in increasing T_c had become so slow in the fifties and sixties that researchers in the field of superconductivity tended to believe that there was some kind of intrinsic upper bound for T_c and that in practice its increase was near saturation. Hence it has been a great shock to this community when in 1986 Georg Bednorz and Alex Müller found that a perovskite¹ in “the La-Ba-Cu-O system” (more precisely with chemical composition $\text{Ba}_x\text{La}_{1-x}\text{CuO}_{3-y}$) is a superconductor with T_c around 35 K. They received the Nobel Prize the next year for this breakthrough. Then things progressed very rapidly, essentially guided by chemical reasoning that leads one to replace an element with a chemically similar one. The following year, $T_c = 93$ K was reached in $\text{YBa}_2\text{Cu}_3\text{O}_7$. More generally, the critical temperature of $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}$ is very sensitive to the “doping” x since $\text{YBa}_2\text{Cu}_3\text{O}_6$ is an antiferromagnetic insulator. It reaches $T_c = 95$ K for $x \simeq 0.07$. Such critical temperatures have represented an essential step in the increase of T_c since they are above the boiling point of liquid nitrogen, which is at 77 K under atmospheric pressure. Liquid nitrogen is routinely obtained in the gas industry, with a typical annual world production of 8 million tons. Hence cooling such a superconductor below its critical temperature is a considerably simpler matter than when helium has to be used.

Further progress with these kinds of compounds has led to the discovery of bismuth compounds of general formula $\text{Bi}_2\text{Sr}_2\text{Ca}_n\text{Cu}_{n+1}\text{O}_{2n+6-\delta}$ with T_c ranging from 95 K to 107 K depending on n , thallium compounds $\text{Tl}_m\text{Ba}_2\text{Ca}_{n-1}\text{Cu}_n\text{O}_{2n+m+2+\delta}$ with a highest T_c reaching 127 K, and finally mercury compounds $\text{HgBa}_2\text{Ca}_2\text{Cu}_3\text{O}_{8+\delta}$ with a highest critical temperature of 135 K found in 1993. The highest critical temperature reported to date in these cuprates superconductors has been 166 K in $\text{HgBa}_2\text{Ca}_2\text{Cu}_3\text{O}_{8+\delta}$ at a pressure of 23 GPa. A remarkable feature of all these compounds is that above the critical temperature, they are fairly bad metals with poor conductivity, in contrast with the earlier superconductors discussed at the beginning of this section. They are also quasi-bidimensional materials since they are essentially stacks of CuO_2 planes, which are their conducting part, with a fairly weak electronic coupling between the planes.

¹ This is a material with the same crystal structure as CaTiO_3 .

The cuprate discovery clearly showed that there was no barrier around 20 K for T_c . Hence this produced an incentive to check the low-temperature properties of various materials. As early as 1988, superconductivity was found in $\text{Ba}_{0.6}\text{K}_{0.4}\text{BiO}_3$ around 30 K. Like the cuprates, this compound is a perovskite, but it does not contain CuO_2 planes and it is tridimensional. Hence, it is in a somewhat different class of materials.

Markedly different are the “doped” fullerenes, with the main ingredient being the fullerene molecule C_{60} , which has the shape of a soccer ball. It can be doped with various alkali, with Cs_3C_{60} reaching in 1995 a surprising $T_c = 40$ K under pressure. These materials could almost be considered as organic superconductors, with other organic compounds having much lower T_c .

A surprising result was then found in 2001, where MgB_2 was discovered to have a critical temperature of 39 K. This compound is similar to the alloys investigated earlier with the hope of finding higher critical temperature, and its late discovery looks like a miss of earlier searches.

More interesting is the discovery of superconductivity in iron-based compounds. Indeed, starting in 2006, superconductivity in these compounds was investigated because they have fairly high T_c , considering that the magnetic properties of Fe were believed to be detrimental to superconductivity. In 2008, a T_c of 26 K was reported in $\text{LaO}_{1-x}\text{F}_x\text{FeAs}$ (called an “oxypnictide”²) with $x = 0.05 - 0.12$, followed the same year by the finding of $T_c = 55$ K in $\text{SmO}_{1-x}\text{F}_x\text{FeAs}$. These iron-based materials form a rich family with several parent compounds.

Finally the quite recent last step in this progress in T_c has been the evidence for superconductivity in various hydrides around 200 K, under high pressure. A first result of $T_c = 203$ K in H_3S at 155 GPa has appeared in 2015. Then, in 2018, $T_c = 215$ K was reached in LaH_{10} (although the stoichiometry in H may be somewhat uncertain in these compounds), followed by a claim for superconductivity at $T_c = 260$ K around 200 GPa in the same compound. Clearly the search for superconductivity in these hydrides is not over. It is already quite close to the long-lasting dream of finding superconductivity at room temperature.

1.1.3 Meissner Effect

We now come to the second defining property of a superconductor. Although it has been found by Meissner and Ochsenfeld [2] in 1933, a fairly long time after the discovery by Kamerlingh Onnes, it turned out to be a fundamental feature of the superconducting state. Meissner and Ochsenfeld cooled a sample of tin in the presence of an applied magnetic field \mathbf{H} . They expected no change of the field when the temperature was going below the critical temperature T_c for tin. Instead, when measuring the field in the vicinity of the superconducting tin, they found strong changes, as if tin behaved as a magnetic material. These modifications were consistent with the magnetic induction \mathbf{B} going to zero inside the superconducting tin sample. As a result the field lines are pushed away from the superconductor, as shown in Fig. (1.2). The field is “expelled” from the superconductor.

² A pnictide element is an element belonging to the nitrogen column in the periodic table: N, P, As, Sb, Bi.

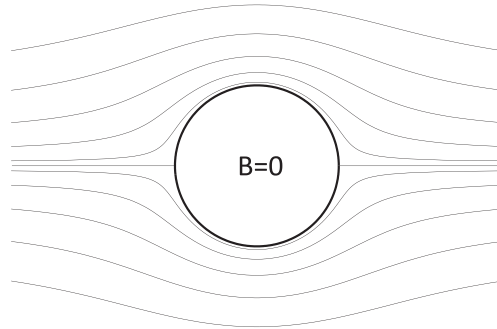


Fig. 1.2

Schematic view of the magnetic field lines for a spherical superconductor, in the presence of an applied magnetic field. Without the superconductor or for a normal metal, the magnetic field lines would just be horizontal parallel lines.

If the temperature was first lowered below the critical temperature T_c with a zero applied magnetic field (zero-field-cooling), and then the magnetic field would be raised at fixed temperature, this experimental result could easily be understood as resulting from the infinite conductivity of the superconductor. Indeed, in this case, raising the magnetic field gives rise to induced currents in the superconductor, and from Lenz's law, they oppose the variation of the induction inside the superconductor. For a standard metal, these induced currents decay by dissipation due to the metal resistivity. However, with the infinite conductivity of the superconductor there is no such dissipation; these currents run forever, and Lenz's law can reach its full effect of maintaining the magnetic induction at its initial value $\mathbf{B} = \mathbf{0}$.

This "freezing" of the induction lines is, for example, well known in plasma physics, where very large conductivity (although not infinite) can be found. Basically the infinite conductivity σ forces the electric field $\mathbf{E} = \mathbf{j}/\sigma$ to be zero inside the superconductor regardless of the current \mathbf{j} . Then for $\mathbf{E} = \mathbf{0}$, Maxwell's equation $\mathbf{curl} \mathbf{E} = -\partial\mathbf{B}/\partial t$ implies that the magnetic induction cannot change. However, in this zero-field-cooled case, we would have reached an out-of-equilibrium situation lasting forever.

By contrast, the Meissner effect cannot be explained by Lenz's law, since it is obtained by merely changing the temperature at a fixed field, so that no induced currents can arise. Rather, one comes to the conclusion that the situation depicted by Fig. (1.2) corresponds to a thermodynamical equilibrium situation for the superconductor since, for given temperature and field, it is the one that is found regardless of the order in which the temperature is lowered and the magnetic field is raised.

Nevertheless, although infinite conductivity cannot fully explain by itself the Meissner effect, it is an essential ingredient of the effect. Indeed the fact that the induction is zero inside the superconductor is physically due to the existence of permanent currents in the superconductor which screen the external field, and they can persist only because conductivity is infinite.

If one does not look for a microscopic understanding of the superconductor and stays only at a macroscopic level, one can summarize the Meissner effect by the fact that it is a magnetic material with the property $\mathbf{B} = \mathbf{0}$ in the superconductor in the presence of an

applied magnetic field \mathbf{H} . By definition of the magnetic susceptibility χ of the material, we have

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi)\mathbf{H} \quad (1.2)$$

since the magnetization \mathbf{M} is related to the field by $\mathbf{M} = \chi\mathbf{H}$. Here μ_0 is the vacuum permeability. Hence we have for a superconductor

$$\chi = -1 \quad (1.3)$$

In other words, a superconductor is perfect diamagnet. In general, diamagnets tend to weaken the external magnetic field. A superconductor does it perfectly by reducing the magnetic induction to zero.

Let us conclude by stressing that the standard procedure for identifying a metallic compound as a superconductor is to show experimentally that it has zero resistivity and displays the Meissner effect. “Zero” resistivity only may just correspond to a very good conductor with a resistivity below the experimental resolution. The Meissner effect is usually checked both when the compound is cooled in a zero field (and then a field is applied) and when it is cooled through the critical temperature in the presence of a magnetic field, to check that the field is properly expelled from the superconductor below the critical temperature. A practical complication is that the Meissner effect may not be complete: some parts of the superconductor may actually stay in the normal state due to inhomogeneities, impurities, and other kinds of defects. As a result, the measured susceptibility may not be as strong as it should be, which may bring some ambiguity in the identification of the compound as a superconductor.

