BULL. AUSTRAL. MATH. SOC. 35J70, 35B65
VOL. 62 (2000) [369–377]

HÖLDER CONTINUITY OF SOLUTIONS OF SOME DEGENERATE ELLIPTIC DIFFERENTIAL EQUATIONS

AHMED MOHAMMED

Weak solutions of the degenerate elliptic differential equation $Lu := -\operatorname{div}\big(A(x)\nabla u\big) + \mathbf{b}\cdot\nabla u + Vu = f$, with $|\mathbf{b}|^2\omega^{-1}$, V, f in some appropriate function spaces, will be shown to be Hölder continuous.

1. INTRODUCTION

The purpose of this note is to study Hölder regularity of solutions of the equation $Lu = f$, where L is the degenerate elliptic operator

$$Lu := -\operatorname{div}\big(A(x)\nabla u\big) + \mathbf{b}(x)\cdot\nabla u + Vu\,, \qquad x \in \Omega.$$

The entries $a_{ij}(x)$ of the $n \times n$ coefficient matrix $A(x) := \big(a_{ij}(x)\big)$ are real-valued measurable functions on a bounded domain $\Omega \subseteq \mathbb{R}^n$ and $A(x)$ is a symmetric matrix that satisfies, for some $\lambda > 1$,

$$\frac{1}{\lambda}\omega(x)|\xi|^2 \leqslant \langle A(x)\xi, \xi\rangle \leqslant \lambda\omega(x)|\xi|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. Here $\langle , \rangle$ indicates the Euclidean inner product. The weight $\omega(x)$ will be a non-negative measurable function on $\mathbb{R}^n$ that satisfies either of the conditions (1.1) or (1.2) below.

$$\omega \in A_2, \text{ that is } \quad \sup_B\left(\fint_B \omega(x)\,dx\right)\left(\fint_B \frac{1}{\omega(x)}\,dx\right) = C_\omega < \infty, \tag{1.1}$$

where we have used the notation (with $d\mu := dx$)

$$\fint_B f(x)\,d\mu, \text{ for the integral average } \frac{1}{\mu(B)}\int_B f(x)\,d\mu \text{ of } f \text{ over } B.$$

The supremum in (1.1) is taken over all balls B in $\mathbb{R}^n$. The constant C_ω in (1.1) will be referred to as the A_2 constant of ω.

(1.2) $\omega(x) := \big|\mathbf{f}'(x)\big|^{1-2/n}$, where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping and $\big|\mathbf{f}'(x)\big|$ is the absolute value of the Jacobian determinant of $\mathbf{f}$.

Received 16th December, 1999

$\mathbf{f} = (f_1, \dots, f_n)$ is quasiconformal if $\mathbf{f}$ is one-to-one, $f_j \in H^{1,n}_{loc}(\mathbb{R}^n)$ for $j = 1, 2, \dots, n$ and there is a constant C_ω, called the dilation constant of $\mathbf{f}$, such that

$$\left(\sum_{i,j=1}^{n} (\partial f_i/\partial x_j)^2 (x)\right)^{1/2} \leqslant C_\omega \left|\mathbf{f}'(x)\right|^{1/n}, \quad \text{almost everywhere on } \mathbb{R}^n.$$

It is known that weights that satisfy (1.1) or (1.2) have the following doubling property:

$$\omega(2B) \leqslant C\omega(B), \tag{1.3}$$

for all balls $B \subseteq \mathbb{R}^n$ and some constant C independent of B. In (1.3) and in the sequel we use the notation $f(B)$ for the Lebesgue integral of f over B. We also write tB for the ball concentric with B but with radius t times as big. Fix a ball B_0 of radius R and consider the principal part L_0 of the operator L:

$$L_0 := -\operatorname{div}\bigl(A(x)\nabla\bigr).$$

In the paper [**3**], Fabes, Jerison and Kenig prove the existence of the Green function $G(x, y)$ of L_0 on B_0 and, among many other important properties, they establish the estimate

$$\frac{1}{C} g(x, y; R) \leqslant G(x, y) \leqslant C g(x, y; R), \qquad x, y \in \frac{1}{4} B_0\,, \tag{1.4}$$

where

$$g(x, y; R) := \int_{|x-y|}^{R} \frac{s^2}{\omega\bigl(B(x, s)\bigr)} \frac{ds}{s}.$$

