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Strictly Singular and Cosingular Multiplications

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Abstract. Let $L(X)$ be the space of bounded linear operators on the Banach space X. We study the strict singularity and cosingularity of the two-sided multiplication operators $S \mapsto ASB$ on $L(X)$, where $A, B \in L(X)$ are fixed bounded operators and X is a classical Banach space. Let $1 < p < \infty$ and $p \neq 2$. Our main result establishes that the multiplication $S \mapsto ASB$ is strictly singular on $L\big(L^p(0,1)\big)$ if and only if the non-zero operators $A, B \in L\big(L^p(0,1)\big)$ are strictly singular. We also discuss the case where X is a $\mathcal{L}^1$- or a $\mathcal{L}^\infty$-space, as well as several other relevant examples.

1 Introduction

Let X and Y be Banach spaces. Recall that the bounded linear operator $U \in L(X,Y)$ is *strictly singular* if the restriction $U|_M$ is not bounded below for any infinite-dimensional subspaces $M \subset X$. Furthermore, $U \in L(X,Y)$ is *strictly cosingular* if

$$Q_M U \colon X \to Y/M$$

is not surjective for any closed subspaces $M \subset Y$ such that $\dim(Y/M) = \infty$. Here $Q_M \colon Y \to Y/M$ is the quotient map. The class of strictly singular operators $X \to Y$ is denoted by $S(X,Y)$ and that of the strictly cosingular operators by $P(X,Y)$. These fundamental classes of operators, which contain the compact operators $K(X,Y)$, were introduced by Kato and Pełczyński, respectively. The classes S and P are of importance, *e.g.*, in the structure theory of Banach spaces and in Fredholm theory. Let $A \in L(X)$ be a fixed bounded linear operator. The left and right multiplication operators L_A and R_A on $L(X)$ are defined by $L_A(S) = AS$ and $R_A(S) = SA$ for $S \in L(X)$. The basic two-sided multiplication operator $L_A R_B \colon L(X) \to L(X)$ corresponding to $A, B \in L(X)$ is given by

$$S \mapsto L_A R_B(S) = ASB, \quad S \in L(X).$$

Substantial studies have been made of qualitative and spectral properties of the operators $L_A R_B$, as well as of their finite sums $\sum_{j=1}^n L_{A_j} R_{B_j}$ (see the surveys [C, F, M1, M2] as well as [ST1, ST2]). This paper focuses on the strict (co)singularity of the two-sided multiplications $L_A R_B$.

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Problem 1.1 *Let X be a given Banach space. Characterize the operators $A, B \in L(X)$ for which $L_A R_B$ is strictly (co)singular $L(X) \to L(X)$.*

The earliest related result is due to Vala [V]: if $A, B \in L(X)$ are non-zero operators, and X is an arbitrary Banach space, then $L_A R_B$ is a compact operator on $L(X)$ if and only if A and B are compact. By contrast, the conditions for $L_A R_B$ to be a weakly compact operator on $L(X)$ depend intrinsically on X, see, *e.g.*, [ST1], [R], [LS].

It is a simple observation that if $L_A R_B$ is strictly singular $L(X) \to L(X)$ and $A, B \neq 0$, then $A \in S(X)$ and $B^* \in S(X^*)$ (*cf.* Fact 2.1(ii) below). An analogous fact holds for strictly cosingular multiplications $L_A R_B$. We are here mainly interested in spaces X where the converse implications hold. (Similar questions can obviously be raised for the restriction $L_A R_B : K(X) \to K(X)$, as well as in many other settings.)

Problem 1.2

(i) *For which Banach spaces X is $L_A R_B$ strictly singular $L(X) \to L(X)$ whenever $A \in S(X)$ and $B^* \in S(X^*)$?*
(ii) *For which spaces X is $L_A R_B$ strictly cosingular $L(X) \to L(X)$ whenever $A \in P(X)$ and $B^* \in P(X^*)$?*

We obtain definitive results related to Problems 1.1 and 1.2 for certain classical Banach spaces X, where $S(X)$ or $P(X)$ admit concrete characterizations. This is the case, *e.g.*, if $X = L^p(0,1)$ $(1 \leq p < \infty)$ or if X is a $C(K)$-space. Let $1 < p < \infty$ and $p \neq 2$. The main result of this paper (Theorem 2.9) shows that for non-zero $A, B \in L\big(L^p(0,1)\big)$ the multiplication $L_A R_B$ is strictly singular on $L\big(L^p(0,1)\big)$ if and only if $A, B \in S\big(L^p(0,1)\big)$. The argument is fairly complicated, and it combines block diagonalization techniques applied in $L\big(L^p(0,1)\big)$ with classical estimates for unconditional basic sequences in $L^p(0,1)$. The delicacy of the situation is partly explained by the fact (see Section 4) that a similar result holds for $X = \ell^p \oplus \ell^q$, but fails for $X = \ell^p \oplus \ell^q \oplus \ell^r$ $(1 < p < q < r < \infty)$, where the latter space embeds into $L^p(0,1)$ for certain combinations of the exponents.

In Section 3 we consider Problems 1.1 and 1.2 for the classes of $\mathcal{L}^1$- and $\mathcal{L}^\infty$-spaces. The non-trivial fact (due to Bourgain [B2]) that certain spaces of bounded operators have the Dunford–Pettis property is a crucial ingredient in these cases. Section 4 contains several examples which illustrate how, in general, the solution to Problem 1.1 depends on X. [LT2], [LT3] are our standard sources for unexplained notation and concepts related to the theory of Banach spaces.

2 Strict Singularity of Multiplications on $L\big(L^p(0,1)\big)$

The main purpose of this paper is to solve Problem 1.1 for strict singularity in the case of $X = L^p(0,1)$, where $1 < p < \infty$ and $p \neq 2$. In fact, in Theorem 2.9 we establish that $L^p(0,1)$ satisfies Problem 1.2(i). We begin with some basic observations. Let E_1, E_2, E_3, E_4 be Banach spaces, and $A \in L(E_3, E_4)$, $B \in L(E_1, E_2)$ be fixed operators. Thus $S \mapsto ASB$ defines a bounded composition operator $L_A R_B \colon L(E_2, E_3) \to L(E_1, E_4)$. The following simple general facts are special cases of [LS, 2.1, 2.2, 2.3].