1.1.4 Critical Field

The Meissner effect makes it clear physically that there must be a critical field, beyond which the superconductor no longer exists at a given temperature. Indeed there is clearly a magnetic energy cost to the distortion of the field from its value in the absence of the superconductor. This will be quantified below in the next subsection. This is compensated by the lowering in energy associated with the spontaneous transition from the normal to the superconducting state. However, this gain in energy is a fixed amount, independent of the field, whereas the magnetic energy cost increases indefinitely with the field. Obviously, if the field is too high, the total energy for going into the superconducting state will be positive and this state will no longer be stable with respect to the normal state of the metal. Hence, for a given temperature T , there is a critical field $H_c(T)$ beyond which superconductivity disappears.

The existence of this critical field was actually discovered experimentally by Kamerlingh Onnes in 1914, not much after his initial discovery of superconductivity, and much earlier than the Meissner effect. Indeed, soon after his discovery, Kamerlingh Onnes was interested in the possibility of producing high magnetic fields by electromagnets with superconducting wires, in which huge currents could in principle be fed. The existence of a critical field puts a fundamental limit to the production of such high currents, since

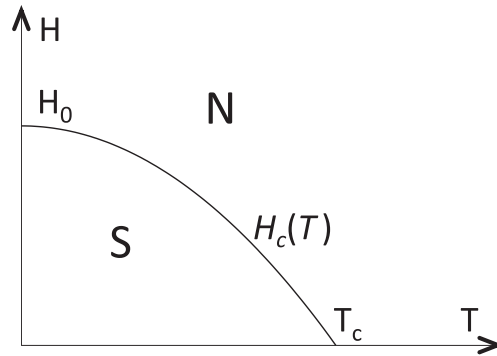


Fig. 1.3

Phase diagram of a type I superconducting metal in the presence of a magnetic field H . In the domain below the critical field curve $H_c(T)$, the metal is in the superconducting state (S), while above this curve it is in the normal state (N).

the magnetic field produced by the current itself destroys superconductivity when it goes beyond the critical field. Hence there is an upper limit for the supercurrent carried by the superconductor, which is called the critical current.

Experimentally the critical field $H_c(T)$ decreases with increasing temperature and goes smoothly to zero at the critical temperature. The resulting phase diagram is pictured in Fig. (1.3). The experimental results turn out to be very close to a parabolic law:

$$H_c(T) = H_0 \left[1 - \left(\frac{T}{T_c} \right)^2 \right] \quad (1.4)$$

Actually this simple situation holds only for one class of superconductors, called type I superconductors, which have been the first to be discovered and studied. A second class of superconductors, called type II superconductors, was discovered by Shubnikov in 1933. In these type II superconductors, instead of having a sudden transition from the superconducting state to the normal state upon increasing the magnetic field, the transition is progressive. More precisely, below a first critical field $H_{c1}(T)$ the situation is exactly the same as in type I superconductors and one has a complete exclusion of the field from the bulk of the superconductor. However, above $H_{c1}(T)$, there is a progressive admission of the field in the bulk of the superconductor, until an upper critical field $H_{c2}(T)$ is reached where superconductivity disappears completely and the metal is in the normal state. The state between $H_{c1}(T)$ and $H_{c2}(T)$ is called the “mixed state.” This more complex situation is depicted in Fig. (1.4).

Since there is a partial admission of the field in the mixed state, it is intuitively clear that the magnetic energy cost is lowered, compared to the situation of full exclusion we have seen for type I superconductors. Hence, in this case, superconductivity can survive to higher magnetic fields than for type I. Indeed type II superconductors are the only ones of interest for applications where superconductivity has to survive very high fields or very high currents.

Finally it is important to stress that if we want to directly apply the above considerations to a real superconducting sample, we have to take it with a shape infinitely elongated in the

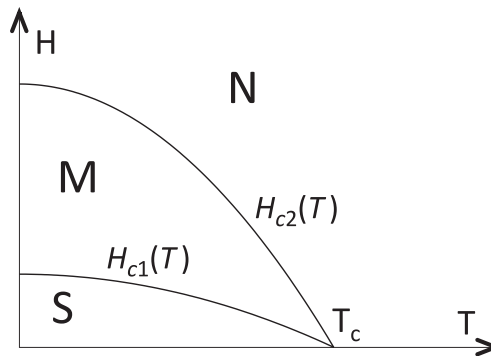


Fig. 1.4

Phase diagram of a type II superconducting metal in the presence of a magnetic field H . In the domain below the critical field curve $H_{c1}(T)$, the metal is in the superconducting state (S); between $H_{c1}(T)$ and $H_{c2}(T)$, it is in the mixed state (M); and above $H_{c2}(T)$, it is in the normal state (N).

direction of the field – for example, a very long cylinder parallel to the field. Otherwise, as for any magnetic material, we have to take into account the demagnetizing field created by the sample itself, which is naturally inhomogeneous and depends on the sample shape. As a result, the strength of the actual field depends on the spatial position. For example, in the situation depicted qualitatively in Fig. (1.2), the field at the equator of the sphere will be stronger than at the poles.

If we take the case of a type I superconductor, this may result in the fact that if the applied field is strong enough, some parts of the metal will have a field larger than the critical field and will accordingly be in the normal state, while some other parts will have a field smaller than the critical field and be superconducting. Naturally the field distribution itself depends on which parts of the sample are normal and which parts are superconducting. Hence one finds a situation where there is a mixture of normal and superconducting domains in the metallic sample. This is called the “intermediate state.” Finding the distribution of domains that minimizes the energy is quite a complicated problem, and the result may be a fairly complex structure. Correspondingly, it is also quite difficult to experimentally determine this structure with good precision.

1.1.5 Thermodynamics

The interpretation of the Meissner effect in magnetic terms, leading to the conclusion that a superconductor can be considered as a perfect diamagnet, allows one to relate in a simple way the thermodynamic properties of a superconductor and its magnetic properties. Actually this holds only for type I superconductors, to which we here restrict ourselves, the case of type II superconductors being somewhat more involved.

Thermodynamics tells us that the appropriate thermodynamical potential, to investigate the properties of a system as a function of its temperature T and its volume V , is its free energy $F(T, V) = E - TS$, rather than its energy E . Indeed the differential of the free energy, in terms of the entropy S and the pressure p of the system, is

$$dF = -SdT - p dV \quad (1.5)$$

whereas we have for the energy $dE = TdS - pdV$. Note that here, since we deal with a system with a fixed number of particles N , we do not have to include a contribution μdN , where μ is the chemical potential, in contrast with situations we will deal with later on.

When a magnetic field is present, this has to be generalized to take into account the magnetic variables. In terms of the magnetic field H and the magnetic induction B , electromagnetism tells us that the energy increase dE , due to a change dB in magnetic induction, is given by $dE = HdB$. Actually, in standard electromagnetism, one deals usually with space-dependent field $H(\mathbf{r})$ and induction $B(\mathbf{r})$, and the local change in energy $H(\mathbf{r})dB(\mathbf{r})$ has to be summed over all space to give the total energy change $\int d\mathbf{r} H(\mathbf{r})dB(\mathbf{r})$. However, we consider here a homogeneous system, so we do not need to take into account space dependence.

To take into account the change in energy due to magnetic variables, this electromagnetic contribution has to be added to the above energy variation, and similarly to the free energy variation, to give

$$dF = -SdT - pdV + HdB \quad (1.6)$$

However, by analogy with the case of a temperature T imposed by an external source, we rather want to consider the superconductor submitted to an external magnetic field H , produced, for example, by external currents in a coil. For this purpose, it is more convenient to perform a Legendre transform and consider the thermodynamical potential $G = F - HB$, which has a differential

$$dG = -SdT - pdV - BdH \quad (1.7)$$

Actually the effects associated with the volume variation of a superconductor, at the low temperature we are interested in, are in practice extremely small, and hence we will disregard them. We will assume that the superconductor has a fixed volume, which we take for convenience equal to the unit volume. Hence, we may omit the pdV term in Eq. (1.7), which reduces merely to

$$dG = -SdT - BdH \quad (1.8)$$

Consider first the compound in its normal state, for which we assume that there are no magnetic properties at all, so that $B = \mu_0 H$ where μ_0 is again the vacuum permeability. Hence $BdH = \mu_0 HdH$. Integrating Eq. (1.8) at fixed temperature with the magnetic field going from 0 to H , we obtain the normal state potential $G_n(T, H)$ in terms of $G_n(T, 0)$ as

$$G_n(T, H) - G_n(T, 0) = \int_0^H dG = -\mu_0 \int_0^H HdH = -\frac{1}{2}\mu_0 H^2 \quad (1.9)$$

In the superconducting state, we may perform the same calculation, which gets even simpler since from the Meissner effect we merely have $B = 0$ in the superconductor. This leads to

$$G_s(T, H) - G_s(T, 0) = \int_0^H dG = -\int_0^H BdH = 0 \quad (1.10)$$

Let us now, for a given temperature T , take the magnetic field equal to $H_c(T)$, which is the field corresponding to the transition line from the superconducting to the normal state. On this line, there is a thermodynamical equilibrium between the superconducting and the normal states, which implies that the corresponding potential G for these two phases are equal:

$$G_s(T, H_c(T)) = G_n(T, H_c(T)) \quad (1.11)$$

Making use of Eq. (1.9) and Eq. (1.10), this leads to

$$G_n(T, 0) - G_s(T, 0) = \frac{1}{2} \mu_0 H_c^2(T) \quad (1.12)$$

Hence we have just to measure the magnetic field corresponding to the normal-superconducting phase transition, and we know the thermodynamic potential $G_s(T, 0)$ in the zero field in the superconducting state as soon as we know the corresponding thermodynamic potential $G_n(T, 0)$ in the normal state. And from Eq. (1.9) and Eq. (1.10), we have the same information for any magnetic field.