The constant C in (1.4) depends on the A_2 constant (or dilation constant) C_ω of ω only. In [**5**], Gutierrez introduced the Kato-Stummel type class K as follows.

$$K := \left\{h : \lim_{r\to 0} \eta(h)(r) = 0\right\},$$

where

$$\eta(h)(r) := \sup_{x\in B_0} \int_{B(x,r)} \bigl|h(y)\bigr| g(x, y; R)\, dy, \qquad 0 < r \leqslant R.$$

REMARK 1.1. K reduces to the usual Kato class if we take $\omega(x) \equiv 1$.

In [**2**], Chiarenza, Fabes and Garofalo proved continuity of weak solutions and Harnack's inequality for non-negative weak solutions of uniformly elliptic equations $-\operatorname{div}\bigl(A(x)\nabla u\bigr) + Vu = 0$, $V \in K$. Their result on the continuity of weak solutions implies that weak solutions are Hölder continuous if $\eta(V)(r) = O(r^\alpha)$ for some positive α. However, under the general assumption that $V \in K$, one cannot prove Hölder continuity (see [**7**]). In [**7**] Kurata generalised the results of [**2**] by showing continuity of weak solutions of the uniformly elliptic equation $Lu = 0$ and that non-negative weak solutions of such equations obey Harnack's principle. The conditions $|\mathbf{b}|^2, V \in K$ were used to obtain these results. These results imply that weak solutions of $Lu = 0$ are Hölder continuous if in addition $\eta(|\mathbf{b}|^2 + |V|)(r) = O(r^\alpha)$ for some $\alpha > 0$. However the method of [7] does not show that weak solutions are Hölder continuous if the assumption that $\eta(|\mathbf{b}|^2)(r) = O(r^\alpha)$ for some $\alpha > 0$ is weakened to requiring that $|\mathbf{b}|^2 \in K$.

Our purpose here is to show that weak solutions of the degenerate equation $Lu = f$ are locally Hölder continuous under the assumption that $|\mathbf{b}|^2\omega^{-1}$, V, $f \in K$ and that $\eta(|V| + |f|)(r) = O(r^\alpha)$ for some positive α. We shall use a new approach to obtain a weak Harnack type inequality from which Hölder continuity will be obtained, (see [**4**, pp.200]). The paper is organised as follows.

In Section 2 we shall state some useful facts that will be used subsequently. In Section 3 we derive a weak Harnack type inequality. In Section 4 we indicate how the Hölder continuity can be obtained from the weak Harnack type inequality.

2. Preliminary Results

The dependence of a constant C, which may vary with different occurrences, on parameters α, β, γ, et cetera, will be denoted by $C(\alpha, \beta, \gamma, \dots)$. A ball centred at x and with radius r will be denoted by $B(x, r)$ or $B_r(x)$.

For a domain $\Omega \subseteq \mathbb{R}^n$, we use $L^2(\Omega, \omega)$ to denote the Lebesgue class with respect to the measure $d\mu := \omega\, dx$ and $H^{1,2}(\Omega, \omega)$ will denote the weighted Sobolev space $H^{1,2}(\Omega, \omega) := \{v \in L^2(\Omega, \omega) : \partial v/\partial x_i \in L^2(\Omega, \omega),\ i = 1, \dots, n\}$ with norm

$$\|v\| := \|v\|_{L^2(\Omega,\omega)} + \sum_{i=1}^{n} \|\partial v/\partial x_i\|_{L^2(\Omega,\omega)}.$$

Finally $H_0^{1,2}(\Omega, \omega)$ will be the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \omega)$ under the above norm.

As the concept of approximate Green function is used in this work let us briefly recall the definition and some of the basic properties. For further details and proofs we refer the reader to the paper [**1**].