Fact 2.1

(i) *Suppose that* $A \in K(E_3, E_4)$ *and* $B^* \in S(E_2^*, E_1^*)$, *or that* $A \in S(E_3, E_4)$ *and* $B \in K(E_1, E_2)$. *Then* $L_A R_B$ *is strictly singular* $L(E_2, E_3) \to L(E_1, E_4)$.

(ii) *If* $A, B \neq 0$ *and* $L_A R_B$ *is strictly singular* $L(E_2, E_3) \to L(E_1, E_4)$, *then* $A \in S(E_3, E_4)$ *and* $B^* \in S(E_2^*, E_1^*)$.

Analogous results hold for the strict cosingularity of $L_A R_B$ (where one replaces S by P in (i) and (ii)). These results are also valid for the restriction $L_A R_B \colon K(E_2, E_3) \to K(E_1, E_4)$.

Fact 2.1 provides the relevant extremal conditions for the strict (co)singularity of $L_A R_B$, and Problem 1.2 asks for spaces X where the maximal conditions are sharp. We also note that the answer to Problem 1.1 follows from known results in the case of the sequence spaces ℓ^p $(1 \leq p < \infty)$ and c_0. Let $W(E, F)$ denote the class of weakly compact operators $E \to F$.

Example 2.2 Let $X = \ell^p$, where $1 \leq p < \infty$, or $X = c_0$. Then the following conditions are equivalent for non-zero $A, B \in L(X)$.

(i) $A, B \in K(X)$,
(ii) $L_A R_B$ is compact on $L(X)$,
(iii) $L_A R_B$ is strictly singular on $L(X)$,
(iv) $L_A R_B$ is strictly cosingular on $L(X)$.

If $X = \ell^1$ or $X = c_0$, then conditions (i)–(iv) are also equivalent to the weak compactness of $L_A R_B$ on $L(X)$. If $1 < p < \infty$ and $A \in K(\ell^p)$ is a non-zero operator, then L_A and R_A are weakly compact on $L(\ell^p)$, but they are neither strictly singular nor strictly cosingular.

Example 2.2 combines several known results, which we briefly outline. The equivalence of (i) and (ii) is a special case of [V]. The other equivalences follow from Fact 2.1 and the classical fact that $K(X) = S(X) = P(X)$ for $X = \ell^p$ $(1 \leq p < \infty)$ and $X = c_0$ (see [Pi, 5.1, 5.2]). The case $X = \ell^1$ also uses Gantmacher's theorem [Wo, II.C.6] and the equalities $W(\ell^1) = K(\ell^1)$ and $S(\ell^\infty) = P(\ell^\infty) = W(\ell^\infty)$ (see [LT2, 2.f.4]). Suppose next that $X = \ell^1$ or c_0, and that $L_A R_B$ is a weakly compact operator on $L(X)$. Then the weakly compact version of Fact 2.1(ii) (see [ST1, 2.1] or [LS]) yields that $A, B \in W(X) = K(X)$, so that $L_A R_B$ is compact. Let $1 < p < \infty$. It follows from [ST1, 3.2] that L_A and R_A are weakly compact operators on $L(\ell^p)$ whenever $A \in K(\ell^p)$.

Example 2.2 is rather exceptional, and in general there are plenty of non-compact strictly (co)singular multipliers $L_A R_B$ on $L(X)$ (see, *e.g.*, Theorems 2.9 and 3.2). Further examples, which illustrate the diversity of the conditions characterizing the strict (co)singularity of $L_A R_B$, are collected in Section 4.

Our study of Problems 1.1 and 1.2 for $X = L^p(0, 1)$ is in part motivated by the following fact:

- $U \notin S\big(L^p(0, 1)\big)$ *if and only if there is* $M \subset L^p(0, 1)$ *so that* $M \approx \ell^p$ *or* $M \approx \ell^2$, U *defines an isomorphism* $M \to UM$, *and* M *as well as* UM *are complemented in* $L^p(0, 1)$ (see [KP, Theorem 2 and Corollary 1] for $2 < p < \infty$ and [W, Theorem] for $1 < p < 2$).

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This fact (vaguely) suggests a similarity between the strict singularity of multiplications on $L\big(L^p(0,1)\big)$ and $L(\ell^p \oplus \ell^2)$. Examples in Section 4 show that the maximal condition from Problem 1.2 is sharp on $L(X)$ for $X = \ell^p \oplus \ell^q$, but that this is not so for $X = \ell^p \oplus \ell^q \oplus \ell^r$ (where $1 < p < q < r < \infty$) or $X = L^p(0,1) \oplus L^q(0,1)$ (where $p, q \in (1,\infty) \setminus \{2\}$ and $p \neq q$). On the other hand, since $\ell^p \oplus \ell^q \oplus \ell^r$ embeds isomorphically into $L^s(0,1)$ for certain combinations of $1 < p < q < r < \infty$ and $s \in (1,\infty)$, one might be tempted to think that $X = L^s(0,1)$ does not satisfy Problem 1.2(i) for $s \neq 2$. However, $L^s(0,1)$ cannot contain any complemented copy of $\ell^p \oplus \ell^q \oplus \ell^r$ (*cf.* [LT1, II.5.5] and [LT2, 2.c.14]), which still leaves open the possibility that the maximal condition holds.

It is convenient to divide the lengthy argument of Theorem 2.9 into several distinct steps. We begin by isolating several auxiliary results, some of which also apply to more general composition operators $L_A R_B \colon L(E_2, E_3) \to L(E_1, E_4)$, where $A \in L(E_3, E_4)$ and $B \in L(E_1, E_2)$. (This flexibility will be useful for the case $\ell^p \oplus \ell^q$ in Theorem 4.1). We always assume in addition that $E_1, \dots, E_4$ are reflexive Banach spaces having the unconditional Schauder bases $(e_n) \subset E_1$, $(f_n) \subset E_2$, $(g_n) \subset E_3$ and $(h_n) \subset E_4$, respectively. The following notation is fixed in this section: let $P_n^{(j)}$ stand for the natural basis projection of E_j onto the first n basis elements, $Q_n^{(j)} = I - P_n^{(j)}$ and $P_{m,n}^{(j)} \equiv P_m^{(j)} - P_n^{(j)}$. Here $m, n \in \mathbf{N}$, $n < m$ and $j = 1, 2, 3, 4$. Recall that the unconditional basis constant of the unconditional basis (e_n) is $\sup\big\{ \|M_\theta\| : \theta = (\theta_j) \in \{-1,1\}^{\mathbf{N}} \big\}$, where $M_\theta \in L(E_1)$ is defined by $M_\theta(\sum_{j=1}^\infty a_j e_j) = \sum_{j=1}^\infty \theta_j a_j e_j$ for $\sum_{j=1}^\infty a_j e_j \in E_1$.

