Canad. Math. Bull. Vol. **47** (4), 2004 pp. 481–491

A New Characterization of Hardy Martingale Cotype Space

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Abstract. We give a new characterization of Hardy martingale cotype property of complex quasi-Banach space by using the existence of a kind of plurisubharmonic functions. We also characterize the best constants of Hardy martingale inequalities with values in the complex quasi-Banach space.

1 Introduction

It is well known, by now, that some special functions are closely related to the inequalities of martingales and the geometric structure of Banach space. Burkholder[Bu1] [Bu2] gave the biconvex function characterization of Hilbert space and UMD space, and the convex function characterization of martingale cotype space. Lee [L] gave the biconcave function characterization of Hilbert space and UMD space. Piasecki [P] obtained the shew-plurisubharmonic function characterization of AUMD space. In this paper we establish a geometric characterization of Hardy martingale cotype space via the plurisubharmonic function.

2 Preliminaries

Let $\Omega = [0, 2\pi]^{\mathbb{N}}$, Σ be Borel σ-algebra on $[0, 2\pi]^{\mathbb{N}}$ and P the product measure of normalized Lebesgue measure on $[0, 2\pi]$. An element $\theta \in \Omega$ is written as $\theta = (\theta_1, \theta_2, \dots)$. Let Σ_n stand for σ-algebra generated by the first n coordinates $\theta_1, \theta_2, \dots, \theta_n$. Where $\Sigma_0 = \{\phi, [0, 2\pi]\}$ and E is the expectation with respect to P. Suppose that X is a complex quasi-Banach space. For simplicity, we assume that the quasi-norm of X is plurisubharmonic, *i.e.*,

$$\|x\| \le \frac{1}{2\pi} \int_0^{2\pi} \|x + ye^{i\theta}\|\, d\theta \quad \forall x, y \in X. \tag{1}$$

Then, by the result of Kalton [K], there is an equivalent quasi-norm which is both plurisubharmonic and ρ-subadditive ($\|x + y\|^\rho \le \|x\|^\rho + \|y\|^\rho, \forall x, y \in X$) for some $0 < \rho \le 1$. So without loss of generality, throughout this paper, we assume that the quasi-norm of X is ρ-subadditive.

A sequence $F = (F_n)$ of X-valued random variables adapted to the sequence of sub-σ-algebras (Σ_n) is called Hardy martingale if

$$F_0 = x, dF_n = F_n - F_{n-1} = \sum_{k=1}^{\infty} \varphi_{n,k}(\theta_1, \dots, \theta_{n-1}) e^{i\theta_n} \quad \text{for } n \ge 1,$$

Received by the editors November 12, 2002; revised April 20,2003.
AMS subject classification: 46B20, 52A07, 60G44.
Keywords: Hardy martingale, Hardy martingale cotype, plurisubharmonic function.

https://doi.org/10.4153/CMB-2004-048-5 Published online by Cambridge University Press

where $x \in X$, $\varphi_{n,k}$ are X-valued (strongly) measurable function of $\theta_1, \dots, \theta_{n-1}$, for $k = 1, 2, \dots$. If additionally $\varphi_{n,k} = 0$ for all $k \geq 2, n = 1, 2, \dots$, then $F = (F_n)$ is called analytic martingale.

We say that X is of Hardy (resp. analytic) martingale cotype $q\,(2 \leq q < \infty)$ if there is a constant C such that

$$\Big(\sum_{n\geq o} \|dF_n\|_q^q\Big)^{\frac{1}{q}} \leq C \sup_{n\geq o} \|F_n\|$$

for all Hardy (resp. analytic) martingales $F = (F_n)$ with values in X. By the renorming theorem of [X1](see also [X2], [LB]), X is of Hardy martingale cotype q iff X has an equivalent quasi-norm $|\cdot|$ whose uniform H_q-convexity modulus is of power type q:

$$h_q^X(\varepsilon) \geq C\varepsilon^q, \quad \forall 0 < \varepsilon \leq 1,$$

where $C > 0$ is a constant, and the so called uniform H_q-convexity modulus is

$$h_q^X(\varepsilon) = \inf\Big\{ \|f\|_{L_q([0,2\pi];(X,|\cdot|))} : |\hat{f}(0)| = 1, \\ \|f - \hat{f}(0)\|_{L_q([0,2\pi];(X,|\cdot|))} > \varepsilon, f \in H_q(X) \Big\}.$$