Let Ω be a bounded domain in $\mathbb{R}^n$. Corresponding to $x_0 \in \Omega$ and $\rho > 0$ with $B_\rho := B_\rho(x_0) \subseteq \Omega$ there is $G^\rho := G^\rho(x_0, .) \in H_0^{1,2}(\Omega, \omega)$ such that

$$\int_\Omega \langle A\nabla G^\rho, \nabla\psi\rangle = \fint_{B_\rho} \psi\omega \quad \text{for any } \psi \in H_0^{1,2}(\Omega, \omega). \tag{2.1}$$

G^ρ will be called an approximate Green function of L_0 on Ω with pole x_0. The following properties of the approximate Green function G^ρ and the Green function G of L_0 on a ball B of radius r will prove useful in our subsequent discussion.

(2.2) $\quad G^\rho$ is non-negative and $G^\rho \in L^\infty(B)$.

(2.3) $\quad G^\rho(x_0, y) \to G(x_0, y)$ almost everywhere on B, if $x_0 \in \frac{1}{4}B_0$.

(2.4) $\quad G^\rho(x_0, y) \leqslant Cg(x_0, y; R)$, almost everywhere on B, if $x_0 \in \frac{1}{4}B_0$, $\rho < \frac{1}{4}r$, where in (2.4) the constant C is independent of R, the centre of B, x_0, and ρ. (2.4) follows from the weak maximum principle by comparing the approximate Green function $G_0^\rho(x_0,.)$ of L_0 on B_0 with $G^\rho(x_0,.)$ on B and then applying [**1**, Lemma 3.5].

Let Ω be a bounded open set in $\mathbb{R}^n$. In the sequel we shall assume that R is big enough so that $\Omega \subseteq (1/4)B_0$.

In this section and Section 3 we consider weak solutions of the equation $Lu = f$ on Ω under the assumption that $|\mathbf{b}|^2\omega^{-1}$, V, $f \in K$. The dependence of a constant C on $\eta\big(|\mathbf{b}|^2\omega^{-1} + |V|\big)$ will be shown by writing $C(\eta)$.

Lemma 2.1. *Let u be a non-negative, locally bounded weak solution of $Lu = f$ on Ω. Given a ball $B \subset\subset \Omega$ of radius r and $k \geqslant \eta(f)(2r)$ there are constants $C := C(\lambda, C_\omega)$ and $r_0 := r_0(\eta)$ such that for $0 < r \leqslant r_0$,*

$$\int_B |\nabla u|^2 G^\rho \omega \leqslant C \sup_B (u+k)^2$$

where G^ρ is an approximate Green function of L_0 on B and $\rho < r/4$.

Proof: Since u, $G^\rho \in L^\infty_{loc}(B)$ we can take $\widetilde{u}G^\rho$ as a test function, where $\widetilde{u} := u+k$, to estimate

$$\begin{aligned}
\frac{1}{\lambda}\int_B |\nabla u|^2 G^\rho \omega &\leqslant \int_B \langle A\nabla u, \nabla \widetilde{u}\rangle G^\rho \\
&= \int_B \big\langle A\nabla u, \nabla(\widetilde{u}G^\rho)\big\rangle - \frac{1}{2}\int_B \big\langle A\nabla(\widetilde{u}^2), \nabla G^\rho\big\rangle \\
&= -\int_B \mathbf{b}\cdot(\nabla u)\widetilde{u}G^\rho - \int_B Vu\widetilde{u}G^\rho + \int_B f\widetilde{u}G^\rho + \frac{1}{2}\fint_{B_\rho}(\zeta - \widetilde{u}^2)\omega\,.
\end{aligned}$$

In the last equation we have taken $\zeta \in H^{1,2}(B,\omega)$ such that $L_0\zeta = 0$ and $\zeta - \widetilde{u}^2 \in H_0^{1,2}(B,\omega)$, (see [**6**, Theorem 3.17]). By the weak maximum principle we see that $\zeta - \widetilde{u}^2 \leqslant \sup_B \widetilde{u}^2$. We use Hölder's inequality followed by the Cauchy-Schwartz inequality to estimate the first integral on the right side of the last inequality. On using (2.4) in the last inequality and rearranging terms we obtain the following.