We state a simple fact that will be used repeatedly in the sequel (and follows easily by finite rank approximation).

Lemma 2.3 *Suppose that E_1 and E_2 are reflexive Banach spaces having Schauder bases, and let $S \in K(E_1, E_2)$. Then $\lim_{m\to\infty} \|Q_m^{(2)} S\| = 0$ and $\lim_{m\to\infty} \|S Q_m^{(1)}\| = 0$.*

Let E_1 and E_2 be reflexive Banach spaces having the unconditional bases (e_n) and (f_n), respectively. By a block-diagonal sequence $(T_k) \subset K(E_1, E_2)$ we mean here that $P^{(2)}_{n_k, n_{k-1}} T_k P^{(1)}_{n_k, n_{k-1}} = T_k$, $k \in \mathbf{N}$, for some fixed strictly increasing sequence $(n_k) \subset \mathbf{N}$. Our next lemma is a first step towards building a special block diagonal sequence of operators associated to certain non-strictly singular multiplications on $L\big(L^p(0,1)\big)$.

Lemma 2.4 *Suppose that $E_1, \dots, E_4$ are reflexive Banach spaces having unconditional bases. Assume that $A \in L(E_3, E_4)$ and $B \in L(E_1, E_2)$ are such that*

$$L_A R_B\big(L(E_2, E_3)\big) \subset K(E_1, E_4).$$

Assume also that there is a sequence $(S_k) \subset L(E_2, E_3)$, an increasing sequence $(n_j) \subset \mathbf{N}$ and constants $c, c_1, c_2 > 0$ so that

(i) $c_1 \le \|S_k\| \le c_2$,
(ii) $Q_{n_k}^{(3)} S_k Q_{n_k}^{(2)} = S_k$,
(iii) $\|L_A R_B(S_k)\| = \|A S_k B\| > c$,

for $k \in \mathbf{N}$. Then there is a subsequence (S_{k_j}) so that $\big(L_A R_B(S_{k_j})\big) = (A S_{k_j} B)$ is equivalent (and as close as we wish) to a block-diagonal sequence $(T_j) \subset K(E_1, E_4)$.

Proof Observe first that

$$L_A R_B(S_k) = \lim_{m\to\infty} P_m^{(4)} [L_A R_B(S_k)] P_m^{(1)},$$

in the operator norm for each fixed S_k by Lemma 2.3, since $L_A R_B(S_k) \in K(E_1, E_4)$ by assumption. The construction of the desired subsequence (S_{k_j}) is by induction. Suppose that we have found $S_{k_1}, \dots, S_{k_{j-1}}$ as well as $1 = m_0 < m_1 < \cdots < m_{j-1}$, so that

$$\|L_A R_B(S_{k_r}) - (P_{m_r}^{(4)} - P_{m_{r-1}}^{(4)})[L_A R_B(S_{k_r})](P_{m_r}^{(1)} - P_{m_{r-1}}^{(1)})\| < \frac{1}{2K} \cdot 2^{-r-1} \tag{2.1}$$

for $r = 1, \dots, j-1$. Here $K > 0$ can be chosen in a uniform manner so that (2.1) then guarantees, by the usual perturbation argument (*cf.* [LT2, 1.a.9]), that the semi-normalized subsequence $\big(L_A R_B(S_{k_j})\big)$ will be equivalent to the block-diagonal sequence (T_j), where

$$T_j = (P_{m_j}^{(4)} - P_{m_{j-1}}^{(4)})[L_A R_B(S_{k_j})](P_{m_j}^{(1)} - P_{m_{j-1}}^{(1)}), \quad j \in \mathbf{N}.$$

Note that $K > 0$ will depend on the bound c and the basis constants of $(e_n) \subset E_1$ and $(h_n) \subset E_4$, but not on the particular sequence (m_j) defining the block-diagonal sequence (T_j) (*cf.* the proof of Lemma 4.3 below). We indicate how to find S_{k_j} and $m_j > m_{j-1}$ so that (2.1) holds for j. Note first that

$$\|P_r^{(4)} A S_k B\| \to 0 \text{ and } \|A S_k B P_r^{(1)}\| \to 0 \quad \text{as } k \to \infty, \tag{2.2}$$

for $r \in \mathbf{N}$. In fact, since $Q_{n_k}^{(3)} S_k Q_{n_k}^{(2)} = S_k$ for all k by (ii), Lemma 2.3 applied to the finite rank operator $P_r^{(4)} A$ yields

$$\|P_r^{(4)} A S_k\| \le \|S_k\| \cdot \|Q_{n_k}^{(3)}\| \cdot \|P_r^{(4)} A Q_{n_k}^{(3)}\| \to 0, \quad k \to \infty.$$

The second claim in (2.2) is seen in a similar manner from Lemma 2.3. In particular, (2.2) implies that the differences

$$\begin{aligned} L_A R_B(S_n) - Q_{m_{k-1}}^{(4)} [L_A R_B(S_n)] Q_{m_{k-1}}^{(1)} = {} & P_{m_{k-1}}^{(4)} [L_A R_B(S_n)] Q_{m_{k-1}}^{(1)} \\ & + Q_{m_{k-1}}^{(4)} [L_A R_B(S_n)] P_{m_{k-1}}^{(1)} \\ & + P_{m_{k-1}}^{(4)} [L_A R_B(S_n)] P_{m_{k-1}}^{(1)} \end{aligned}$$

can be made arbitrarily small by picking $n = k_j$ big enough. Since $L_A R_B(S_{k_j}) \in K(E_1, E_4)$ by assumption, we again use Lemma 2.3 to get $m_j > m_{j-1}$ so that

$$\|Q_{m_{j-1}}^{(4)} [L_A R_B(S_{k_j})] Q_{m_{j-1}}^{(1)} - P_{m_j}^{(4)} Q_{m_{j-1}}^{(4)} [L_A R_B(S_{k_j})] Q_{m_{j-1}}^{(1)} P_{m_j}^{(1)}\|$$

is as small as we like. This yields our claim. ■

We next record, for reader convenience, the version of a block diagonalization principle for operators between Banach spaces having unconditional bases which will be used repeatedly. A proof is contained in [LT2, 1.c.8 and Remark 1, p. 21].