Several other equivalent conditions for the Hardy martingale cotype can be found in [LB, X1, X2, X3]. For convenience we state the following criteria (see [LB]) that will be applied below. We use the customary notations

$$F_n^* = \sup_{k\leq n} \|F_k\|, \quad F^* = \sup_{n\geq 0} \|F_k\|, \quad \|F\|_p = \sup_{k\geq 0} \|F_k\|_p,$$

$$S_n^{(p)}(F) = (\sum_{k=0}^{n} \|dF_k\|^p)^{\frac{1}{p}}, \quad S^{(p)}(F) = (\sum_{k=0}^{\infty} \|dF_k\|^p)^{\frac{1}{p}}.$$

Theorem A *Let $2 \leq q < \infty$, X be a quasi-Banach space, the following statements are equivalent:*

(i) *X is of Hardy martingale cotype q.*
(ii) *If $\|F\|_\infty < \infty$, then $S^q(F) < \infty$ a.e. for every X-valued Hardy martingale $F = (F_n)$.*
(iii) *For $0 < p < \infty$ there is a constant C_p such that*

(2) $$\|S^{(q)}(F)\|_p \leq C_p \|F\|_p$$

for every X-valued Hardy martingale $F = (F_n)$.

We recall a classical fact about lower semi-continuous functions (see [R 2.1.3]).

Lemma B *Let u be a lower semi-continuous real-valued function defined on a metric space X, such that u is bounded below on X. Then there exist uniformly continuous functions $\phi_n \colon X \to \mathbf{R}$ such that the sequence ϕ_n is increasing and $\lim_{n\to\infty} \phi_n = u$ on X.*

https://doi.org/10.4153/CMB-2004-048-5 Published online by Cambridge University Press

3 Main Theorems and Their Proofs

Let X be a quasi-Banach space. An X-valued Hardy(resp. analytic) martingale $F = (F_n)$ is called simple if there is n such that $F_m = F_n$ for all $m \geq n$, and every $\varphi_{l,k}$ (resp. $\varphi_{l,1}$) is X-valued simple function of $\theta_1, \ldots, \theta_{l-1}$ for $l = 1, \ldots, n, k = 1, 2, \ldots$. One may check that such martingales are dense in the space of Bochner integrable X-valued Hardy(resp. analytic) martingales.

Let $2 \leq q < \infty$ and $v \colon X \times [0, \infty) \to \mathbb{R}$ a function satisfying

$$v(0,0) > 0, \tag{3}$$

$$v(x,t) \leq \|x\|^\rho \quad \text{if } t \geq 1, \tag{4}$$

$$v(x,t) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x + \sum_{k=1}^n x_k e^{ik\theta}, t + \| \sum_{k=1}^n x_k e^{ik\theta} \|^q) d\theta, \tag{5}$$

for all $x, x_k \in X$, $(k = 1, 2, \ldots, n)$, $n \geq 1$ and $t \geq 0$.

If $(x,t) \in X \times [0, \infty)$, let $L(x,t)$ be the set of all X-valued simple Hardy martingales $F = (F_n)$ such that $F_0 = x$ and

$$P(t - \|x\|^q + (S^{(q)}(F))^q \geq 1) = 1.$$

It is clear that $L(x,t)$ is nonempty. Set

$$u(x,t) = \inf \left\{ \|F\|_\rho^\rho : F \in L(x,t) \right\}. \tag{6}$$

Lemma 1 *Let X be a complex quasi-Banach space. Then u is the greatest plurisubharmonic function $X \times [0, \infty) \to \mathbb{R}$ which satisfies* (4) *and* (5).

