Notes

107.01 A simple integral representation of the Fibonacci numbers

The sequence $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ where each number is the sum of the two preceding ones corresponds to the famous *Fibonacci sequence*. As is well known, if n is a non-negative integer, the nth Fibonacci number F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with $F_0 = 0$ and $F_1 = 1$. Known properties for the Fibonacci numbers are vast, and if anyone is in any doubt about this one need look no further than the two volumes devoted to these numbers by Koshy [1, 2].

In this Note I give a simple* integral representation for these numbers. I begin by stating the result before proceeding to give a proof. Some consequences of the result are then discussed and an application making use of the result considered.

For $n \geqslant 1$ an integral representation of the Fibonacci numbers is given by

$$F_n = \frac{n}{2^n}\int_{-1}^{1}\left(1 + x\sqrt{5}\right)^{n-1} dx. \qquad (1)$$

To prove the result we recall Catalan's formula for the Fibonacci numbers [1, p. 197]

$$F_n = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n}{2k+1}5^k.$$

Here $\lfloor x \rfloor$ denotes the floor function, that is, the greatest integer less than or equal to x. Since the binomial coefficient can be expressed as

$$\binom{n}{2k+1} = \frac{n}{2k+1}\binom{n-1}{2k},$$

Catalan's formula can be rewritten as

$$F_n = \frac{n}{2^n}\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}5^k\frac{2}{2k+1}.$$

Reindexing the sum by $k \to \frac{k}{2}$ produces

$$F_n = \frac{n}{2^n}\sum_{\substack{k=0 \\ k \in \text{even}}}^{2\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{k}5^{k/2}\frac{2}{k+1}. \qquad (2)$$

Noting that

$$\int_{-1}^{1} x^k\, dx = \begin{cases} 0, & k \text{ odd}, \\ \dfrac{2}{k+1}, & k \text{ even}, \end{cases}$$

* Some might say obvious, but I have not seen it before and failed to find it anywhere including in Koshy's two volume set of texts which would be the obvious first place to look.

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we may rewrite (2) as

$$F_n = \frac{n}{2^n} \sum_{\substack{k=0 \\ k \in \text{even}}}^{2\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1}{k} 5^{k/2} \int_{-1}^{1} x^k \, dx. \tag{3}$$

Now when k is odd, the integral appearing in (3) is zero causing all such terms in the summation to be zero. The upper limit of the summation may therefore be changed from $2\left\lfloor \frac{n-1}{2} \right\rfloor$ to $(n - 1)$ without it affecting the value of the sum. With the changed upper limit for the sum, interchanging the summation with the integration produces

$$F_n = \frac{n}{2^n} \int_{-1}^{1} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(x\sqrt{5}\right)^k dx = \frac{n}{2^n} \int_{-1}^{1} \left(1 + x\sqrt{5}\right)^{n-1} dx.$$

Here we have made use of the binomial theorem, and completes the proof.

An immediate consequence of this result is Binet's formula for the Fibonacci numbers [1, p. 90, Theorem 5.5]. To see this, as the integral appearing in (1) is elementary, integrating gives

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + x\sqrt{5}}{2} \right)^n \right]_{-1}^{1} = \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5} + 1}{2} \right)^n - (-1)^n \left(\frac{\sqrt{5} - 1}{2} \right)^n \right]$$

$$= \frac{1}{\sqrt{5}} \left(\varphi^n - \frac{(-1)^n}{\varphi^n} \right),$$

which is Binet's formula for the nth Fibonacci numbers. Here φ denotes the golden ratio defined by $\left(\sqrt{5} + 1\right)/2$. Indeed, (1) can be seen as a thinly disguised form of Binet's formula for F_n with the connection becoming obvious if we write

$$F_n = \frac{n}{\sqrt{5}} \int_{-1/\varphi}^{\varphi} t^{n-1} dt,$$

and substitute $t = \left(1 + x\sqrt{5}\right)/2$.

From the integral representation of the Fibonacci numbers we used to arrive at Binet's formula, using this as a guide one can almost guess what the integral representation of the Fibonacci numbers for even orders ought to be. The clue here is to be found in the value for the square of the golden ratio, namely $\varphi^2 = \left(3 + \sqrt{5}\right)/2$. From this we conjecture

$$F_{2n} = \frac{n}{2^n} \int_{-1}^{1} \left(3 + x\sqrt{5}\right)^{n-1} dx, \tag{4}$$

for $n \geqslant 1$. In proving the conjecture true, replacing n with $2n$ in (1) yields

$$F_{2n} = \frac{2n}{2^{2n}} \int_{-1}^{1} \left(1 + x\sqrt{5}\right)^{2n-1} dx.$$

Substituting $t = \frac{1}{2\sqrt{5}} \left[\left(1 + x\sqrt{5}\right)^2 - 6 \right]$ produces $dx = dt / \sqrt{6 + 2t\sqrt{5}}$ while the limits of integration remain unchanged. Thus

$$F_{2n} = \frac{2n}{2^{2n}} \int_{-1}^{1} \left(\sqrt{6 + 2t\sqrt{5}} \right)^{2n-1} \frac{dt}{\sqrt{6 + 2t\sqrt{5}}} = \frac{n}{2^n} \int_{-1}^{1} \left(3 + t\sqrt{5}\right)^{n-1} dt,$$

as required to prove.