We make use of Eq. (1.12) to investigate the order of the normal-superconducting phase transition. From Eq. (1.8), the entropy of any phase is given by $S = -\partial G/\partial T|_H$. Having the entropy of the two phases, the latent heat L is obtained from $L = T(S_n - S_s)$, to be evaluated on the transition line. From Eq. (1.9), Eq. (1.10), and Eq. (1.12), we obtain

$$\begin{aligned} S_n - S_s &= -\left. \frac{\partial(G_n(T, H) - G_s(T, H))}{\partial T} \right|_H \\ &= -\frac{\partial(G_n(T, 0) - G_s(T, 0))}{\partial T} = -\mu_0 H_c(T) \frac{dH_c(T)}{dT} \end{aligned} \quad (1.13)$$

and the latent heat is given by

$$L = -\mu_0 T H_c(T) \frac{dH_c(T)}{dT} \quad (1.14)$$

We see from Fig. (1.3) that experimentally the transition field $H_c(T)$ decreases with increasing temperature, so that $dH_c(T)/dT < 0$ and the latent heat Eq. (1.14) is positive. This means that the transition from superconducting to normal state is first order. The only exception is at the zero field, where $H_c(T) = 0$, so that $L = 0$ and the transition is second order at the standard critical temperature T_c of the superconductor.

For this second-order phase transition at zero field, we can obtain the specific heat jump $C_s - C_n$ from the experimental knowledge of $H_c(T)$. Since $C = T \partial S/\partial T$, we have at temperature T

$$C_s - C_n = T \left. \frac{d(S_s - S_n)}{dT} \right|_{T_c} = \frac{1}{2} \mu_0 T \left. \frac{d^2 H_c^2(T)}{dT^2} \right|_{T_c} \quad (1.15)$$

At $T = T_c$ where $H_c(T) = 0$, this quantity reduces to $\mu_0 T_c (dH_c(T)/dT)^2$, which is positive. But without further microscopic knowledge, there is nothing more to be said about this. However, it is interesting to try to go further by introducing some phenomenological considerations.

We first take the very good parabolic approximation for the critical field $H_c(T) = H_0[1 - (T/T_c)^2]$ and insert it in Eq. (1.15). Furthermore, we assume the temperature to

be low enough for the standard linear dependence of the normal state specific heat $C_n(T)$ to be valid. Hence $C_n(T) = \gamma T$, where γ is the Sommerfeld constant. We extrapolate this expression below T_c , corresponding to a possible metastable normal state for the metal. All this gives

$$C_s(T) = \gamma T - 2\mu_0 H_0^2 \frac{T}{T_c^2} \left[1 - 3 \left(\frac{T}{T_c} \right)^2 \right] \quad (1.16)$$

Hence the temperature dependence of $C_s(T)$ is a combination of a linear law and a cubic law. However, for most standard low-temperature superconductors, there is experimentally no hint of a linear dependence, whereas a cubic dependence describes fairly well the general behavior of $C_s(T)$ (although such a law is by no means exact, as we will later see). In order to eliminate the linear contribution, this leads us to conclude that

$$\frac{\gamma T_c^2}{\mu_0 H_0^2} = 2 \quad (1.17)$$

Here we have found quite easily a very simple relation linking a thermal energy γT_c^2 to a magnetic energy $\mu_0 H_0^2$. It will be much more difficult to obtain this dimensionless ratio from BCS theory.

Taking into account this relation, the superconducting specific heat Eq. (1.16) reads

$$C_s(T) = 3\gamma T \left(\frac{T}{T_c} \right)^2 \quad (1.18)$$

which implies $C_s(T_c) = 3\gamma T_c$, to be compared with the normal state specific heat $C_n(T_c) = \gamma T_c$ at the same temperature. This leads for the specific heat jump to

$$\frac{C_s - C_n}{C_n} \Big|_{T_c} = 2 \quad (1.19)$$

This will again have to be compared to the BCS theory result. A schematic view of the normal state and superconducting state specific heat is displayed in Fig. (1.5).

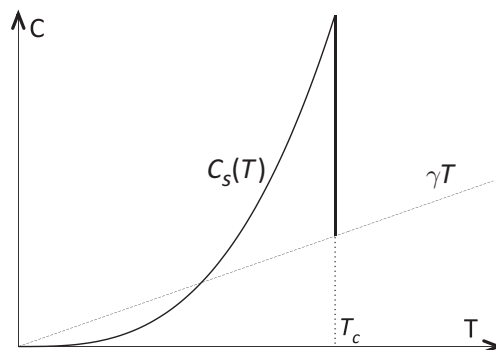


Fig. 1.5

Schematic view of the normal state $C_n(T) = \gamma T$ and superconducting state specific heat $C_s(T)$, together with the specific heat jump at T_c .

1.2 London Theory

The equation proposed by brothers Fritz and Heinz London in 1935 is the first major step in understanding superconductivity. It is in essence a phenomenological equation, without a well-structured demonstration. Hence we first present this equation, discuss it, and consider its consequences before later on coming back to the justification given by London.

The London equation aims at describing the reaction of the superconductor to the application of a magnetic field, as it occurs for example in the Meissner effect. The magnetic field $\mathbf{B}(\mathbf{r})$ is characterized by the vector potential $\mathbf{A}(\mathbf{r})$, related to the field by $\mathbf{B}(\mathbf{r}) = \text{curl } \mathbf{A}(\mathbf{r})$. The superconductor reacts by the creation of currents, characterized by the current density $\mathbf{j}_s(\mathbf{r})$. The London equation reads

$$\mathbf{j}_s(\mathbf{r}) = -\frac{n_s e^2}{m} \mathbf{A}(\mathbf{r}) \quad (1.20)$$

relating at equilibrium the current density $\mathbf{j}_s(\mathbf{r})$ to the vector potential $\mathbf{A}(\mathbf{r})$ at the same location \mathbf{r} in the superconductor. This equation may be seen as a case of “linear response”: when a physical system is subject to a small perturbation, it is usually slightly modified. The modification (the “response”) is quite often proportional to the perturbation; hence the response is linear. In the present case, the perturbation is characterized by $\mathbf{A}(\mathbf{r})$ and the superconductor response by $\mathbf{j}_s(\mathbf{r})$; hence the linear relation between these two quantities given by Eq. (1.20).

In the coefficient entering Eq. (1.20), e is the electronic charge, m is the electronic mass, and n_s is some electronic density. This expression for the coefficient can mostly be seen as resulting from dimensional considerations. This can be seen by looking at the units corresponding to the various physical quantities entering the equation. But it is easier to realize that, since $\mathbf{p} - e\mathbf{A}$ is entering the kinetic energy in the expression of the Hamiltonian of the electron in the presence of the magnetic field, the momentum $\mathbf{p} = m\mathbf{v}$, where \mathbf{v} is the electronic velocity, has the same dimension as $e\mathbf{A}$. Since we have $\mathbf{j}_s = n_s e\mathbf{v}$ for the current density, this shows that the coefficient in Eq. (1.20) has indeed the correct dimension.

On the other hand, the precise value of the coefficient is unknown since we have not specified exactly what is n_s . We expect it to be related to the electronic density n of the conduction electrons, which participate in the conduction process in the normal metal (the electrons making up the ions of the crystal lattice are expected to stay out of the superconductivity phenomenon). And indeed when we will need to evaluate n_s below, we will take $n_s = n$ as a reasonable guess. But we will see later on in the text, where we will obtain specific expressions for n_s , that things are more complicated and that most of the time this simple guess is not the appropriate one, although we expect it to give the correct order of magnitude. Finally this uncertainty on n_s makes it so that we do not need to worry about introducing the (bare) mass m of the electron in Eq. (1.20). Indeed we could wonder if some other electronic mass, related to the band structure of the metal, would not be the correct one. But we may just consider at this stage that all these uncertainties are included in n_s .

Even if we have argued that Eq. (1.20) describes a linear response, this equation is very strange since it relates a physical quantity, namely the current density $\mathbf{j}_s(\mathbf{r})$, to an unphysical one, the vector potential $\mathbf{A}(\mathbf{r})$. From gauge invariance, we could add to $\mathbf{A}(\mathbf{r})$ the gradient $\text{grad}\chi(\mathbf{r})$ of any scalar function $\chi(\mathbf{r})$ without modifying the field $\mathbf{B}(\mathbf{r})$. From Eq. (1.20) this would modify the current density without changing the field, which is physically absurd. Hence, as such, Eq. (1.20) cannot be correct. What is missing becomes clear when one realizes that the continuity equation $e \partial n(\mathbf{r})/\partial t + \text{div} \mathbf{j}_s(\mathbf{r}) = 0$ (which expresses electronic charge conservation) implies $\text{div} \mathbf{j}_s(\mathbf{r}) = 0$, since we are at equilibrium so that $\partial n(\mathbf{r})/\partial t = 0$. From Eq. (1.20) this implies $\text{div} \mathbf{A}(\mathbf{r}) = 0$. Hence the gauge is not free. In addition to Eq. (1.20), the vector potential must satisfy

$$\text{div} \mathbf{A}(\mathbf{r}) = 0 \quad (1.21)$$

The vector potential is said to be in the London gauge when this condition is satisfied. One can see that, with this condition, together with appropriate boundary conditions, the vector potential $\mathbf{A}(\mathbf{r})$ is uniquely determined from the magnetic field $\mathbf{B}(\mathbf{r})$ (if one starts with a vector potential which is not in the London gauge, the above scalar function $\chi(\mathbf{r})$ to obtain a vector potential satisfying Eq. (1.21) is uniquely determined). Hence $\mathbf{A}(\mathbf{r})$ becomes a physical quantity directly related to $\mathbf{B}(\mathbf{r})$, and the above physical problem disappears.