$$\int_B |\nabla u|^2 G^\rho \omega \leqslant C\Big(1 + \eta\big(|\mathbf{b}|^2\omega^{-1} + |V|\big)(2r)\Big) \sup_B (u+k)^2,$$

provided that $k \geqslant \eta(f)(2r)$. If we now pick $r_0 := r_0(\eta)$ such that $\eta\big(|\mathbf{b}|^2\omega^{-1} + |V|\big)(2r_0) \leqslant 1$ then, for $0 < r \leqslant r_0$, we obtain the desired inequality. ∎

Lemma 2.2. *Let u be a non-negative weak solution of $Lu = f$ on Ω. Let $B \subseteq \Omega$ be a ball of radius r. Then there are positive constants $C := C(\lambda, C_\omega)$, $r_0 := r_0(\eta)$ and $0 < \alpha < 1$ such that for $0 < r \leqslant r_0$ and $k \geqslant \eta(f)(2r)$ we have*

$$\int_{B(x,2s)} (u+k)^\alpha \omega \leqslant C \int_{B(x,s)} (u+k)^\alpha \omega,$$

for all balls with $B(x,4s) \subseteq B$.

https://doi.org/10.1017/S0004972700018888 Published online by Cambridge University Press

PROOF: Let $B(x,2s) \subseteq B$ and $\phi \in C_0^\infty(B(x,2s))$ such that $0 \leqslant \phi \leqslant 1$, $\phi \equiv 1$ on $B(x,s)$ and $|\nabla\phi| \leqslant Cs^{-1}$. Then, with $\widetilde{u} := u + k$, we have

$$\begin{aligned}\int_B \langle A\nabla u, \nabla u\rangle \phi^2 \widetilde{u}^{-2} &= \int_B (\mathbf{b}\cdot\nabla u + Vu - f)\phi^2\widetilde{u}^{-1} + 2\int_B \langle A\nabla u, \nabla\phi\rangle\phi\widetilde{u}^{-1} \\ &\leqslant \lambda \int_{B(x,2s)} \Big(|\mathbf{b}|^2\omega^{-1} + |V| + \frac{1}{k}|f|\Big) + 4\lambda\int_{B(x,2s)} |\nabla\phi|^2\omega \\ &\quad + \frac{1}{2}\int_B \langle A\nabla u, \nabla u\rangle\phi^2\widetilde{u}^{-2}.\end{aligned} \tag{2.5}$$

Since ω is doubling we see that $g(x,y;R) \geqslant Cs^2\omega(B(x,s))^{-1}$; $y \in B(x,2s)$ so that

$$\int_{B(x,2s)} |h(y)|\,dy \leqslant Cs^{-2}\omega(B(x,s))\int_{B(x,2s)} |h(y)|g(x,y;R)\,dy.$$

Using this in (2.5) and choosing $r_0 := r_0(\eta)$ appropriately we obtain, for $k \geqslant \eta(f)(2r)$ and $0 < r \leqslant r_0$,

$$\fint_{B(x,s)} \big|\nabla \log(u+k)\big|^2\omega \leqslant Cs^{-2}.$$

A weighted Poincare inequality then implies that $\log(u+k) \in BMO(B,\omega)$, (see [**6**, pp.307]). By the weighted version of the John-Nirenberg Lemma, (see [**6**, pp.341]), there is $0 < \alpha < 1$ such that

$$\Big(\fint_{B(x,s)} (u+k)^\alpha\omega\Big)\Big(\fint_{B(x,s)} (u+k)^{-\alpha}\omega\Big) \leqslant C,$$

whenever $B(x,4s) \subseteq B$. This implies the desired doubling property, (see [**6**, pp.299]). ∎