Fact 2.5 (Unconditional Operator Blocking Principle) *Let E_1 and E_2 be Banach spaces having unconditional bases, and let (m_k) and (n_k) be strictly increasing sequences of natural numbers (where $m_0 = n_0 = 0$). Then there is $K > 0$ (depending only on the unconditional basis constants), so that*

$$\Big\| \sum_{k=1}^{\infty} P^{(2)}_{n_k,n_{k-1}} S P^{(1)}_{m_k,m_{k-1}} \Big\| \leq K\|S\|, \quad S \in L(E_1, E_2),$$

where the sum $\sum_{k=1}^{\infty} P^{(2)}_{n_k,n_{k-1}} S P^{(1)}_{m_k,m_{k-1}}$ converges in the strong operator topology.

We only formulate the remaining steps of the argument for $L^p(0,1)$. The Haar basis (h_n) is an unconditional basis for $L^p(0,1)$ (see [LT3, 2.c.5]), which will be the fixed basis in our argument. Let P_j and $Q_j = I - P_j$ stand for the related basis projections for $j \in \mathbf{N}$. The following consequence of Mazur's lemma will be our basic tool of approximation related to certain multiplications $L_A R_B$ on $L\big(L^p(0,1)\big)$.

Lemma 2.6 *Suppose that $A, B \in S\big(L^p(0,1)\big)$, where $1 < p < \infty$. Then for any $S \in L\big(L^p(0,1)\big)$ and $\varepsilon > 0$ there is a convex combination $\theta_1, \dots, \theta_r \geq 0$, $\sum_{j=1}^{r} \theta_j = 1$, and indices $m_1 < \cdots < m_r$, so that*

$$\Big\| ASB - A\Big(\sum_{j=1}^{r} \theta_j P_{m_j} S P_{m_j}\Big) B \Big\| < \varepsilon.$$

Proof Recall that $UV \in K(L^p(0,1))$ whenever $U, V \in S(L^p(0,1))$, see [Mi2, Teor. 7]. Thus the range $L_A R_B\big(L(L^p(0,1))\big) \subset K(L^p(0,1))$, so that $L_A R_B$ is a weakly compact operator on $L(L^p(0,1))$ by [ST1, Corollary 2.4]. Let $S \in L(L^p(0,1))$. It follows that $(AP_m SP_m B) = (L_A R_B(P_m SP_m))$ has a weakly convergent subsequence $(AP_{m_k} SP_{m_k} B)$ in $L(L^p(0,1))$. Note that the WOT-limit of $(AP_{m_k} SP_{m_k} B)$ is ASB, since

$$\langle x^*, AP_{m_k} SP_{m_k} Bx\rangle = \langle S^* P^*_{m_k} A^* x^*, P_{m_k} Bx\rangle \to \langle x^*, ASBx\rangle, \quad k \to \infty,$$

for $x \in L^p(0,1)$ and $x^* \in L^{p'}(0,1)$. Conclude that $AP_{m_k} SP_{m_k} B \xrightarrow{w} ASB$ in $L(L^p(0,1))$ as $k \to \infty$. The claim now follows from Mazur's lemma. ∎

We need a definition: The operator $\psi\colon L\big(L^p(0,1)\big) \to L\big(L^p(0,1)\big)$ is said to *reside* in the square $(m,n] \otimes (m,n] \subset \mathbf{N}^2$ if there is a convex combination $\theta_1, \dots, \theta_k \geq 0$, $\sum_{j=1}^{k} \theta_j = 1$, as well as $m < r_1 < \cdots < r_k \leq n$ so that

$$\psi(S) = \sum_{j=1}^{k} \theta_j P_{r_j,m} S P_{r_j,m}, \quad S \in L(L^p(0,1)). \tag{2.3}$$

Here $(m, n] = \{m+1, \dots, n\}$ and $P_{n,m} = P_n - P_m$ for $m < n$. We will say that ψ is a *convex projector* that resides in $(m, n] \otimes (m, n]$ (note that ψ is not a projection on $L(L^p(0,1))$ if there is more than one term in (2.3)). The following lemma isolates a technical ingredient needed for Proposition 2.8 below.

Lemma 2.7 *Let $1 < p < \infty$. Assume that $(S_j) \subset L\big(L^p(0,1)\big)$ is a normalized sequence so that*

(i) *(S_j) is an unconditional basic sequence in $L(L^p(0,1))$,*
(ii) *there is an increasing sequence $(n_j) \subset \mathbf{N}$ for which $P_{n_{j-1}} S_j P_{n_{j-1}} = 0$, $j \in \mathbf{N}$.*

Then for any sequence (ψ_j) of convex projectors where, for every $j \in \mathbf{N}$, ψ_j resides in $(n_{j-1}, n_j] \times (n_{j-1}, n_j]$, there is a constant $d > 0$ so that

$$\Big\| \sum_{j=1}^{\infty} a_j \psi_j(S_j) \Big\| \le d \Big\| \sum_{j=1}^{\infty} a_j S_j \Big\| \tag{2.4}$$

whenever $\sum_{j=1}^{\infty} a_j S_j$ converges in norm in $L(L^p(0,1))$.