Another way to have an equation relating only physical quantities is to get rid of the vector potential by merely taking the curl of Eq. (1.20). This leads to

$$\text{curl} \mathbf{j}_s(\mathbf{r}) = -\frac{n_s e^2}{m} \mathbf{B}(\mathbf{r}) \quad (1.22)$$

This tells us which currents $\mathbf{j}_s(\mathbf{r})$ the field $\mathbf{B}(\mathbf{r})$ is generating in the superconductor. On the other hand, these currents will produce themselves magnetic fields, the precise relation between them being given by the Maxwell–Ampère equation

$$\text{curl} \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{j}_s(\mathbf{r}) \quad (1.23)$$

where μ_0 is the vacuum permeability (we assume that the superconducting metal is non-magnetic, as is very often the case). Taken together, Eq. (1.22) and Eq. (1.23) give us the space dependence of $\mathbf{j}_s(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. If we want to obtain an equation for $\mathbf{B}(\mathbf{r})$, we take the curl of Eq. (1.23) and make use of Eq. (1.22) for $\text{curl} \mathbf{j}_s(\mathbf{r})$. This leads us to

$$\text{curl} \text{curl} \mathbf{B}(\mathbf{r}) = \text{grad} \text{div} \mathbf{B}(\mathbf{r}) - \Delta \mathbf{B}(\mathbf{r}) = -\frac{\mu_0 n_s e^2}{m} \mathbf{B}(\mathbf{r}) \quad (1.24)$$

where the first equality is just an identity from vector calculus, valid for any vector $\mathbf{B}(\mathbf{r})$, which can be easily checked by taking the Cartesian components of both sides of the equality. Making use of Maxwell's equation $\text{div} \mathbf{B}(\mathbf{r}) = 0$, we end up with

$$\Delta \mathbf{B}(\mathbf{r}) = \frac{\mu_0 n_s e^2}{m} \mathbf{B}(\mathbf{r}) \quad (1.25)$$

The physical implication of Eq. (1.25) is most easily seen by considering a one-dimensional situation. Specifically we take the case where the superconductor occupies the whole $x > 0$ half-space while there is a vacuum in the $x < 0$ half-space, as seen in Fig. (1.6)a). Clearly in this case all physical quantities depend only on the x variable. In

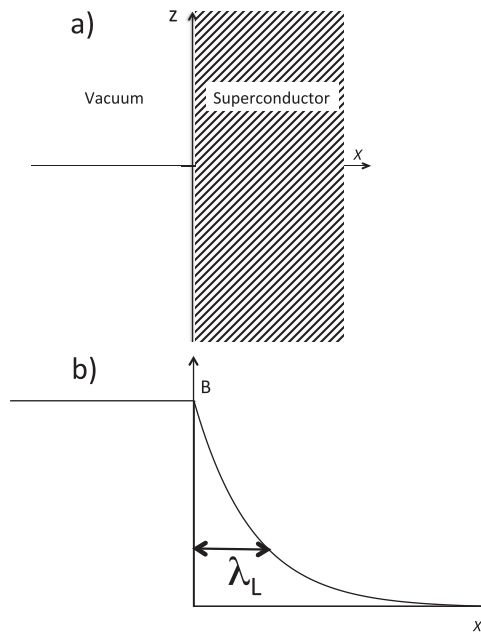


Fig. 1.6

(a) Sketch of a simple one-dimensional situation: superconductor for $x > 0$ and vacuum for $x < 0$. (b) Exponential decrease of the magnetic field inside the superconductor.

order to have the electronic charges staying in the superconductor, the currents have to flow parallel to the vacuum-superconductor interface. If the y -axis is along these currents, the magnetic field is along the z -axis from Eq. (1.22). Then Eq. (1.25) reduces to

$$\frac{d^2}{dx^2} B_z(x) = \frac{\mu_0 n_s e^2}{m} B_z(x) \quad (1.26)$$

The solution of this differential equation is a simple exponential. Comparing the two sides of the equation, it is clear that the coefficient on the right-hand side has the dimension of the inverse of a length squared, since it compares to a second derivative with respect to length on the left-hand side. Hence we introduce the London penetration depth λ_L defined as

$$\lambda_L = \left(\frac{m}{\mu_0 n_s e^2} \right)^{1/2} \quad (1.27)$$

Then the solution of Eq. (1.26) is

$$B_z(x) = B_0 \exp\left(-\frac{x}{\lambda_L}\right) \quad (1.28)$$

where B_0 is the value of the magnetic field at the superconductor-vacuum interface $x = 0$. This is depicted in Fig. (1.6b). We have discarded the exponentially growing solution of

Eq. (1.26) as unphysical since one would get a very large energy for the superconductor, due to the very large magnetic field for large x . Numerical (or analytical) solutions of the three-dimensional equation Eq. (1.25) give the same physical behavior of a magnetic field decreasing essentially exponentially over a typical length λ_L when one goes from the surface into the superconductor. Hence the magnetic field $\mathbf{B}(\mathbf{r})$ is vanishingly small inside the superconductor, which is just the Meissner effect. Therefore the London equation leads immediately to the Meissner effect, obviously a remarkable success of this theory.

Nevertheless the magnetic field penetrates the superconductor over a typical length λ_L , and it is of interest to evaluate it quantitatively. This can be done by plugging into the definition in Eq. (1.27) values of the various physical quantities typical of a standard metal. But it is more interesting to notice that it is directly related to an important physical characteristic of the metal, namely its plasma frequency³ ω_p , defined by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0} \quad (1.29)$$

where ϵ_0 is the vacuum permittivity. From standard electromagnetism $\epsilon_0\mu_0c^2 = 1$, where c is the speed of light, so from Eqs. (1.27) and (1.29) we have merely

$$\lambda_L = \frac{c}{\omega_p} \quad (1.30)$$

if we take $n_s = n$.

Let us recall that ω_p is the frequency of spontaneous small oscillations of the homogeneous electron gas in the metal, with equilibrium density n , under the restoring force due to the electric field created by the electronic charge oscillation itself. This plasma frequency is linked to the phenomenon of “ultraviolet transparency”: the metal is transparent to electromagnetic waves with a frequency beyond the plasma frequency, while electromagnetic waves with a frequency below the plasma frequency are absorbed. Standard metals are opaque to visible light; therefore the plasma frequency is located beyond this optical domain, indeed in the ultraviolet range. From Eq. (1.30) the London penetration depth is just the wavelength of an electromagnetic wave with a frequency equal to the plasma frequency. Hence we have typically $\lambda_L \sim 100$ nm in standard metals.

This typical value for λ_L is quite interesting. Indeed it is quite large compared to atomic sizes. This justifies the above large-scale treatment, where the physical quantities correspond to averages over many elementary cells of the metal. On the other hand, λ_L is quite small compared to a macroscopic scale, so for a standard size sample, the magnetic field is zero over essentially the whole sample, which corresponds to the experimental observation in the Meissner effect. Accordingly, London theory provides a fully coherent account of the Meissner effect.

Although quite small, the value of the penetration depth given by the London theory provided for experiment a definite goal that was not out of reach. Indeed experimentalists have succeeded in showing that the magnetic field indeed penetrates slightly in a superconductor. They were able to measure λ_L . The results were in rough agreement with the estimate from the London theory. However, discrepancies were found,

³ This plasma frequency is derived in Section 3.7.1.

which are in part at the origin of the generalization of London theory proposed by Pippard, the subject of the next section.

Let us finally come to the justification⁴ given by London of Eq. (1.20). London was deeply convinced that quantum mechanics is required to explain superconductivity, that superconductivity is a macroscopic manifestation of quantum mechanics. As we will see, this remarkable physical intuition has been fully vindicated by the following developments in our understanding of superconductivity. This is all the more impressive from London because at the time of the paper publication, a coherent formulation of quantum mechanics itself had been achieved only very recently, in 1927 at the Solvay conference, with the Copenhagen interpretation.

Accordingly, London envisioned a superconductor as being described by a single multi-electronic wave function. Then the current is given by its quantum mechanical expression. In the current $e\mathbf{v}$ carried by a single electron, we have to express the velocity in terms of the electron momentum \mathbf{p} by $\mathbf{v} = \mathbf{p}/m$, which becomes in the presence of a magnetic field $\mathbf{v} = (\mathbf{p} - e\mathbf{A})/m$ from classical electrodynamics. In quantum mechanics, the momentum \mathbf{p} becomes an operator, and to obtain the current, we need to calculate the following average in the electronic wavefunction, described by $|\psi\rangle$:

$$\mathbf{j}_s = \frac{e}{m} \langle \psi | \mathbf{p} - e\mathbf{A} | \psi \rangle = \frac{e}{m} \langle \psi | \mathbf{p} | \psi \rangle - \frac{e^2}{m} \langle \psi | \psi \rangle \mathbf{A} \quad (1.31)$$

where a summation over all the electrons is implicitly understood.