3. A Weak Harnack Inequality

THEOREM 3.1. *Let u be a locally bounded, non-negative weak solution of $Lu = f$ in Ω and $B \subseteq \Omega$ be a ball of radius r with $5B \subseteq \Omega$. Then there are positive constants $C := C(n,\lambda,C_\omega)$ and $r_0 := r_0(\eta)$ such that for any $0 < \varepsilon < 1$ we have*

$$\Big(\fint_B u^\alpha\omega\,dx\Big)^{1/\alpha} \leqslant C\Big[\big(\vartheta(2r) + \varepsilon\big)\sup_B u + \inf_{(1/2)B} u + \frac{1}{\varepsilon}\eta(f)(2r)\Big],$$

where $\vartheta(r) := \sqrt{\eta\big(|\mathbf{b}|^2\omega^{-1}\big)(2r)} + \eta(V)(2r)$, and α is the constant in Lemma 2.2.

PROOF: Let $B := B(x_1, r) \subseteq \Omega$ be a ball of radius r and let $x_0 \in (1/2)B$. For $t > 0$, put

$$\Omega_t^\rho(x_0) = \{x \in B : G^\rho(x) > t\}, \text{ and } \Omega_t(x_0) = \{x \in B : G(x_0,x) > t\},$$

where G^ρ is the approximate Green function of L_0 on B with pole x_0 and G is the Green function of L_0 on B. We shall write Ω_t^ρ and Ω_t for these sets respectively. In the sequel we shall use the notation Γ_t^ρ for the function

$$\Big(\frac{G^\rho}{t} - 1\Big)^+ - \log^+\Big(\frac{G^\rho}{t}\Big).$$

https://doi.org/10.1017/S0004972700018888 Published online by Cambridge University Press

Since $(\log^2 s)/2 \leqslant s - 1 - \log s$ for $s \geqslant 1$, let us first observe that

$$\frac{1}{2}\left[\log^+(G^\rho/t)\right]^2 \leqslant \Gamma_t^\rho \leqslant G^\rho/t \tag{3.1}$$

and that Γ_t^ρ is supported on Ω_t^ρ for all $t > 0$. Let $0 < \tau < 1$ and $\widetilde{u} := u + k$, with $k > 0$ to be specified later. In the definition (2.1) of the approximate Green function we take

$$\phi := \left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ \widetilde{u}^\tau$$

as a test function. Since G^ρ, $u \in L^\infty_{loc}(B)$ note that ϕ is a legitmate test function. Then we find that

$$\begin{aligned}\int_{\Omega_t^\rho} \langle A\nabla G^\rho, \nabla G^\rho\rangle \frac{\widetilde{u}^\tau}{(G^\rho)^2} + \tau \int_B \langle A\nabla G^\rho, \nabla u\rangle \left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ \widetilde{u}^{\tau-1} \\ = \fint_{B_\rho(x_0)} \left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ \widetilde{u}^\tau \omega.\end{aligned} \tag{3.2}$$

Using the identity

$$\nabla(\Gamma_t^\rho \widetilde{u}^{\tau-1}) + (1-\tau)\widetilde{u}^{\tau-2}\Gamma_t^\rho \nabla u = \widetilde{u}^{\tau-1}\left(\frac{1}{t} - \frac{1}{G^\rho}\right)^+ \nabla G^\rho$$

in (3.2), followed by an application of (3.1), we find that

$$\begin{aligned}\int_{\Omega_t^\rho} \langle A\nabla G^\rho, \nabla G^\rho\rangle \frac{\widetilde{u}^\tau}{(G^\rho)^2} + \frac{2(1-\tau)}{\tau} \int_{\Omega_t^\rho} \langle A\nabla(\widetilde{u}^{\tau/2}), \nabla(\widetilde{u}^{\tau/2})\rangle \left[\log^+(G^\rho/t)\right]^2 \\ \leqslant \fint_{B_\rho(x_0)} \frac{u^\tau}{t} - \tau \int_B \langle A\nabla u, \nabla(\Gamma_t^\rho \widetilde{u}^{\tau-1})\rangle.\end{aligned}$$