Proof If $t \geq 1$ and $F_n = x$ for all $n \geq 0$, then $F = (F_n) \in L(x,t)$ and $\|F\|_\rho^\rho = \|x\|^\rho$, which implies that $u(x,t) \leq \|x\|^\rho$.

We next show that u has the property (5). Let $L_k(x,t)$ be the set of all X-valued simple Hardy martingales $F = (F_n)$ such that $F_0 = x$ and

$$P\left(t - \|x\|^q + (S^{(q)}(F))^q \geq 1 + \frac{1}{k}\right) = 1.$$

Define $u_k(x,t) = \inf \left\{ \|F\|_\rho^\rho : F \in L_k(x,t) \right\}$ for $k = 1, 2, \ldots$. Then it is clear that

$$L_k(x,t) \subseteq L_{k+1}(x,t), \quad u_{k+1}(x,t) \leq u_k(x,t)$$

and

$$u(x,t) = \inf_{k \geq 1} u_k(x,t). \tag{7}$$

In fact, $L_k(x,t) \subseteq L(x,t)$ and $u(x,t) \leq u_k(x,t)$, so

$$u(x,t) \leq \inf_{k \geq 1} u_k(x,t). \tag{8}$$

On the other hand, for arbitrary $\varepsilon > 0$, there is a simple Hardy martingale $F = (F_n) \in L(x,t)$ which satisfies

$$\|F\|_\rho^\rho \le u(x,t) + \varepsilon. \tag{9}$$

Choose k and $y \in X$ so that $(\frac{1}{k})^{\rho/q} \le \varepsilon, \|y\|^q = \frac{1}{k}$. We introduce a new Hardy martingale $G = (G_n)$ by

$$G_0 = x, G_{n+1} = F_n + e^{i\theta} y, \quad \text{for } \theta \in [0, 2\pi],\ n \ge 0.$$

Notice that

$$\begin{aligned} t - \|x\|^q + (S^{(q)}(G))^q &= t - \|x\|^q + (S^{(q)}(F))^q + \|y\|^q \\ &= t - \|x\|^q + (S^{(q)}(F))^q + \frac{1}{k}, \end{aligned}$$

therefore, $G = (G_n) \in L_k(x,t)$. Hence, by (9),

$$u_k(x,t) \le \|G\|_\rho^\rho \le \|F\|_\rho^\rho + \|y\|^\rho \le \|F\|_\rho^\rho + (\frac{1}{k})^{\frac{\rho}{q}} \le u(x,t) + 2\varepsilon$$

and

$$\inf_{k \ge 1} u_k(x,t) \le u(x,t) + 2\varepsilon.$$

We deduce that

$$\inf_{k \ge 1} u_k(x,t) \le u(x,t), \tag{10}$$

since $\varepsilon > 0$ is arbitrary. By (8) and (10), we obtain (7).

To show the function u_k $(k \ge 1)$ is continuous, it suffices to prove that

$$|u_k(x,t) - u_k(x',t')| \le \|x - x'\|^\rho + |t - t'|^{\frac{\rho}{q}} \quad \text{if } (x,t), (x',t') \in X \times [0, \infty). \tag{11}$$

To see this, for $t' = t$ and $\varepsilon > 0$ take $F = (F_n) \in L_k(x,t)$ such that $\|F\|_\rho^\rho \le u_k(x,t) + \varepsilon$. We define a new Hardy martingale $G = (G_n)$ by $G_0 = x'$, $G_n = (F_n - F_0) + G_0$. Notice that $G = (G_n) \in L_k(x',t)$, then

$$u_k(x',t) \le \|G\|_\rho^\rho \le \|F\|_\rho^\rho + \|x - x'\|^\rho \le u_k(x,t) + \|x - x'\|^\rho + \varepsilon.$$