The integral representation of the Fibonacci numbers for even orders given by (4) is known in the literature. It first appears to have been given by Dilcher [3, p. 358, Eq. (10.2)] who obtained it using an approach that relied on the (Gaussian) hypergeometric function. Dilcher's result is presented as a trigonometric integral that is obtained by using a substitution of $x = \cos t$ in (4). For alternative integral representations of the Fibonacci numbers see [4, Eq. (1.2)], [5, Eq. (4.29), p. 132] and for the Fibonacci numbers for even orders see [4, Eq. (3.12)].

As an application of the integral representation given by (1) we will use it to show the well-known result for the generating function of the Fibonacci numbers of [1, pp. 236-237, Example 13.8]

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}, \qquad |x| < \frac{1}{\varphi}.$$

To prove this result, replacing F_n in the series with its integral representation we have

$$\sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} \frac{n}{2^n} \int_{-1}^{1} \left(1 + t\sqrt{5}\right)^{n-1} x^n dt = \frac{x}{2} \int_{-1}^{1} \sum_{n=0}^{\infty} n \left(\frac{x\left(1 + t\sqrt{5}\right)}{2}\right)^{n-1} dt.$$

The interchange made here between the integration and the summation is permissible due to Fubini's theorem [6, p. 55, Theorem 2.25]. Recalling the result

$$\sum_{n=0}^{\infty} n z^{n-1} = \frac{1}{(1 - z)^2}, \qquad |z| < 1,$$

we see that

$$\sum_{n=0}^{\infty} n \left(\frac{x\left(1 + t\sqrt{5}\right)}{2}\right)^{n-1} = \frac{1}{\left(1 - \frac{x\left(1 + t\sqrt{5}\right)}{2}\right)^2},$$

where $|x| < 1/\varphi$ for $t \in (-1, 1)$. Thus

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{2} \int_{-1}^{1} \frac{dt}{\left(1 - \frac{x\left(1 + t\sqrt{5}\right)}{2}\right)^2} = \frac{x}{2} \left[\frac{-4}{x\sqrt{5}\left(xt\sqrt{5} + x - 2\right)}\right]_{-1}^{1}$$

$$= -\frac{2}{\sqrt{5}} \left[\frac{1}{x + x\sqrt{5} - 2} - \frac{1}{x - x\sqrt{5} - 2}\right] = \frac{x}{1 - x - x^2},$$

as required to show. If integral representation (4) is used instead of (1), in a similar manner it can be shown that

$$\sum_{n=0}^{\infty} F_{2n} x^n = \frac{x}{x^2 - 3x + 1}, \qquad |x| < \frac{1}{\varphi^2},$$

a generating function for the even order Fibonacci numbers.

We end this Note by giving an integral representation of the Lucas numbers. The Lucas numbers are closely connected with the Fibonacci numbers. Recall if n is a non-negative integer, the nth Lucas number L_n is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$ with $L_0 = 2$ and $L_1 = 1$. One particularly useful relation between the Lucas numbers and the Fibonacci numbers is: $L_n = F_n + 2F_{n-1}$, $n \geqslant 1$ (see [1, p. 117, Ex. 5.44]). Combining this relation with the result given in (1) leads to a relatively simple integral representation of the Lucas numbers. It is

$$L_n = \frac{n}{2^n}\int_{-1}^{1}\left(5 + x\sqrt{5} - \frac{4}{n}\right)\left(1 + x\sqrt{5}\right)^{n-2}dx,$$

and is valid for $n \geqslant 1$. Many other integral representations of the Lucas numbers can be found by employing other known relations between the two numbers L_n and F_n which the reader may care to find.

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107.02 Collatz conjecture: coalescing orbits and conditions on a minimum counterexample

Introduction

Originally proposed by Lothar Collatz in the 1930s, the Collatz Conjecture, also known as the Collatz Problem, Syracuse Problem, and $3n + 1$ Conjecture, has become a notoriously difficult unsolved problem in mathematics. Much of its appeal is in the simplicity of the problem statement. The conjecture states that for every positive integer n, iterating