That is

$$\int_{\Omega_t^\rho} \left|\nabla\left(\widetilde{u}^{\tau/2}\log^+(G^\rho/t)\right)\right|^2 \omega \leqslant C(\lambda,\tau)\left[\int_B (\mathbf{b}\cdot\nabla u + Vu - f)\,\Gamma_t^\rho \widetilde{u}^{\tau-1} + \fint_{B_\rho} \frac{u^\tau}{t}\right].$$

By the Hölder inequality and Lemma 2.1, we can find $r_0 > 0$ such that for $0 < r \leqslant r_0$ and $k \geqslant \eta(f)(2r)$ we have

$$\begin{aligned}\int_B \mathbf{b}\cdot\nabla u \Gamma_t^\rho \widetilde{u}^{\tau-1} &\leqslant \frac{1}{t}\left(\int_B |\mathbf{b}|^2\omega^{-1}G^\rho \widetilde{u}^{2(\tau-1)}\right)^{1/2}\left(\int_B |\nabla u|^2 G^\rho \omega\right)^{1/2} \\ &\leqslant \frac{C}{t}\sup_B \widetilde{u}^\tau \left(\int_B |\mathbf{b}|^2\omega^{-1}G^\rho\right)^{1/2}.\end{aligned}$$

Thus from the last two inequalities we obtain

$$\begin{aligned}\int_{\Omega_t^\rho} \left|\nabla\left(\widetilde{u}^{\tau/2}\log^+(G^\rho/t)\right)\right|^2 \omega \leqslant \frac{C}{t}\sup_B \widetilde{u}^\tau \left(\int_B |\mathbf{b}|^2\omega^{-1}G^\rho\right)^{1/2} \\ + \frac{C}{t}\sup_B \widetilde{u}^\tau \int_B \left(|V| + \frac{1}{k}|f|\right)G^\rho + C\fint_{B_\rho(x_0)} \frac{\widetilde{u}^\tau}{t}.\end{aligned}$$

Recall from (2.4) that for sufficiently small ρ we have $G^\rho(y) \leqslant Cg(x_0, y; R)$ for any $y \in B$. Therefore using this in the last inequality above we obtain

$$\int_{\Omega_t^\rho} \left|\nabla\left(\widetilde{u}^{\tau/2}\log^+(G^\rho/t)\right)\right|^2 \omega \leqslant \frac{C}{t}\left(\vartheta(r) + \frac{1}{k}\eta(f)(2r)\right)\sup_B \widetilde{u}^\tau + C \fint_{B_\rho(x_0)} \frac{\widetilde{u}^\tau}{t},$$

where $\vartheta(r) := \sqrt{\eta\left(|\mathbf{b}|^2\omega^{-1}\right)(2r)} + \eta(V)(2r)$.

Notice that, by a weighted version of Sobolev's inequality, (see [**6**, pp.304]), we have (one can replace Ω_t^ρ by the ball B of radius r)

$$\frac{C}{r^2}\int_{\Omega_t^\rho} \left|\widetilde{u}^{\tau/2}\log^+(G^\rho/t)\right|^2\omega \leqslant \int_{\Omega_t^\rho}\left|\nabla\left(\widetilde{u}^{\tau/2}\log^+(G^\rho/t)\right)\right|^2\omega.$$

Hence, on noting that $\Omega_{2t}^\rho \subseteq \Omega_t^\rho$ and that $\log^+(G^\rho/t) \geqslant \log 2$ on Ω_{2t}^ρ, we obtain

$$\frac{Ct}{r^2}\int_{\Omega_{2t}^\rho} \widetilde{u}^\tau\omega \leqslant C\left(\vartheta(r) + \frac{1}{k}\eta(f)(2r)\right)\sup_B \widetilde{u}^\tau + C\fint_{B_\rho(x_0)} \widetilde{u}^\tau.$$

As a result of (2.3) we observe that $\chi_{\Omega_t} \leqslant \liminf_\rho \chi_{\Omega_t^\rho}$. On taking the limit as $\rho \to 0$ and applying Fatou's Lemma we obtain, for $k \geqslant \eta(f)(2r)$ and $0 < r \leqslant r_0$,

$$\frac{Ct}{r^2}\int_{\Omega_{2t}} \widetilde{u}^\tau\omega \leqslant C\left(\vartheta(r) + \frac{1}{k}\eta(f)(2r)\right)\sup_B \widetilde{u}^\tau + C\widetilde{u}^\tau(x_0), \tag{3.3}$$

if x_0 is in the Lebesgue set of $\widetilde{u}^\tau$ with respect to the measure $d\mu = \omega(x)\,dx$. We should point out that $d\mu$ and dx are mutually absolutely continuous so that they have the same zero sets.