This gives $u_k(x',t) - u_k(x,t) \le \|x - x'\|^\rho$. Similarly we have $u_k(x,t) - u_k(x',t) \le \|x - x'\|^\rho$, so $|u_k(x,t) - u_k(x',t)| \le \|x - x'\|^\rho$ and (11) holds for the special case $t' = t$. If $t' > t$ and $y \in X$ is chosen to satisfy $\|y\| = (t' - t)^{\frac{1}{q}}$, we take $F = (F_n) \in L_k(x,t')$ such that $\|F\|_\rho^\rho \le u_k(x,t') + \varepsilon$ and define a new Hardy martingale $G = (G_n)$ by $G_0 = x, G_{n+1} = F_n + ye^{i\theta}$. Then $G = (G_n) \in L_k(x,t)$,

$$u_k(x,t) \le \|G\|_\rho^\rho \le \|F\|_\rho^\rho + \|y\|^\rho \le u_k(x,t') + |t' - t|^{\frac{\rho}{q}} + \varepsilon.$$

Hence, we get $u_k(x,t) - u_k(x,t') \le |t - t'|^{\rho/q}$ or $|u_k(x,t) - u_k(x,t')| \le |t - t'|^{\rho/q}$. That is to say (11) holds for the special case $x = x'$. Combining these two special cases with the triangle inequality, we derive (11).

Now suppose that

$$f(\theta) = \sum_{l=1}^{n} x_l e^{il\theta},\ \theta \in [0, 2\pi], x_l \in X,\ l = 1, \dots, n,\ n \ge 1$$

and $\varepsilon > 0,\ k \ge 1$. A continuity argument gives $J > 0$ such that

$$\|f(\tfrac{j}{J}2\pi) - f(\theta)\|^{\rho} < \varepsilon, \quad \big|\,\|f(\tfrac{j}{J}2\pi)\|^q - \|f(\theta)\|^q\big| < \frac{1}{2k}, \tag{12}$$

$$\Big|u_k(x + f(\theta), t + \|f(\theta)\|^q) - u_k\Big(x + f\big(\tfrac{j}{J}2\pi\big), t + \|f\big(\tfrac{j}{J}2\pi\big)\|^q\Big)\Big| < \varepsilon, \tag{13}$$

whenever $\frac{j-1}{J}2\pi < \theta \le \frac{j}{J}2\pi$ for $1 \le j \le J$. Clearly,

$$\sum_{j=1}^{J} \frac{1}{2\pi} \int_{\frac{j-1}{J}2\pi}^{\frac{j}{J}2\pi} u_k(x + f(\theta), t + \|f(\theta)\|^q)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} u_k(x + f(\theta), t + \|f(\theta)\|^q)\, d\theta. \tag{14}$$

For each $1 \le j \le J$, there exists $F^{(j)} \in L_k(x + f(\frac{j}{J}2\pi), t + \|f(\frac{j}{J}2\pi)\|^q)$ with

$$\|F^{(j)}\|_{\rho}^{\rho} \le u_k(x + f(\tfrac{j}{J}2\pi), t + \|f(\tfrac{j}{J}2\pi)\|^q) + \varepsilon. \tag{15}$$

We now define a Hardy martingale $F = (F_n)$ by

$$F_0 = x,\ F_n(\theta, \theta_1, \dots, \theta_{n-1}) = F_{n-1}^{(j)}(\theta_1, \dots, \theta_{n-1}) + f(\theta) - f(\tfrac{j}{J}2\pi)$$

for $\frac{j-1}{J}2\pi < \theta \le \frac{j}{J}2\pi,\ 1 \le j \le J$ and $n \ge 1$. If $\theta \in (\frac{j-1}{J}2\pi, \frac{j}{J}2\pi]$, we have

$$t - \|x\|^q + (S^{(q)}(F))^q = t + \|f(\theta)\|^q + \sum_{l=1}^{\infty} \|dF_l^{(j)}\|^q.$$