In the absence of a magnetic field $\mathbf{A} = \mathbf{0}$, we expect, at least for symmetry reasons, that the electronic system in its ground state has a zero average momentum $\langle \psi | \mathbf{p} | \psi \rangle = \mathbf{0}$. This leads to $\mathbf{j}_s = \mathbf{0}$, as physically expected. In the presence of a magnetic field $\mathbf{A} \neq \mathbf{0}$, London argued that the electronic ground state could display some “rigidity” so that $|\psi\rangle$ would be unchanged and one would still have $\langle \psi | \mathbf{p} | \psi \rangle = \mathbf{0}$. This sounds strange, but it is actually similar to what happens in an insulating compound: in the absence of electric field $\mathbf{E} = \mathbf{0}$, all the electronic states of the valence band are full, those of the conduction band are empty, and there is no net current. When a small electric field $\mathbf{E} \neq \mathbf{0}$ is applied, the situation is unchanged because there is a finite gap separating the top of the valence band from the bottom of the conduction band. A weak electric field is unable to transfer an electron from the valence to the conduction band, in much the same way as an isolated atom cannot be ionized by a small electric field. London thought that, in a superconductor, a gap could similarly exist in the energy spectrum, between the ground state and the lowest energy excited states, which would account for the rigidity of the wave function. Finally, for a single electron, we would have from normalization $\langle \psi | \psi \rangle = 1$. Hence, taking into

⁴ In their original work [3], F. and H. London obtained Eq. (1.20) by an argument equivalent to the following. The equation of motion for a free electron $m d\mathbf{v}/dt = e\mathbf{E}$ leads for the electronic current $\mathbf{j} = ne\mathbf{v}$ to $m d\mathbf{j}/dt = ne^2\mathbf{E}$, in the linear regime where the departure of the electronic density from its equilibrium value n is small enough to be negligible. Assuming that the electric field derives entirely from a time-dependent vector potential, $\mathbf{E} = -d\mathbf{A}/dt$ leads to $m d\mathbf{j}/dt = -ne^2 d\mathbf{A}/dt$. Integrating with respect to time leads to Eq. (1.20), assuming the integration constant to be zero. The London brothers realized that Eq. (1.20) leads to a natural explanation of the Meissner effect. On the other hand, they knew that the above argument is by no means a derivation of Eq. (1.20), since Eq. (1.20) contains only time-independent physical quantities for which all time derivatives are zero. They understood that Eq. (1.20) should be considered as a constitutive equation for a superconductor, requiring a specific justification which they sketched at the end of their paper.

account all the involved electrons in the wave function, it is natural to take $\langle \psi | \psi \rangle = n_s$ for the properly normalized wave function. All this makes it so that Eq. (1.31) indeed reduces to Eq. (1.20).

This justification is somewhat imprecise, partly because we did not want to go into too much detail. Nevertheless we will see that a number of its ingredients have been validated later on. This makes the Londons' intuition quite impressive.

1.3 Electromagnetic Response

1.3.1 Pippard Theory

We now come to an extension of the London equation, which in addition to providing a better agreement between experiment and theory, introduces important physical ideas. It was proposed by Pippard [4] in 1953 while he was experimentally studying the effect of impurities on the penetration depth. Pippard was also motivated theoretically by his knowledge of the physics of normal metals. Indeed electrons located at \mathbf{r} in a metal are influenced by electrons located nearby at \mathbf{r}' ; their physical properties are correlated. In this respect, the London equation Eq. (1.20) is somewhat peculiar, since the current at \mathbf{r} depends only on the vector potential \mathbf{A} at \mathbf{r} and not at nearby location \mathbf{r}' . In other words, it is a purely local relation.

If \mathbf{r}' is too far away from \mathbf{r} , we naturally do not expect that physical properties at \mathbf{r} are influenced by what happens at \mathbf{r}' ; there is surely some limit to the range of correlation for the physical properties. A fairly obvious origin for the limitation of this range, well known in normal metals, is the existence of imperfections, which is a departure from the perfect arrangement of ions or atoms in the ideal crystal. A frequent source of imperfection is the existence of atoms foreign to the metal, which are called impurities. But a departure from the perfect crystalline arrangement of atoms belonging to the metal has the same kind of effect. In the following, we use the term "impurity" as a generic term for any kind of imperfection.

The existence of impurities provides a clear origin to the limitation of the correlation range. A classical view is to see the electron bouncing back and forth on all the impurities around it, and as a result unable to go far away to carry information. In the corresponding quantum view, the electron in the perfect metal is a propagating wave extending throughout the crystal. Impurities in the metal scatter this wave, which introduces dephasing. The destructive interferences between all the possible paths from these scattering processes make it impossible for an electron to carry information far away with a sizeable probability. Whatever the physical picture, the net result is the existence of a mean free path ℓ for the electron, which is the typical distance on which it can propagate before having a first encounter or scattering with an impurity. Clearly this mean free path sets the order of magnitude for the electronic correlation range.

Pippard proposed to replace the local London equation Eq. (1.20) with the following non-local relation

$$\mathbf{j}_s(\mathbf{r}) = -\frac{3}{4\pi\xi_0} \frac{n_s e^2}{m} \int d\mathbf{r}' \frac{\mathbf{R}[\mathbf{R} \cdot \mathbf{A}(\mathbf{r}')]]}{R^4} e^{-R/\xi} \quad (1.32)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and ξ_0 is by definition the value of ξ in the pure superconductor. Let us consider the various ingredients in this formula. The essential physical point is that the current \mathbf{j}_s at \mathbf{r} is now determined by the value of the vector potential \mathbf{A} in a region of typical size ξ around \mathbf{r} , since R cannot be much larger than ξ due to the factor $e^{-R/\xi}$. The choice of the exponential as a cut-off function is made for simplicity, since this formula is anyway a phenomenology.

Eq. (1.32) is designed to reduce to the London equation in the “local” limit, where ξ is very small. Indeed, in this case, since \mathbf{r}' must be very close to \mathbf{r} to give a sizeable contribution, we have $\mathbf{A}(\mathbf{r}') \simeq \mathbf{A}(\mathbf{r})$. Hence $\mathbf{A}(\mathbf{r}')$ can be taken outside the integral, and we are left with the calculation of $\int d\mathbf{R} (R_i R_j / R^4) e^{-R/\xi}$, where we have changed the integration variable from \mathbf{r}' to \mathbf{R} , and we have introduced the Cartesian component R_i of \mathbf{R} . For symmetry reasons, this integral is zero when $i \neq j$, and when $i = j$ the result is the same for $i = x, y$, and z . Adding these three results gives $\int d\mathbf{R} (R_x^2 + R_y^2 + R_z^2) e^{-R/\xi} / R^4 = 4\pi \int_0^\infty dR e^{-R/\xi} = 4\pi\xi$. So $\int d\mathbf{R} (R_i R_j / R^4) e^{-R/\xi} = 4\pi \delta_{ij} \xi / 3$, where δ_{ij} is the Kronecker delta and Eq. (1.32) becomes

$$\mathbf{j}_s(\mathbf{r}) = -\frac{\xi}{\xi_0} \frac{n_s e^2}{m} \mathbf{A}(\mathbf{r}) \quad (1.33)$$

Hence, in the case of a pure superconductor where $\xi = \xi_0$, Eq. (1.32) reduces to the London equation, Eq. (1.20). Note that a simpler kernel in the integral Eq. (1.32), such as $e^{-R/\xi} / R^2$, could lead to the same result. Pippard made his specific choice Eq. (1.32) by analogy with the anomalous skin effect in normal metals.

In the presence of impurities, one expects the range ξ to be smaller since, as we have seen, impurities reduce the correlation range. When the mean free path ℓ is very short, a reasonable assumption is that the correlation range ξ cannot extend much beyond ℓ , so $\xi \simeq \ell$. To evaluate ξ between the two limits of very pure ($\ell \rightarrow \infty$) and very dirty ($\ell \ll \xi_0$) superconductor, Pippard proposed the simple interpolation formula

$$\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{\ell} \quad (1.34)$$

which indeed gives the expected result in the two above limits, and that agreed reasonably well with his experimental results. Hence Eq. (1.32) together with Eq. (1.34) accounted for the increase in penetration depth Pippard observed when the impurity content was increased.

A very important physical point in the Pippard theory is the introduction of the length ξ , which characterizes the non-locality of the superconductor response and is called “coherence length” in superconductivity. However, this concept is the same as the “correlation length,” which comes in for phase transitions. Pippard understood that ξ_0 has to be related to a characteristic length emerging from the microscopic theory of superconductivity, but it will be easier to be more specific when we will deal with BCS theory.

The Pippard formula has the form of a convolution product, namely the three-dimensional equivalent of a one-dimensional integral expression $h(y) = \int dx f(y-x)g(x)$.

It is well known that if one goes to the Fourier transforms of the various functions, $f_F(q) = \int dx e^{-iqx} f(x)$ with $f(x) = (1/2\pi) \int dq e^{iqx} f_F(q)$, and so on, the Fourier transform of $h(y)$ is merely the simple product of the Fourier transforms of $f(x)$ and $g(x)$: $h_F(q) = f_F(q)g_F(q)$. The same result holds for three-dimensional integrals. We introduce for example the Fourier transform of the vector potential $\mathbf{A}(\mathbf{q}) = \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \mathbf{A}(\mathbf{r})$. Here, for simplicity, we drop the index F and do not write explicitly $\mathbf{A}_F(\mathbf{q})$ for the Fourier transform, relying on the standard variable name \mathbf{q} to make it clear that we deal with Fourier transforms.