Let us notice that, for some constant C (the constant in (1.4)),

$$\frac{1}{C}\int_{r/4}^r \frac{s^2}{\omega\left(B(x_0,s)\right)}\frac{ds}{s} \leqslant \frac{1}{C}\int_{|x_0-y|}^r \frac{s^2}{\omega\left(B(x_0,s)\right)}\frac{ds}{s} \leqslant G(x_0,y),$$

for $y \in B(x_0, r/4)$. So if we take

$$t := \frac{1}{2C}\int_{r/4}^r \frac{s^2}{\omega\left(B(x_0,s)\right)}\frac{ds}{s},$$

we notice, by the doubling condition (1.3) of ω, that $t \geqslant Cr^2\omega\left(B(x_0,r/4)\right)^{-1}$. Using this in (3.3) and noting that $B(x_0,r/4) \subseteq \Omega_{2t}$ we obtain

$$\fint_{B(x_0,r/4)} \widetilde{u}^\tau\omega \leqslant C\left(\vartheta(r) + \frac{1}{k}\eta(f)(2r)\right)\sup_B \widetilde{u}^\tau + C\widetilde{u}^\tau(x_0).$$

Let us now take $\tau := \alpha$, where α is the constant in Lemma 2.2. Since ω is doubling, by Lemma 2.2 (recall that $k \geqslant \eta(f)(2r)$), we obtain

$$\fint_B u^\alpha\omega \leqslant \fint_{B(x_0,2r)} \widetilde{u}^\alpha\omega \leqslant C\fint_{B(x_0,r/4)} \widetilde{u}^\alpha\omega \leqslant C\left(\left(\vartheta(r) + \frac{1}{k}\eta(f)(2r)\right)\sup_B \widetilde{u}^\alpha + \inf_{(1/2)B}\widetilde{u}^\alpha\right).$$

The claimed result follows on taking $k := \eta(f)(2r)/\varepsilon$. ∎

https://doi.org/10.1017/S0004972700018888 Published online by Cambridge University Press

4. Hölder Regularity

In what follows we shall write $\mathrm{Osc}(f;E)$ for the oscillation $\sup_E f - \inf_E f$ of f on E.

Theorem 4.1. *Let u be a locally bounded weak solution of $Lu = f$ on Ω. Then there are positive constants $C = C(n,\lambda,C_\omega)$, $\kappa = \kappa(n,\lambda,C_\omega)$, and $r^* = r^*(n,\lambda,\eta)$ such that if $0 < \mu < 1$, $B(x_0,r^*) \subseteq \Omega$ and $0 < r < r^*$ then*

$$Osc\big(u\,;\,B(x_0,r)\big) \leqslant C\Big(\frac{r}{r^*}\Big)^{\kappa} Osc\big(u\,;\,B(x_0,r^*)\big) + C\eta\big(|f|+|V|\big)\big(5r^{\mu}(r^*)^{1-\mu}\big)M^0,$$

where $M^0 := 1 + \sup_{B(x_0,r^*)} u$. Furthermore, if $\eta\big(|f|+|V|\big)(r) = O(r^{\alpha})$ for some $\alpha > 0$, then u is locally Hölder continuous on Ω.