We use $F^{(j)} \in L_k(x + f(\frac{j}{J}2\pi), t + \|f(\frac{j}{J}2\pi)\|^q)$, *i.e.*,

$$t + \|f(\tfrac{j}{J}2\pi)\|^q - \|x + f(\tfrac{j}{J}2\pi)\|^q + (S^{(q)}(F^{(j)}))^q = t + \|f(\tfrac{j}{J}2\pi)\|^q + \sum_{l=1}^{\infty} \|dF_l^{(j)}\|^q \ge 1 + \frac{1}{k} \text{ a.e.}$$

https://doi.org/10.4153/CMB-2004-048-5 Published online by Cambridge University Press

and (12) to obtain that

$$t - \|x\|^q + (S^{(q)}(F))^q = t + \|f(\theta)\|^q + \sum_{l=1}^{\infty} \|dF_l^{(j)}\|^q \geq 1 + \frac{1}{2k} \text{ a.e.}$$

when $\theta \in (\frac{j-1}{J}2\pi, \frac{j}{J}2\pi]$. So $F = (F_n) \in L_{2k}(x, t)$. From (12–15), it follows that

$$\begin{aligned} u_{2k}(x,t) &\leq \|F\|_\rho^\rho \leq \sum_{j=1}^{J} \frac{1}{2\pi} \int_{\frac{j-1}{J}2\pi}^{\frac{j}{J}2\pi} \|F^{(j)}\|_\rho^\rho \, d\theta + \varepsilon \\ &\leq \sum_{j=1}^{J} \frac{1}{2\pi} \int_{\frac{j-1}{J}2\pi}^{\frac{j}{J}2\pi} \Big[u_k\big(x + f(\frac{j}{J}2\pi), t + \|f(\frac{j}{J}2\pi)\|^q\big) + \varepsilon \Big] \, d\theta + \varepsilon \\ &\leq \sum_{j=1}^{J} \frac{1}{2\pi} \int_{\frac{j-1}{J}2\pi}^{\frac{j}{J}2\pi} u_k\big(x + f(\theta), t + \|f(\theta)\|^q\big) \, d\theta + 3\varepsilon \\ &= \frac{1}{2\pi} \int_0^{2\pi} u_k\big(x + f(\theta), t + \|f(\theta)\|^q\big) \, d\theta + 3\varepsilon, \end{aligned}$$

this implies

$$u_{2k}(x,t) \leq \frac{1}{2\pi} \int_0^{2\pi} u_k\big(x + f(\theta), t + \|f(\theta)\|^q\big) \, d\theta.$$

Now take limits to obtain

$$u(x,t) \leq \frac{1}{2\pi} \int_0^{2\pi} u\big(x + f(\theta), t + \|f(\theta)\|^q\big) \, d\theta,$$

which shows that u satisfies (5).

To see that u is the greatest function, let v satisfy (4), (5), $F = (F_n) \in L(x, t)$ and choose n so that $P(t - \|x\|^q + (S_n^{(q)}(F))^q \geq 1) = 1$. Then, by (4) and (5), we have

$$\begin{aligned} \|F\|_\rho^\rho \geq E\|F_n\|^\rho &\geq Ev(F_n, t - \|x\|^q + (S_n^{(q)}(F))^q) \\ &\geq Ev(F_0, t - \|x\|^q + (S_0^{(q)}(F))^q) = v(x,t), \end{aligned}$$

which implies that $u \geq v$.