In this way, introducing the Cartesian components, the Fourier transform of Eq. (1.32) can be written as

$$\mathbf{j}_{s,i}(\mathbf{q}) = -K_{ij}(\mathbf{q})\mathbf{A}_j(\mathbf{q}) \quad (1.35)$$

where a summation over j is understood and

$$K_{ij}(\mathbf{q}) = \frac{3}{4\pi\xi_0} \frac{n_s e^2}{m} \int d\mathbf{R} \frac{R_i R_j}{R^4} e^{-i\mathbf{q}\cdot\mathbf{R}} e^{-R/\xi} \quad (1.36)$$

Here the direction of \mathbf{q} provides a specific direction, and we take the z -axis along this direction. Just as above, for symmetry reasons, $K_{ij}(\mathbf{q})$ is zero when $i \neq j$. When $i = j$ the result is the same for $i = x, y$, and we call it $K_{\perp}(q)$, with $q = |\mathbf{q}|$, but it is different from the result for $i = z$, which we call $K_{\parallel}(q)$. On the other hand, just as for the London equation, we have to satisfy the continuity equation $\text{div} \mathbf{j}_s(\mathbf{r}) = 0$, which implies for the Fourier transform $\mathbf{q} \cdot \mathbf{j}_s(\mathbf{q}) = 0$. From Eq. (1.35) and the above symmetries, this implies $\mathbf{q} \cdot \mathbf{A}(\mathbf{q}) = 0$. This is the London gauge condition, which must be satisfied by the Fourier transform $\mathbf{A}(\mathbf{q})$ of the vector potential for Eq. (1.35) to be valid. But this condition implies that we do not need to know $K_{\parallel}(q)$; it is irrelevant for our purpose.

The expression for $K_{\perp}(q)$ is not difficult to obtain from Eq. (1.36). We just give the result without the explicit calculation, in order to discuss the physical implications,

$$\mu_0 K_{\perp}(q) = \frac{1}{\lambda_L^2} \frac{\xi}{\xi_0} \frac{3}{2(q\xi)^2} \left[\left(1 + (q\xi)^2 \right) \frac{\arctan(q\xi)}{q\xi} - 1 \right] \quad (1.37)$$

where the London penetration depth λ_L is given by Eq. (1.27). One checks that for $q \rightarrow 0$, $\mu_0 K_{\perp}(q)$ goes to $(1/\lambda_L^2)(\xi/\xi_0)$, which is the London result (for $\xi = \xi_0$). This is actually obvious from Eq. (1.36). In the limit of large q , $K_{\perp}(q)$ decreases as $1/q$. This makes sense physically, since one expects the superconductor response to be weaker for a rapidly oscillating magnetic field than for a constant one. This is also clear directly from Eq. (1.36), where for large q the exponential $e^{-i\mathbf{q}\cdot\mathbf{R}}$ is rapidly oscillating as a function of R , which makes the overall integral small by cancellation of positive and negative contributions.

Hence Pippard theory explicitly displays the physically reasonable fact that the superconductor response depends on the wavevector q , in contrast with London theory, which provides a response independent of the wavevector. The characteristic length for this dependence is the coherence length ξ . This length comes in addition to the London penetration depth λ_L , which is a typical length scale for the variation of the vector potential and the supercurrents in the superconductor.

However, if this penetration depth is much larger than the coherence length, this means that the vector potential is essentially constant on the scale of the coherence length. In this

case, as we have seen, the Pippard equation, Eq. (1.32), reduces to the London equation, Eq. (1.20). This physical situation turns out, as we will see later, to be the one found in type II superconductors, as introduced in Section 1.1.4 (they are sometimes called London superconductors).

On the other hand, in the opposite physical situation, where the coherence length is much larger than the penetration depth, we naturally need to apply the Pippard equation. This physical situation is the one found in type I superconductors (sometimes called Pippard superconductors). The transition from type I to type II superconductors will be discussed in more detail within the discussion of the Ginzburg–Landau theory in Chapter 6.

We finally note that for type I superconductors, the calculation of the actual penetration depth is more subtle since we have to use a complicated response function like Eq. (1.37) instead of a simple constant, and the simple exponential decay found in Eq. (1.28) is no longer valid. This point is discussed more explicitly in the Further Reading section at the end of this chapter.

1.3.2 Response Function, the Link Between Infinite Conductivity and the Meissner Effect

In the preceding subsection we have seen, as an outcome of the Pippard theory, that the response function $K(q)$ has a natural dependence on the wavevector q (from now on, we drop the subscript \perp , which is unimportant once we know that $\mathbf{j}_s(\mathbf{q})$ and $\mathbf{A}(\mathbf{q})$ must be orthogonal to \mathbf{q}). We have considered only time-independent situations corresponding physically to the superconductor in its ground state, in the presence of a static magnetic field. However, we may generalize the concept of response function to the case where the potential vector is time-dependent. Going again to the Fourier transform, we consider a vector potential $\mathbf{A}(\mathbf{q}, \omega)$ having a given wave vector \mathbf{q} and frequency ω . Generalizing Eq. (1.35), the supercurrent \mathbf{j}_s produced by the superconductor is

$$\mathbf{j}_s(\mathbf{q}, \omega) = -K(\mathbf{q}, \omega)\mathbf{A}(\mathbf{q}, \omega) \quad (1.38)$$

which serves as a definition for the response function $K(\mathbf{q}, \omega)$. This response function provides a full knowledge of the reaction of a superconductor to an electromagnetic perturbation. In particular, we may use it to specify when the response corresponds to an infinite conductivity or to a Meissner effect.

Indeed the conductivity characterizes the response to an electric field, but we can produce a time-dependent electric field $\mathbf{E}(t) = -\partial\mathbf{A}(t)/\partial t$ by applying a time-dependent vector potential $\mathbf{A}(t)$. For a vector potential $\mathbf{A}(t) = e^{-i\omega t}\mathbf{A}(\omega)$ with given frequency ω , we will have a corresponding electric field $\mathbf{E}(t) = e^{-i\omega t}\mathbf{E}(\omega)$, given by $\mathbf{E}(\omega) = i\omega\mathbf{A}(\omega)$. Here again $\mathbf{E}(\omega)$ and $\mathbf{A}(\omega)$ are shorthands for $\mathbf{E}_F(\omega)$ and $\mathbf{A}_F(\omega)$, and so on. On the other hand, for the conductivity, we have in mind the response to a uniform electric field with no space variation. In terms of the Fourier transform, this corresponds to the case where the wave vector is zero $\mathbf{q} = \mathbf{0}$. From the response, Eq. (1.38), $\mathbf{j}_s(\mathbf{0}, \omega) = -K(\mathbf{0}, \omega)\mathbf{A}(\mathbf{0}, \omega)$ and the definition $\mathbf{j}_s(\omega) = \sigma(\omega)\mathbf{E}(\omega)$ of the conductivity at frequency ω , we obtain

$$\sigma(\omega) = i \frac{K(\mathbf{0}, \omega)}{\omega} \quad (1.39)$$

Finally when we say that the conductivity is infinite, this is only under static conditions, which means in the limit of zero frequency. Hence we have infinite conductivity provided $K(\mathbf{0}, \omega)/\omega$ goes to infinity when the frequency goes to zero. To obtain this result, it is enough that $K(\mathbf{0}, \omega)$ has a nonzero limit when ω goes to zero. This is what will happen in regular cases. Hence, barring more exotic situations, we have infinite conductivity provided the response function satisfies

$$\lim_{\omega \rightarrow 0} K(\mathbf{0}, \omega) \equiv K_0 \neq 0 \quad (1.40)$$

On the other hand, we have seen that the Meissner effect occurs under static conditions, which means at zero frequency $\omega = 0$. When London theory applies, we have a simple exponential decay of the magnetic field upon penetration in the superconductor. However, the decay is not that simple when we have to use Pippard theory, but in order to obtain the physics of the Meissner effect, we merely want a rapid decay (in practice exponential) of the field when, starting from the surface, we go deep enough in the superconductor. When we go in Fourier transform, a large distance from the surface translates into a small wave vector. Hence only in the limit $q \rightarrow 0$ do we want to recover the behavior resulting from the London theory (this is discussed in more detail in the “Further Reading” section). This is obtained as soon as the response is nonzero, just as in the London theory. Accordingly, we obtain the Meissner effect when

$$\lim_{q \rightarrow 0} K(\mathbf{q}, 0) \equiv K'_0 \neq 0 \quad (1.41)$$

Comparing conditions Eq. (1.40) and Eq. (1.41), one sees that both involve $\lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} K(\mathbf{q}, \omega)$. The only difference is that the respective orders of the two limits, $\lim_{q \rightarrow 0}$ and $\lim_{\omega \rightarrow 0}$, are exchanged when we go from one condition to the other. It is very tempting to believe that there is no singularity at $q = 0$ and $\omega = 0$, that the order of these two limits is unimportant, and that $K_0 = K'_0$, which would imply that infinite conductivity and the Meissner effect are automatically linked. However, this is incorrect, as it is easily recognized by considering the case of free electrons: they display infinite conductivity but no Meissner effect, as is well known. We will see this explicitly in Chapter 4. Hence there is in general a singularity at $q = 0$ and $\omega = 0$ for the response function, so infinite conductivity and the Meissner effect are not linked in principle. However, we will see that this singularity is removed in superconductivity, in the framework of BCS theory.