Proof: The proof uses the techniques in [**4**, pp.200]. Let $r^* > 0$ be such that $B(x_0,5r^*) \subseteq \Omega$. For $0 < r \leqslant r^*$ let M, M', and M_0 be the essential supremum of u on $B(x_0,5r)$, $B(x_0,r/2)$, and $B(x_0,r^*)$ respectively. Likewise, let m and m' be the infimum of u on $B(x_0,5r)$ and $B(x_0,r/2)$ respectively. $M-u$ and $u-m$ are locally bounded non-negative functions on $B(x_0,5r)$ that satisfy $L(M-u) = -f + MV$ and $L(u-m) = f - mV$. Thus by Theorem 3.1 we find constants C and r_0 such that for $0 < r < r_0$ we have

$$\Big(\fint_{B(x_0,r)} (M-u)^{\alpha}\omega\Big)^{1/\alpha} \leqslant C\Big(\big(\vartheta(2r)+\varepsilon\big)(M-m) + (M-M') + \frac{1}{\varepsilon}\eta\Big),$$
$$\Big(\fint_{B(x_0,r)} (u-m)^{\alpha}\omega\Big)^{1/\alpha} \leqslant C\Big(\big(\vartheta(2r)+\varepsilon\big)(M-m) + (m'-m) + \frac{1}{\varepsilon}\eta\Big),$$

where we have used $\eta := \eta\big(|f|+|V|\big)(2r)(1+M_0)$. We have also used the obvious facts that $\sup_{B(x_0,r)} u \leqslant M$ and $\inf_{B(x_0,r)} u \geqslant m$. Adding the last two inequalities leads to

$$M - m \leqslant 2C\big(\vartheta(2r)+\varepsilon\big)(M-m) + 2C\big(M - m - (M'-m')\big) + \frac{2}{\varepsilon}\eta.$$

We now choose $r^* \leqslant r_0$ and ε small enough so that $2C\vartheta(2r) + \varepsilon \leqslant 1/2$ for all $0 < r < r^*$. Then we have

$$M - m \leqslant C\big(M - m - (M'-m')\big) + C'\eta\big(|f|+|V|\big)(2r)(1+M_0)$$

for some positive constants C, C'. Thus we have

$$\varpi\Big(\frac{1}{2}r\Big) \leqslant \gamma\varpi(5r) + C'(1+M_0)\eta\big(|f|+|V|\big)(2r),$$

where $\gamma := (C-1)/C$, and $\varpi(\rho) := \mathrm{Osc}\big(u\,;\,B(x_0,\rho)\big)$. Appealing to [**4**, Lemma 8.23, pp.201] gives the desired result. ∎

Remark 4.2. In the uniformly elliptic case, $\omega \equiv$ Constant, u is known to be locally bounded on Ω, (see [**7**]). In fact the proof in [7] can be repeated to show that u is locally bounded on Ω even in the degenerate case that was considered here.

https://doi.org/10.1017/S0004972700018888 Published online by Cambridge University Press

References

[1] S. Chanillo and R. Wheeden, 'Existence and estimates of Green's function for degenerate elliptic equations', *Ann. Scoula Norm. Sup. Pisa Cl. Sci.* **15** (1988), 309–340.

[2] F. Chiarenza, E. Fabes and N. Garofalo, 'Harnack's inequality for Schrödinger operators and continuity of solutions', *Proc. Amer. Math. Soc.* **98** (1986), 415–425.

[3] E. Fabes, D. Jerison and C. Kenig, 'The Wiener test for degenerate elliptic equations', *Ann. Inst. Fourier (Grenoble)* **32** (1982), 151–182.

[4] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, (2nd edition) (Springer-Verlag, Berlin, Heidelberg, New York, 1983).

[5] C.E. Gutierrez, 'Harnack's inequality for degenerate Schrödinger operators', *Trans. Amer. Math. Soc.* **312** (1989), 403–419.

[6] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations* (Oxford University Press Inc., Oxford, 1993).

[7] K. Kurata, 'Continuity and Harnack's inequality for solutions of elliptic partial differential equations of second order', *Indiana Univ. Math.* **54** (1995), 511–550.

Department of Mathematics & Computer Science
Faculty of Science
Kuwait University
P.O.Box 5969
Safat 13060
Kuwait
e-mail: ahmedm@sun490.sci.kuniv.edu.kw