Now we have

Corollary *If u satisfies* (3), (4) *and* (5), *and $F = (F_n)$ is a X-valued Hardy martingale, then, for all $\lambda > 0$,*

$$P(S^{(q)}(F) \geq \lambda) \leq \frac{\|F\|_\rho^\rho}{\lambda^\rho u(0,0)} \tag{16}$$

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Proof It suffices to prove (16), for u as in Lemma 1. We assume that $\lambda = 1$. For X-valued Hardy martingale $F = (F_n), F_0 = 0$, by (4) and Chebyshev's inequality,

$$P(S_n^{(q)}(F) \geq 1) \leq P\big(\|F_n\|^\rho - u(F_n, (S_n^{(q)}(F))^q) + u(0,0) \geq u(0,0)\big)$$
$$\leq \frac{E\big[\|F_n\|^\rho - u(F_n, (S_n^{(q)}(F))^q) + u(0,0)\big]}{u(0,0)}.$$

On the other hand, by (5),

$$\begin{aligned} u(0,0) &= Eu(F_0, (S_0^{(q)}(F))^q) \\ &\leq Eu(F_1, (S_1^{(q)}(F))^q) \\ &\quad \cdots \\ &\leq Eu(F_{n-1}, (S_{n-1}^{(q)}(F))^q) \\ &\leq Eu(F_n, (S_n^{(q)}(F))^q). \end{aligned}$$

Hence, we have

$$P\big(S_n^{(q)}(F) \geq 1\big) \leq \frac{E\|F_n\|^\rho}{u(0,0)}. \tag{17}$$

Now we use homogeneity and take limits to obtain

$$P\big(S^{(q)}(F) \geq \lambda\big) \leq \frac{\|F\|_\rho^\rho}{\lambda^\rho u(0,0)}. \tag{18}$$

If X-valued Hardy martingale $F = (F_n), F_0 = x \neq 0$, we define a Hardy martingale $G = (G_n)$ by

$$G_0 = 0, G_{n+1} = G_n + e^{i\theta} dF_n, \quad \text{for } \theta \in [0, 2\pi], \quad n \geq 0.$$

Then $S_n^{(q)}(F) = S_{n+1}^{(q)}(G), \|F_n\| = \|G_{n+1}\|$; thus (18) yields (16).

Theorem 1 *Let $2 \leq q < \infty$, X be a quasi-Banach space. Then X is of Hardy martingale cotype q iff there is a plurisubharmonic function $u\colon X \times [0, \infty) \to \mathbb{R}$ such that* (3), (4) *and* (5) *hold.*

Proof Suppose that X is of Hardy martingale cotype q. Theorem A implies that there is a constant $C > 0$ such that $\|F\|_\rho^\rho \geq C$ whenever $F = (F_n) \in L(0,0)$. Let u be defined by (6), then $u(0,0) \geq C > 0$ *i.e.*, u satisfies (3). From Lemma 1 we know that (4) and (5) hold.

Conversely, suppose that there is a plurisubharmonic function $u\colon X \times [0, \infty) \to \mathbb{R}$ such that (3), (4) and (5) hold, from the corollary of Lemma 1 and Theorem A, we obtain that X is of Hardy martingale cotype q.

Let $\gamma_{p,q}^H$ (resp. $\gamma_{p,q}^A$) be the least $\gamma < \infty$ such that

$$\|S^q(F)\|_p \leq \gamma \|F\|_p \tag{19}$$

for all Hardy (resp. analytic) martingales $F = (F_n)$ with values in X.

Theorem 2 *Suppose that X is a complex quasi-Banach space,* $p \in (0,\infty)$, $q \in [2,\infty)$ *and* $\gamma \in [1,\infty)$. *Then*

$$\gamma^{H}_{p,q} \le \gamma \tag{20}$$

iff there is a lower semi-continuous function $u\colon X \times [0,\infty) \to [-\infty,\infty]$ *such that, for all* $x, x_k \in X$ $(k = 1,2,\dots,n)$, $n \ge 1$ *and* $t \ge 0$,

$$u(x,t) \ge \phi(x,t), \tag{21}$$

$$u(x,t) \ge \frac{1}{2\pi}\int_0^{2\pi} u\Big(x + \sum_1^n x_k e^{ik\theta}, t + \|\sum_1^n x_k e^{ik\theta}\|^q\Big)\, d\theta \tag{22}$$

where $\phi(x,t) = t^{\frac{p}{q}} - \gamma^p \|x\|^p$.