1.4 Physical Ideas, Microscopic Origin of Superconductivity

This last section is an introduction to the various physical ideas leading to the microscopic theory of superconductivity, which will be considered in the following chapters. It is often said that the lapse of time between the discovery of superconductivity in 1911 and the publication of the microscopic theory of Bardeen, Cooper, and Schrieffer (BCS) in 1957 has been quite long. Considering the difficulty of the matter, together with the ten years of world wars that to some extent kept people away from basic research, one could easily argue contrarily that the pace of progress has been remarkable. For example, the matter of

high T_c superconductivity in cuprates is still quite controversial, even though the discovery goes back to 1986.

One main difficulty in elaborating a microscopic theory has been the proper treatment of electronic interactions in metals. Indeed superconductivity is clearly the result of inter-electronic interactions that cooperate to produce a state in which dissipation has disappeared. However, it is a difficult matter to properly handle the strong Coulomb interaction between electrons in metals. The elaborations of the proper theoretical frameworks, including the random phase approximation (RPA) and the Landau Fermi liquid theory [5, 6, 7], appeared essentially at the same time as the BCS theory.

A main ingredient in the progress of ideas has been the strong similarity between superconductivity and the superfluidity of liquid ^4He , discovered in 1937 by Kapitza [8] and by Allen and Misener [9]. At atmospheric pressure, ^4He goes from a gas to a liquid phase at 4.2 K. A remarkable property of this liquid is that for low enough pressure, it is still present down to the lowest temperature, that is $T = 0$, instead of going to a solid phase, which happens for all other known liquids. This is due to the very light mass of the helium atom and the very weak interactions between helium atoms (except for the hard core repulsion, which comes in at a short distance). This liquid ^4He behaves as an ordinary “normal” liquid until, when temperature is lowered, its viscosity suddenly vanishes below 2.17 K.

Viscosity is a universal property of liquids corresponding physically to the transfer of momentum from some part of the liquid to the neighboring ones. This transfer occurs as the result of collisions between atoms or molecules making up the liquid. Qualitatively it describes the rubbing of different parts of the liquid against each other. As a result of viscosity, different parts of a liquid tend to flow at the same velocity. Similarly scattering of the particles making up the liquid against the wall of the container makes the liquid have a vanishing velocity near an immobile wall. In particular, in capillaries, a liquid has a hard time moving at all since it stands still against the wall, and viscosity makes its velocity to be barely different from zero in the middle of the capillary. For example, we are used to the fact that, in ordinary life, water does not flow in very small apertures. As a result, a very small hole in a pipe does not give rise to a leak. This is the physical principle behind standard waterproof plumbing.

In contrast, quite the opposite occurs with liquid ^4He when, by going below 2.17 K, it loses its viscosity and becomes superfluid. As a result, superfluid ^4He can run in extremely small pores and capillaries. This is immediately noticed by an experimentalist who sees many leaks appearing in the vessel containing liquid ^4He when it goes below 2.17 K, even though it was a perfectly fine tight vessel above 2.17 K. Similarly superfluid ^4He can run in the extremely thin ^4He films covering the walls of a vessel and leak out efficiently in this way. There are many other spectacular experimental manifestations⁵ of the superfluidity of ^4He .

A simple qualitative way to characterize this superfluid behavior is to say that ^4He behaves as a perfect fluid. Its flow is fully coherent, with all the atoms moving in accordance with its single velocity field and no apparent role for inter-atomic collisions. This is to be contrasted with the behavior of a standard fluid, where collisions occur all over the

⁵ Many videos of such experiments can be found by searching the Internet.

fluid, which results in its viscous properties, with no relation between the atomic collisions occurring in various places of the fluid. These processes are not directly related, and they are fully incoherent.

This contrast between the coherent behavior of the superfluid and the incoherence of the standard fluid was rapidly ascribed to a direct manifestation of quantum mechanics [10]. Naturally such an interpretation of superfluidity was by no means obvious, and there were competing proposals, but it is crucial that it turned out to be basically correct. The link between coherence and quantum mechanics is clear if we consider the naive picture of the electron orbiting around the proton in the ground state of the hydrogen atom. The “motion” of the electron is described by its wave function, which is a coherent field, and there is naturally no dissipation in this motion, which is everlasting. If we could pile up a large number of electrons in the state described by this wave function, we would have something very much like a superfluid current of electrons. Naturally such a simple picture does not work, first of all because electrons are fermions and it is forbidden to put more than a single fermion in a specific state.

However, ^4He atoms are bosons, and for particles following Bose statistics [42], there is no such exclusion principle, so in principle one can put any number of bosons in a specific state. Actually, not so long before the discovery of superfluidity in ^4He , Einstein [11] had studied the peculiar condensation of a gas of non-interacting particles following this bosonic statistics. In particular, in the ground state of this system obtained at zero temperature, all the particles are indeed in the same one-particle state, corresponding to the lowest kinetic energy. However, the link with experimental reality was quite unclear. London went on to suggest [10], with caution due to the interactions between ^4He atoms, that the transition of liquid ^4He from its normal to its superfluid phase could be related to this Bose–Einstein condensation. Nevertheless ^4He atoms have a very strong hard-core repulsion, which makes liquid ^4He quite different from an ideal non-interacting gas. Hence the link with the Bose–Einstein condensation was rejected by some physicists. Landau [12], for example, went on to build a theory for the hydrodynamics of superfluid ^4He which, while relying directly on quantum mechanics, had nothing to do with Bose–Einstein condensation. We will return to Landau’s theory later on in Chapter 7.

Unexpectedly, a side effect of the Second World War brought a clear answer to the question of the link between superfluidity and Bose–Einstein statistics. Indeed, as a by-product of the nuclear activities started at that time, a sizeable quantity of the isotope ^3He became available (it comes from tritium disintegration), so that low-temperature liquid ^3He could be produced and its physical properties explored. This showed unambiguously [13] that in the temperature range where liquid ^4He becomes superfluid, nothing of this sort occurs in liquid ^3He . On the other hand, the only difference between the two isotopes is that a neutron is missing in the ^3He nucleus with respect to the ^4He nucleus. This means that ^3He atoms, being made of an odd number of fermions (three nucleons and two electrons), are themselves fermions while ^4He atoms, having an additional neutron, are bosons (as already emphasized). Otherwise, just as for any isotopes, ^3He and ^4He have the same electronic structure and accordingly the same chemical and interaction properties.

Hence it is natural to ascribe the marked difference in physical properties between the two liquids to the only qualitative difference⁶ between them, namely the different statistics they obey. This is a very strong argument in favor of a link between superfluidity in ^4He and Bose-Einstein statistics, and hence a possible relation with Bose-Einstein condensation. However, this did not improve the microscopic understanding of superconductivity. Indeed, although superconductivity is very much like electronic superfluidity in metals, one cannot merely argue that electrons behave as ^4He atoms in superfluid ^4He , since electrons are fermions and cannot undergo anything like Bose-Einstein condensation.

Another very important step was made about the same time, pointing toward an essential role played by lattice vibrations in the occurrence of superconductivity. On one hand, Fröhlich [14] pointed out that an indirect attractive interaction between two electrons could result from the exchange of lattice vibrations. This is completely analogous to the phenomena explored at the same period of time in quantum electrodynamics, where a proper physical description of the interaction between two charges has to include electromagnetic fluctuations, namely the possibility of emission or absorption of (virtual) photons. Such processes give rise, for example, to the Lamb shift in atomic physics. The quanta of the electromagnetic field are photons. Similarly the quanta of the field of lattice displacements are phonons, and the indirect interaction pointed out by Fröhlich comes from the exchange of phonons between the two electrons.

On the other hand, almost simultaneously, experiments [15] with the various isotopes of mercury available at the time showed that the critical temperature had a clear dependence on the isotope mass, with T_c increasing with decreasing mass. This definitively proved that lattice vibrations played a role in the appearance of superconductivity. Indeed, if the mercury nuclei are assumed to be fixed, the electronic properties of the various isotopes are identical, as we have already stressed above in the case of helium. Accordingly, the superconducting properties of these isotopes should also be identical. A dependence on the nucleus mass can only appear because the motion⁷ of the nucleus comes into play.

The idea of an attractive interaction between electrons drew attention to the possible existence of a bound state of two electrons [16]. Such a pair of electrons, made of two fermions, would be a boson. This would make possible a Bose-Einstein condensation of these charged bosonic pairs. But it is not easy to obtain a quantitative microscopic description from such ideas, and it is hard to reconcile them with the existence of the electronic Fermi sea in the normal metal. Since a quantitative evaluation of the thermodynamics shows that the energy of the superconducting state is not very different from that of the normal state, the superconducting ground state cannot be very different from the normal-state Fermi sea.

The decisive breakthrough occurred in 1956, when Cooper [17] found that, in the presence of the electronic Fermi sea, an attractive interaction between two electrons leads to a bound state, however weak the attraction between the two electrons. This made

⁶ Actually, ^3He is also markedly lighter than ^4He , but it is hard to ascribe a qualitative difference such as the absence of superfluidity in ^3He to the mere quantitative mass difference.