Proof By approximation and WOT-convergence it will be enough to prove that there is $d > 0$ so that

$$\Big\| \sum_{j=1}^{N} a_j \psi_j(S_j) \Big\| \le d \Big\| \sum_{j=1}^{N} a_j S_j \Big\| \tag{2.5}$$

holds uniformly in $a_1, \dots, a_N$ for each N under the assumptions of the lemma. Observe next that it will be enough to establish (2.5) with a uniform constant $d > 0$ in the particular case, where each convex projector ψ_j has only one term in the representation (2.3). In fact, suppose that $\psi_j(S) = \sum_{k=1}^{l_j} \theta_{j,k} P_{r_{j,k};n_{j-1}} S P_{r_{j,k};n_{j-1}}$ for $S \in L\big(L^p(0,1)\big)$, where $j = 1, \dots, N$. We may then actually write

$$\sum_{j=1}^{N} a_j \psi_j(S_j) = \sum_{m_1=1}^{l_1} \cdots \sum_{m_N=1}^{l_N} \theta_{1,m_1} \theta_{2,m_2} \cdots \theta_{N,m_N} \Big[\sum_{j=1}^{N} a_j P_{r_{j,m_j};n_{j-1}} S_j P_{r_{j,m_j};n_{j-1}} \Big]. \tag{2.6}$$

To check (2.6) just note that

$$\sum_{m_1=1}^{l_1} \cdots \sum_{m_N=1}^{l_N} \theta_{1,m_1} \cdots \theta_{N,m_N} a_j P_{r_{j,m_j};n_{j-1}} S_j P_{r_{j,m_j};n_{j-1}} = a_j \sum_{m_j=1}^{l_j} \theta_{j,m_j} P_{r_{j,m_j};n_{j-1}} S_j P_{r_{j,m_j};n_{j-1}}$$

for each $j = 1, \dots, N$, by summing the coefficients θ_{k,m_k} for which $k \ne j$. By using the convex combination (2.6) one sees that it suffices to show that

$$\Big\| \sum_{j=1}^{N} a_j P_{r_{j,m_j};n_{j-1}} S_j P_{r_{j,m_j};n_{j-1}} \Big\| \le d \Big\| \sum_{j=1}^{N} a_j S_j \Big\|,$$

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where $d > 0$ is independent of the indices (m_j) and (r_j). Hence it follows, after applying Fact 2.5, that it will be enough to consider the case where each ψ_j has only one term.

We have thus reduced the verification of (2.5) to establishing that

$$\Big\| \sum_{j=1}^{N} a_j P_{n_j,n_{j-1}} S_j P_{n_j,n_{j-1}} \Big\| \leq d \Big\| \sum_{j=1}^{N} a_j S_j \Big\|$$

for all scalars $a_1, \dots, a_N$ and $N \in \mathbf{N}$. Put $T_{j,k} = P_{n_j,n_{j-1}} S_k P_{n_j,n_{j-1}}$ for $j,k \in \{1,\dots,N\}$, and note that $T_{j,k} = 0$ whenever $j < k$ by assumption (ii). We first apply the diagonal blocking map $S \mapsto \sum_{j=1}^{N} P_{n_j,n_{j-1}} S P_{n_j,n_{j-1}}$ to the operators $S(\varepsilon) = \sum_{k=1}^{N} \varepsilon_k a_k S_k$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{-1,1\}^N$ is any sign sequence. The uniform blocking principle (Fact 2.5), combined with the unconditionality of (S_j), yield that

$$\Big\| \sum_{k=1}^{N} a_k \varepsilon_k \Big(\sum_{j=1}^{N} T_{j,k} \Big) \Big\| \leq d_1 \Big\| \sum_{k=1}^{N} a_k S_k \Big\|, \tag{2.7}$$

where $d_1 > 0$ depends only on the unconditional basis constants of (S_j) and the Haar basis. Above $\sum_{k=1}^{N} a_k \varepsilon_k (\sum_{j=1}^{N} T_{j,k}) = \sum_{k=1}^{N} a_k \varepsilon_k (\sum_{j=k}^{N} T_{j,k})$. Moreover, the left hand operator norm in (2.7) only changes by a uniform constant if we multiply each $T_{j,k} = P_{n_j,n_{j-1}} S_k P_{n_j,n_{j-1}}$ by ε_j for $j = 1, \dots, N$, since the Haar basis is unconditional in $L^p(0,1)$. Hence it follows from (2.7) that

$$\Big\| \sum_{k=1}^{N} a_k \varepsilon_k \Big(\sum_{j=k}^{N} \varepsilon_j T_{j,k} \Big) \Big\| \leq d_2 \Big\| \sum_{k=1}^{N} a_k S_k \Big\| \tag{2.8}$$

for each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{-1,1\}^N$, where $d_2 > 0$ is a uniform constant. By averaging (2.8) over the signs $\varepsilon \in \{-1,1\}^N$, and noting that $\sum_{\varepsilon \in \{-1,1\}^N} \varepsilon_j \varepsilon_k = 0$ whenever $j \neq k$, we get the desired inequality $\| \sum_{k=1}^{N} a_k T_{k,k} \| \leq d_2 \| \sum_{k=1}^{N} a_k S_k \|$ for scalars $a_1, \dots, a_N$ and $N \in \mathbf{N}$. ■

Our next result provides an important reduction in our argument for Theorem 2.9. Here we reduce the study of certain $L_A R_B$ on $L\big(L^p(0,1)\big)$ to their restrictions to subspaces spanned by block-diagonal sequences. Its proof is based on Lemmas 2.4, 2.6 and 2.7.

Proposition 2.8 *Let $1 < p < \infty$. Assume that $A, B \in S(L^p(0,1))$ are such that the multiplication $L_A R_B$ is a non-strictly singular operator $L(L^p(0,1)) \to L(L^p(0,1))$. Then there is a normalized block-diagonal sequence $(S_k) \subset K(L^p(0,1))$, for which*

- *$L_A R_B$ is bounded below on $[S_k : k \in \mathbf{N}]$,*
- *$(L_A R_B(S_k)) = (A S_k B)$ is equivalent (and as close as we wish) to a block-diagonal sequence $(U_k) \subset K(L^p(0,1))$.*

Proof By assumption, there exist $c > 0$ and a closed infinite-dimensional subspace $M \subset L(L^p(0,1))$ so that

$$\|L_A R_B(S)\| = \|ASB\| \geq c\|S\|, \quad S \in M.$$

We first construct by induction a sequence $(V_k) \subset M$ so that

$$\|V_k\| = 1, \tag{2.9}$$

$$P_k V_k P_k = 0, \tag{2.10}$$

$$\|L_A R_B(V_k) - L_A R_B(Q_k V_k Q_k)\| \leq b \cdot 2^{-k} \tag{2.11}$$

for $k \in \mathbf{N}$. Here b satisfies $0 < b < \min\{\frac{c}{24K_p^2}, \frac{d}{8\|A\|\cdot\|B\|K_p^2}\}$, where $d > 0$ is the constant from (2.4) in Lemma 2.7 and $K_p > 0$ is the unconditional basis constant of the Haar basis in $L^p(0,1)$. Suppose that we have chosen operators $V_1, \dots, V_{k-1}$ satisfying (2.9)–(2.11). Consider the closed infinite-dimensional subspace