Proof Assume that (20) holds. Let $x \in X$, $L(x)$ be the set of all X-valued simple Hardy martingales $F = (F_n)$ satisfying $F_0 = x$. Set

$$u(x,t) = \sup\Big\{ E\phi\big(F_\infty, t - |x|^q + (S^{(q)}(F))^q\big) : F \in L(x)\Big\} \tag{23}$$

where F_∞ denotes the pointwise limit of the simple martingale F. Through considering the martingale $F \in L(x)$ with $F_n = x$, $n \ge 0$, we deduce that u satisfies (21).

From the definition of u, it is straightforward to verify that

$$u(x,t) = \sup\Big\{ E\phi\big(x + F_\infty, t + (S^{(q)}(F))^q\big) : F \in L(0)\Big\}. \tag{24}$$

In the following we will show that u is lower semi-continuous. Notice that for fixed $F = (F_n) \in L(0)$, the map

$$(x,t) \to E\phi\big(x + F_\infty, t + (S^{(q)}(F))^q\big)$$

is continuous. Indeed, if $x_k \to x$, $t_k \to t$ then we have

$$\lim_{k\to\infty} \phi\big(x_k + F_\infty(\theta), t_k + (S^{(q)}(F))^q(\theta)\big) = \phi\big(x + F_\infty(\theta), t + (S^{(q)}(F))^q(\theta)\big)$$

for all $\theta \in \Omega$. So

$$\lim_{k\to\infty} E\phi\big(x_k + F_\infty, t_k + (S^{(q)}(F))^q\big) = E\phi\big(x + F_\infty, t + (S^{(q)}(F))^q\big).$$

Hence, u is lower semi-continuous.

To show that u satisfies (22), let

$$f(s) = \sum_{k=1}^n x_k e^{iks},\ s \in [0,2\pi], x_k \in X,\ k = 1,\dots,n,\ n \ge 1.$$

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Let $m(s)$ be a continuous function on $[0, 2\pi]$ and

$$u\big(x + f(s), t + \|f(s)\|^q\big) \geq m(s), s \in [0, 2\pi].$$

For each fixed $s \in [0, 2\pi]$ and $\varepsilon > 0$, there exists $F^{(s)} \in L(0)$ with

$$E\phi\big(x + f(s) + F^{(s)}, t + (S^{(q)}(F^{(s)}))^q\big) > m(s) - \varepsilon. \tag{25}$$

Let

$$g_s(r) = E\phi\big(x + f(r) + F^{(s)}, t + (S^{(q)}(F^{(s)}))^q\big) - m(r) + \varepsilon.$$

Since $E\phi(x + f(r) + F^{(s)}, t + (S^{(q)}(F^{(s)}))^q)$ and $m(s)$ are continuous, $g_s(r)$ is continuous function. By (25) it follows that $g_s(s) > 0$. Hence there exists an open interval I_s such that $s \in I_s$ and $g_s(r) > 0$ for $r \in I_s$. From compactness of $[0, 2\pi]$, we obtain that there are finitely many disjoint semi-open intervals $I_{s_1}, I_{s_2}, \dots, I_{s_J}$ covering $(0, 2\pi] \subseteq [0, 2\pi], s_j \in [0, 2\pi],\ j = 1, 2, \dots, J$ and corresponding martingales $F^{(s_j)},\ j = 1, 2, \dots, J$ such that the following inequality

$$E\phi\big(x + f(r) + F^{(s_j)}, t + (S^{(q)}(F^{(s_j)}))^q\big) > m(r) - \varepsilon \text{ for } r \in I_{s_j}$$

holds. We now define a Hardy martingale $F = (F_n)$ by

$$F_0 = 0,\ F_n(s, \theta_1, \dots, \theta_{n-1}) = F^{s_j}_{n-1}(\theta_1, \dots, \theta_{n-1}) + f(s)$$