⁷ Actually there is, in principle, the effect of zero point motion of the nucleus on the electronic properties, but for such a heavy element as mercury, this is an exceedingly small effect.

electronic pairing compatible with the presence of other electrons, obeying Fermi-Dirac statistics. The next year, in 1957, the fundamental paper [18] of BCS theory by Bardeen, Cooper, and Schrieffer⁸ extended this idea of electronic pairing to all the electrons of the metal. As we will see, the resulting physical picture is quite strange and very far from a gas of charged bosons, since there is actually a huge overlap between all the electronic pairs.

1.5 Further Reading: Magnetic Field Penetration in the General Case

Here we take over the simple one-dimensional situation considered in Section 1.2 and extend the treatment to the case of a general response function $K(q)$, instead of a constant one. However, handling exactly this situation is not very convenient because the result for the field is asymmetrical, as displayed in Fig. (1.6) b). Actually we are only interested in the field distribution in the superconductor – that is, for $x > 0$. Hence it is more convenient to have a (anti)symmetrical situation where on the $x < 0$ side we also have a superconductor, as displayed in Fig. (1.7)a), with a field distribution $\mathbf{B}(-x) = -\mathbf{B}(x)$ antisymmetrical of the one for $x > 0$, as seen in Fig. (1.7)b). This can be produced by an external current sheet at $x = 0$. Hence we consider a space filled by a superconductor in the presence of uniform current sheet $\mathbf{j}_e(\mathbf{r}) = \mathbf{J}_e\delta(x)$, where for example \mathbf{J}_e is along the y -axis and $\delta(x)$ is the Dirac function.

In the absence of a superconductor, the current sheet creates a uniform magnetic field, with opposite values for $x > 0$ and $x < 0$. For the present geometry with $\mathbf{B}(x) = \text{curl } \mathbf{A}(x)$ along z , and $\mathbf{A}(x)$ together with $\mathbf{j}(x)$ along y , the Maxwell–Ampère equation

$$\text{curl } \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{j}(\mathbf{r}) \quad (1.42)$$

gives

$$-\frac{d^2}{dx^2} A_y(x) = \mu_0 j_y(x) \quad (1.43)$$

where the total current $\mathbf{j}(x)$ contains both the current sheet and the supercurrent $\mathbf{j}(x) = \mathbf{j}_e(x) + \mathbf{j}_s(x)$. Going to the Fourier transform with respect to the variable x , Eq. (1.43) gives for the Fourier transform $A(q)$ of $A_y(x)$

$$q^2 A(q) = \mu_0 (J_e + j_s(q)) \quad (1.44)$$

where the Fourier transform $j_e(q) = J_e$ of $j_{e,y}(x) = J_e\delta(x)$ is independent of q . Now the supercurrent is given by Eq. (1.35), $j_s(q) = -K(q)A(q)$, in terms of the vector potential, and Eq. (1.44) leads to the explicit solution:

$$A(q) = \mu_0 \frac{J_e}{q^2 + \mu_0 K(q)} \quad (1.45)$$

⁸ Bardeen, Cooper, and Schrieffer received the 1972 Nobel Prize for their work. This reward came fairly late because Bardeen had already received the Nobel Prize in Physics in 1956 for his work on semiconductors, and it was against the principles of the Nobel Committee to award the same prize to the same recipient twice in a given field.

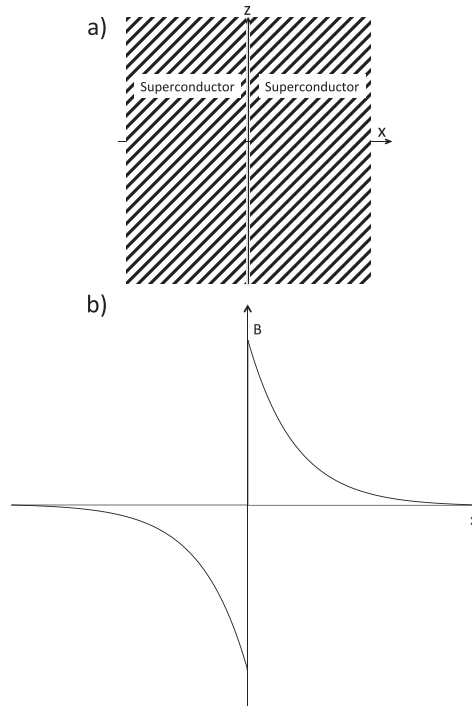


Fig. 1.7

- (a) Sketch of a simple symmetrical one-dimensional situation: superconductor for $x > 0$ and for $x < 0$.
 (b) Antisymmetrical behavior of the magnetic field in the superconductor.

Performing the inverse Fourier transform and going back to the magnetic field yields

$$B_z(x) = \frac{\mu_0 J_e}{2\pi} \int_{-\infty}^{\infty} dq \frac{iq}{q^2 + \mu_0 K(q)} e^{iqx} = -\frac{\mu_0 J_e}{2\pi} \int_{-\infty}^{\infty} dq \frac{q \sin(qx)}{q^2 + \mu_0 K(q)} \quad (1.46)$$

where the imaginary part of $B_z(x)$ is zero because it is the integral of an odd function of q .

In the absence of superconductor $K(q) = 0$, and since $\int_{-\infty}^{\infty} du \sin(u)/u = \pi$, Eq. (1.46) gives $B_z(x) = -(\mu_0 J_e/2) \operatorname{sgn}(x)$, which takes indeed constant opposite values for $x > 0$ and $x < 0$.

For a London superconductor where $K(q) = K_0 = n_s e^2/m = 1/(\mu_0 \lambda_L^2)$ is constant, the result is conveniently obtained from the first expression in Eq. (1.46) by residue integration, as shown in Fig. (1.8). For $x > 0$, the integration contour from $-\infty$ to $+\infty$ can be closed by the half-circle at infinity in the upper complex plane of the q variable, which brings no contribution since e^{iqx} is zero, for $x > 0$, when $\operatorname{Im} q$ is positive and goes to $+\infty$. The only singularity of the integrand in the upper complex plane is a pole at $q = i\sqrt{\mu_0 K_0} = i/\lambda_L \equiv iq_0$; the corresponding residue is $ie^{-q_0 x}/2$. The integral is equal to $2i\pi$ times this residue, which gives for the field $B_z(x) = -(\mu_0 J_e/2)e^{-x/\lambda_L}$. This is the expected result identical to Eq. (1.28).

In the general case, we may proceed in the same way, adding the half-circle at infinity in the upper complex plane to the integration contour and then trying to deform continuously

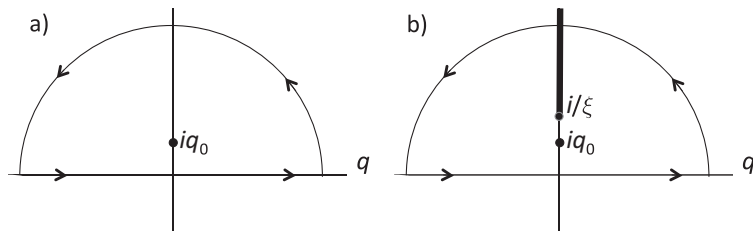


Fig. 1.8

(a) Sketch of the contour for the q integration, with a simple pole in the upper complex plane, corresponding to the case where $K(q)$ is a constant. (b) Sketch of the contour for the q integration, with a simple pole and a cut in the upper complex plane, corresponding to the case where $K(q)$ is given by Eq. (1.37) (the figure assumes $\lambda_L > \xi$).

the resulting closed contour. However, we no longer know the singularities of the integrand in the upper complex plane. In the case where $K(q = 0) \neq 0$, the integrand has a regular expansion in powers of q at $q = 0$. This is in contrast with the case $K(q = 0) = 0$, where there is singularity at $q = 0$. Hence (barring some physically unexpected singularity coming from $K(q)$ itself) the integrand is regular on the real q -axis. So we can shift, parallel to itself, this part of the integration contour toward the upper complex plane, until it meets the singularity nearest to the real axis. If iq_0 is the imaginary part of this singular point, we can make the change of variable $q = iq_0 + q'$. In this way, a factor $e^{-q_0 x}$ comes out of the integral in Eq. (1.46) and the field decreases at least as fast as $e^{-q_0 x}$. This means that there is a Meissner effect. From this kind of procedure, we see that when x gets very large, the decrease of the magnetic is fully controlled by the singularity that is nearest from the real q -axis, with the contributions from singularities located farther in the complex being exponentially smaller.

Clearly the actual calculation of the field in the superconductor is more complicated when $K(q)$ is not constant, and generally it has to be done numerically. If we take the specific case of Eq. (1.37) (with $\xi = \xi_0$ to be definite), there is, as singularities in the upper complex plane, a branch cut starting from $q = i/\xi$ due to the $\arctan(q\xi)$. Additionally there is a pole located at iq_0 with $q_0^2 = \mu_0 K(iq_0)$. When $\lambda_L \gg \xi$, $K(q)$ is small and the nearest singularity from the real axis is the pole at iq_0 . Since $q_0 \xi \ll 1$, $K(iq_0)$ can be approximated by its London limit $K(0)$. Hence the pole is merely given by $q_0 = 1/\lambda_L$. We recover, as expected, a simple exponential decay with λ_L as the penetration depth. On the other hand, when $\lambda_L \ll \xi$, the dominant singularity nearest to the real axis is the branch point at i/ξ . The decrease of the field is now more complicated, but it has a dominant $e^{-x/\xi}$ dependence, corresponding to an actual penetration depth ξ . Between these two limits, the actual penetration depth will go from λ_L to ξ . Hence we see that generally the actual penetration depth is a complicated function of λ_L and ξ .