$$N_k = \{U \in M : P_k U P_k = 0\} \subset M.$$

Conditions (2.9) and (2.10) are satisfied if we agree to choose a normalized operator V_k from N_k. In addition, we may ensure that $V_k \in N_k$ satisfies

$$\|L_A R_B(V_k) - L_A R_B(Q_k V_k Q_k)\| \leq b \cdot 2^{-k}.$$

This is possible, since $N_k \subset M$ is an infinite-dimensional subspace and the operator

$$L_A R_B - L_{AQ_k} R_{Q_k B} = L_{AP_k} R_{Q_k B} + L_{AQ_k} R_{P_k B} + L_{AP_k} R_{P_k B}$$

is strictly singular on $L\big(L^p(0,1)\big)$ in view of Fact 2.2(i), the assumption $A, B \in S\big(L^p(0,1)\big)$, and the fact that $U \in S\big(L^p(0,1)\big)$ if and only if $U^* \in S\big(L^{p'}(0,1)\big)$ (see [Mi2, p. 19] and [W, Corollary 2]).

We next apply Lemma 2.4 to $(T_k) \subset L\big(L^p(0,1)\big)$, where $T_k = Q_k V_k Q_k$ for $k \in \mathbf{N}$. This is allowed, since $L_A R_B\big(L\big(L^p(0,1)\big)\big) \subset K\big(L^p(0,1)\big)$ whenever $A, B \in S\big(L^p(0,1)\big)$ (see the proof of Lemma 2.6). Moreover, $\|L_A R_B(Q_k V_k Q_k)\| > c/2$ by (2.11) and the choice of $b > 0$, so that $\|Q_k V_k Q_k\| > \frac{c}{2\|A\|\cdot\|B\|}$ for $k \in \mathbf{N}$. Hence Lemma 2.4 gives a subsequence (T_{m_k}) of (T_k) so that

(2.12) $P_{m_k} V_{m_k} P_{m_k} = 0$ for $k \in \mathbf{N}$, and $(L_A R_B(T_{m_k}))$ is equivalent (and as close as we wish) to a block-diagonal sequence $(U_k) \subset K(L^p(0,1))$.

We may assume above that $\|U_k\| > \frac{c}{2}$ for $k \in \mathbf{N}$. Note that the block-diagonal sequence (U_k) is an unconditional basic sequence (since the Haar basis is unconditional in $L^p(0,1)$), and that the basis constant of (U_k) is at most K_p^2 (*cf.* the proof of Lemma 4.3 below). If we assume that $\sum_{k=1}^{\infty} \|AT_{m_k}B - U_k\| < c/8K_p^2$, then by following the proof of [LT2, 1.a.9] it is seen that the basis constant of $\big(AT_{m_k}B\big)$ is at most $3K_p^2$ (note that here (U_k) is a semi-normalized basic sequence). By applying the argument in [LT2, 1.a.9] once more, it follows from (2.11) and the choice of $b > 0$ that $(AV_{m_k}B)$ is equivalent to $(AT_{m_k}B)$, and hence also to (U_k). Moreover, since $(V_{m_k}) \subset M$ and L_AR_B is bounded from below on M, we get from (2.12) that

$$(V_{m_k}) \text{ is an unconditional basic sequence in } L\big(L^p(0,1)\big). \tag{2.13}$$

We get the desired final sequence (S_j) from (V_{m_k}) with the help of Lemma 2.6. To this end we first inductively choose a subsequence $(R_j) = (V_{m_{k_j}})$ and a sequence (ψ_j) of convex projectors, where ψ_j resides in $(m_{k_j}, m_{k_{j+1}}] \times (m_{k_j}, m_{k_{j+1}}]$ for $j \in \mathbf{N}$, so that for $j \in \mathbf{N}$,

$$\|R_j\| = 1, \tag{2.14}$$

$$(R_j) \text{ is an unconditional basic sequence in } L\big(L^p(0,1)\big), \tag{2.15}$$

$$P_{m_{k_j}} R_j P_{m_{k_j}} = 0, \tag{2.16}$$

$$\big\| L_AR_B(R_j) - L_AR_B\big(\psi_j(R_j)\big) \big\| < b \cdot 2^{-j}. \tag{2.17}$$

Here the constant $b > 0$ is as above.

We outline the inductive choice of the subsequence (R_j). Condition (2.14) is clear, and (2.15) will be satisfied, by (2.13). Suppose that we have chosen $R_1, \dots, R_{j-1}$, indices $k_1 < \cdots < k_j$ and convex projectors $\psi_1, \dots, \psi_{j-1}$ satisfying (2.16) and (2.17). We put $R_j = V_{m_{k_j}}$, so that (2.16) holds. We next apply Lemma 2.6 to the operator $S = Q_{m_{k_j}} R_j Q_{m_{k_j}}$. We get an index $k_{j+1} > k_j$ and a convex projector ψ_j that resides in $(m_{k_j}, m_{k_{j+1}}] \times (m_{k_j}, m_{k_{j+1}}]$ so that

$$\big\| L_AR_B(Q_{m_{k_j}} R_j Q_{m_{k_j}}) - L_AR_B\big(\psi_j(Q_{m_{k_j}} R_j Q_{m_{k_j}})\big) \big\| < b \cdot 2^{-j-1}.$$

Since $\|L_AR_B(R_j) - L_AR_B(Q_{m_{k_j}} R_j Q_{m_{k_j}})\| < b \cdot 2^{-j-1}$ by (2.11), and one clearly has $\psi_j(Q_{m_{k_j}} R_j Q_{m_{k_j}}) = \psi_j(R_j)$, it follows that (2.17) holds for j.