for $s \in I_{s_j}, 1 \leq j \leq J$ and $n \geq 1$, then it is clear that $F = (F_n) \in L(0)$. Hence

$$\begin{aligned} u(x, t) &\geq E\phi(x + F_\infty, t + (S^{(q)}(F))^q) \\ &= \sum_{j=1}^{J} \frac{1}{2\pi} \int_{I_{s_j}} E\phi\big(x + f(s) + F^{(s_j)}, t + \|f(s)\|^q + (S^{(q)}(F^{(s_j)}))^q\big)\, ds \\ &\geq \sum_{j=1}^{J} \frac{1}{2\pi} \int_{I_{s_j}} m(s)\, ds - \varepsilon = \frac{1}{2\pi} \int_0^{2\pi} m(s)\, ds - \varepsilon \end{aligned}$$

Then $u(x, t) \geq \frac{1}{2\pi} \int_0^{2\pi} m(s)\, ds$. Hence, using Theorem B, we derive that

$$u(x, t) \geq \frac{1}{2\pi} \int_0^{2\pi} u\big(x + f(s), t + \|f(s)\|^q\big)\, ds,$$

so u satisfies (22).

u is the least function satisfying (21) and (22). To see this, let v satisfy (21), (22), $F = (F_n) \in L(x)$. Then, by (21) and (22), it follows that

$$\begin{aligned} EF\big(F_\infty, t - |x|^q + S^q(F)^q\big) &\leq Ev\big(F_\infty, t - |x|^q + S^q(F)^q\big) \\ &\leq Ev\big(F_0, t - \|x\|^q + (S^{(q)}_0(F))^q\big) = v(x, t)\,, \end{aligned}$$

which implies that $u \le v$.

Conversely, without loss of generality, we can assume that u as in (23). Then

$$u(\alpha x, |\alpha|^q t) = |\alpha|^p u(x,t), \quad \forall \alpha \in \mathbb{R}, \tag{26}$$

To see this, consider $v\colon X \times [0,\infty) \to [-\infty,\infty]$ defined by

$$v(x,t) = \inf_{\lambda \neq 0} \frac{u(\lambda x, |\lambda|^q t)}{|\lambda|^p}.$$

Then v satisfies (21), (22) and $v \le u$. Using the minimality of u, we obtain that $u = v$, this gives (26). To show (20) for the γ in the definition of ϕ, we need to prove that (19) holds for all Hardy martingales $F = (F_n)$ with values in X. To do this, we can assume that Hardy martingale $F = (F_n)$ is simple and $F_0 = 0$. Then, from (21), (22) and (26), we derive that

$$E\phi(F_n, (S_n^{(q)}(F))^q) \le Eu(F_n, (S_n^{(q)}(F))^q) \le \cdots \le Eu(F_0, (S_0^{(q)}(F))^q) = u(0,0) = 0$$

so $\|S^q(F)\|_p^p - \gamma^p \|F\|_p^p \le 0$ and (19) follows.

Theorem 3 *Suppose that X is a complex quasi-Banach space, $p \in (0,\infty)$, $q \in [2,\infty)$ and $\gamma \in [1,\infty)$. Then*

$$\gamma_{p,q}^A \le \gamma \tag{27}$$

iff there is a lower semi-continuous function $u\colon X \times [0,\infty) \to [-\infty,\infty]$ such that, for all $x, y \in X$ and $t \ge 0$,

$$u(x,t) \ge \phi(x,t), \tag{28}$$

$$u(x,t) \ge \frac{1}{2\pi} \int_0^{2\pi} u\big(x + ye^{i\theta}, t + \|y\|^q\big)\, d\theta \tag{29}$$

where $\phi(x,t) = t^{\frac{p}{q}} - \gamma^p \|x\|^p$.

The proof of Theorem 3 is the same as the proof of Theorem 2, therefore we omit it.

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https://doi.org/10.4153/CMB-2004-048-5 Published online by Cambridge University Press