We define $\hat{S}_j = \psi_j(R_j)$ for $j \in \mathbf{N}$. Note that $(\hat{S}_j) \subset K\big(L^p(0,1)\big)$ is a block-diagonal sequence. It remains to verify that L_AR_B is bounded below on the subspace $[\hat{S}_j : j \in \mathbf{N}]$. For this purpose we use Lemma 2.7, condition (2.17) and the fact that $[R_j : j \in \mathbf{N}] \subset M$, to obtain that

$$\begin{aligned}
\Big\| \sum_{j=1}^{\infty} a_j \hat{S}_j \Big\| &= \Big\| \sum_{j=1}^{\infty} a_j \psi_j(R_j) \Big\| \le d \Big\| \sum_{j=1}^{\infty} a_j R_j \Big\| \le \frac{d}{c} \Big\| \sum_{j=1}^{\infty} a_j L_AR_B(R_j) \Big\| \\
&\le \frac{2d}{c} \Big\| \sum_{j=1}^{\infty} a_j L_AR_B\big(\psi_j(R_j)\big) \Big\| = \frac{2d}{c} \Big\| L_AR_B\Big(\sum_{j=1}^{\infty} a_j \hat{S}_j \Big) \Big\|
\end{aligned}$$

whenever $\sum_{j=1}^{\infty} a_j \hat{S}_j$ converges in $K\big(L^p(0,1)\big)$. The next to last estimate follows from (2.17) by a standard perturbation argument (*cf.* [LT2, 1.a.9]), and the constant $d > 0$ is the one from (2.4). Finally,

$$\frac{c}{2\|A\|\,\|B\|} \le \|\psi_j(R_j)\| \le K_p^2$$

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for each j by construction. We obtain a norm-1 sequence (S_j) after normalizing $(\hat{S}_j)$ (this does not affect the convergence, since the above basic sequences are unconditional). ■

Let $K_p > 0$ be the unconditional basis constant of the Haar basis (h_n) in $L^p(0,1)$. Clearly any normalized block basic sequence (f_n) of (h_n) is unconditional with $K \leq K_p$, where K is the unconditional basis constant of (f_n). Let $2 < p < \infty$. We will need the following classical estimates: there are $A_p, B_p > 0$, so that any normalized block basic sequence (f_n) of (h_n) satisfies

$$\Big\| \sum_{n=1}^{\infty} a_n f_n \Big\| \leq A_p \Big(\sum_{n=1}^{\infty} |a_n|^2 \Big)^{1/2} \quad \text{for } (a_n) \in \ell^2, \tag{2.18}$$

$$B_p \Big(\sum_{n=1}^{\infty} |a_n|^p \Big)^{1/p} \leq \Big\| \sum_{n=1}^{\infty} a_n f_n \Big\| \quad \text{for } \sum_{n=1}^{\infty} a_n f_n \in [f_n : n \in \mathbf{N}]. \tag{2.19}$$

Above (2.18) can be deduced, *e.g.*, from [KP, Theorem 1f], while (2.19) can be seen, *e.g.*, by modifying the proof of an analogous fact [Ro, pp. 209–210] for $L^r(0,1)$ in the case $1 < r < 2$.

We are now ready for the main result of this paper, which characterizes the strictly singular multiplications $L_A R_B$ on $L\big(L^p(0,1)\big)$ for $1 < p < \infty$. The easy case $p = 2$ is contained in Example 2.2, so that we will assume here that $p \neq 2$. The proof of the implication (ii) $\Rightarrow$ (i) will require considerable work, even with Proposition 2.8 available. The fact which characterizes $U \notin S\big(L^p(0,1)\big)$ is not useful as such for this purpose.

Theorem 2.9 *Let $1 < p < \infty$ and $p \neq 2$. Then the following conditions are equivalent for non-zero $A, B \in L\big(L^p(0,1)\big)$.*

(i) *$L_A R_B$ is strictly singular $L(L^p(0,1)) \to L(L^p(0,1))$,*
(ii) *$A, B \in S(L^p(0,1))$.*

Proof (i) $\Rightarrow$ (ii). Fact 2.1(ii) implies that $A \in S(L^p(0,1))$ and $B^* \in S(L^{p'}(0,1))$, where p' is the conjugate exponent of p. It then follows from [W, Corollary 2] that also $B \in S(L^p(0,1))$.

(ii) $\Rightarrow$ (i). We will argue by contradiction. We begin by observing that it is enough to prove that $L_A R_B$ is a strictly singular operator $L(L^p(0,1)) \to L(L^p(0,1))$ whenever $A, B \in S(L^p(0,1))$ in the case $p > 2$. In fact, then the same result holds also in the case $1 < p < 2$. This is checked by using the linear isometry $S \mapsto S^*$ from $L(L^p(0,1))$ onto $L(L^{p'}(0,1))$, which transforms $L_A R_B$ to $L_{B^*} R_{A^*}$, and the fact that $U^* \in S(L^{p'}(0,1))$ if and only if $U \in S(L^p(0,1))$ (see [W, Corollary 2]). Thus we may (and will) assume that $2 < p < \infty$ in the remainder of the argument. The Haar basis (h_n) will be our fixed unconditional basis for $L^p(0,1)$, and $K_p > 0$ will denote its unconditional basis constant.

Assume that $A, B \in S(L^p(0,1))$ and suppose to the contrary that $L_A R_B$ is a non-strictly singular operator $L(L^p(0,1)) \to L(L^p(0,1))$. Proposition 2.8 implies that there is a normalized block-diagonal sequence $(S_k) \subset K(L^p(0,1))$ so that $L_A R_B$ defines a linear isomorphism $[S_k : k \in \mathbf{N}] \to [AS_kB : k \in \mathbf{N}]$, and $(L_A R_B(S_k)) = (AS_kB)$ is equivalent to (and as close as we wish to) some block-diagonal sequence $(U_k) \subset K(L^p(0,1))$. The above block-diagonal sequences are with respect to (h_n).

Fix $c > 0$ so that $\|L_A R_B(U)\| \geq c\|U\|$ for $U \in [S_k : k \in \mathbf{N}]$. We next combine the strict singularity of A and B with (2.18), (2.19), and the Kadec–Pełczyński dichotomy, in order to deduce the crucial observation that $[S_k : k \in \mathbf{N}] \subset K(L^p(0,1))$ is unique up to isomorphism in our situation.

Claim 1 *There is a subsequence of (S_k), still denoted by (S_k) for simplicity, and constants $C_p, C'_p > 0$ so that*

$$C_p\|(c_k)\|_s \leq \Big\| \sum_{k=1}^{\infty} c_k S_k \Big\| \leq C'_p\|(c_k)\|_s, \quad (c_k) \in \ell^s, \tag{2.20}$$

where s satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$ (that is, $s = \frac{2p}{p-2}$).

Proof of Claim 1 Observe first that there is a subsequence (S_{k_j}) of (S_k) and a block sequence $(x_j) \subset L^p(0,1)$ (with respect to (h_n)) so that $\|x_j\| = 1$ and $\|AS_{k_j}Bx_j\| \geq \frac{c}{2K_p}$ for $j \in \mathbf{N}$. The simple induction is based on Lemma 2.3. Indeed, suppose that we have found operators $S_{k_1}, \dots, S_{k_n}$ and blocks $x_1, \dots, x_n$ as desired. Fix $r \in \mathbf{N}$ so that $Q_r x_j = 0$ for $j = 1, \dots, n$. Note that $AS_jBP_r = AS_j(Q_{j-1}BP_r)$, where $\|Q_{j-1}BP_r\| \to 0$ as $j \to \infty$ by Lemma 2.3 (applied to the compact operator BP_r). Hence there is an index $k_{n+1} > k_n$ so that $\|AS_{k_{n+1}}BQ_r\| \geq c - \|AS_{k_{n+1}}BP_r\| \geq \frac{3c}{4}$. Pick a norm-1 element $y \in L^p(0,1)$ so that $\|AS_{k_{n+1}}BQ_r y\| > \frac{c}{2}$. By truncating the vector $Q_r y$, where $\|Q_r y\| \leq K_p$, in the Haar basis (h_n) we find after normalization a norm-1 block vector $x_{n+1} \in L^p(0,1)$ satisfying $\|AS_{k_{n+1}}Bx_{n+1}\| \geq \frac{c}{2K_p}$. For simplicity we retain the notation (S_j) for the subsequence (S_{k_j}) in the sequel.

Observe that $x_k \xrightarrow{w} 0$ in $L^p(0,1)$ as $k \to \infty$, since (x_k) are block vectors of (h_n) in $L^p(0,1)$. Similarly, (S_kBx_k) and (AS_kBx_k) are weak-null sequences in $L^p(0,1)$, where $\|Bx_k\| \geq \|S_kBx_k\| \geq \frac{c}{2K_p\|A\|}$ for $k \in \mathbf{N}$. By applying, if necessary, the Bessaga–Pełczyński selection theorem [LT2, 1.a.12], we may pass to a further subsequence of (x_k) (and consequently also of (S_k)), still denoted by (x_k), so that (x_k), (Bx_k), (S_kBx_k) and (AS_kBx_k) are basic sequences in $L^p(0,1)$.

We next invoke the Kadec–Pełczyński dichotomy [KP, Theorems 2 and 3, Corollary 1]: Any normalized basic sequence (f_n) of $L^p(0,1)$, where $2 < p < \infty$, has a subsequence (f_{n_k}), so that $[f_{n_k} : k \in \mathbf{N}] \subset L^p(0,1)$ is complemented, and (f_{n_k}) is either equivalent to the unit vector basis in ℓ^p or in ℓ^2. By repeated applications of the dichotomy we may ensure that the following properties hold (again by passing to further subsequences).

(2.21) If (y_k) stands for any one of the sequences (x_k), (Bx_k), (S_kBx_k) or (AS_kBx_k), then either (y_k) is equivalent to the unit vector basis in ℓ^2 or equivalent to the unit vector basis in ℓ^p. For simplicity we denote the above by $(y_k) \approx \ell^2$ or $(y_k) \approx \ell^p$.

(2.22) $[S_kBx_k : k \in \mathbf{N}]$ is complemented in $L^p(0,1)$.

The strict singularity of the restrictions $B\colon [x_k : k \in \mathbf{N}] \to [Bx_k : k \in \mathbf{N}]$ and $A\colon [S_kBx_k : k \in \mathbf{N}] \to [AS_kBx_k : k \in \mathbf{N}]$ enables us to reduce the number of possibilities in (2.21). In fact, we claim that

$$(x_k) \approx \ell^2,\ (Bx_k) \approx \ell^p,\ (S_kBx_k) \approx \ell^2, \text{ and } (AS_kBx_k) \approx \ell^p.$$

Indeed, observe first that if $(x_k) \approx \ell^2$ and $(Bx_k) \approx \ell^2$, or if $(x_k) \approx \ell^p$ and $(Bx_k) \approx \ell^p$, then B cannot define a strictly singular operator $[x_k : k \in \mathbf{N}] \to [Bx_k : k \in \mathbf{N}]$. Moreover, if $(x_k) \approx \ell^p$ and $(Bx_k) \approx \ell^2$, then B is compact $[x_k : k \in \mathbf{N}] \to [Bx_k : k \in \mathbf{N}]$ by Pitt's theorem (see [LT2, 2.c.3] and recall that $p > 2$). This would then imply the contradiction that $\|Bx_k\| \to 0$ as $k \to \infty$, since (x_k) is weakly null. A similar argument applies to the sequences (S_kBx_k) and (AS_kBx_k).

We next show that the resulting (sub)sequence (S_k) satisfies Claim 1. Let $(c_k) \in \ell^s$, where s satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$. We first verify that the right hand inequality in (2.20) follows from (2.18) and (2.19). (Actually, the argument shows that the upper ℓ^s-estimate in (2.20) holds for any normalized sequence of block-diagonal operators in $K(L^p(0,1))$.) This inequality also implies that the norm convergent sum $\sum_{k=1}^{\infty} c_kS_k$ defines a compact operator on $L^p(0,1)$ for $(c_k) \in \ell^s$.

Let (R_k) stand for a fixed sequence of disjoint basis projections onto the supports (with respect to (h_n)) in $L^p(0,1)$ of the block-diagonal operators (S_k), in the sense that $S_kR_k = S_k$ for $k \in \mathbf{N}$. Suppose that $x \in L^p(0,1)$. Since (h_n) is an unconditional basis for $L^p(0,1)$ we get from unconditionality and (2.19) that

$$\Big(\sum_{k=1}^{\infty} \|R_kx\|^p\Big)^{1/p} \le B_p^{-1}K_p\|x\|, \quad x \in L^p(0,1). \tag{2.23}$$

Since $S_k = S_kR_k$ for each k, we get from (2.23) and (2.18) together with Hölder's inequality (with $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$) that

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} c_kS_kx\Big\| &= \Big\|\sum_{k=1}^{\infty} c_kS_kR_kx\Big\| \le A_p\Big(\sum_{k=1}^{\infty} |c_k|^2\|S_kR_kx\|^2\Big)^{1/2} \\ &\le A_p\Big(\sum_{k=1}^{\infty} |c_k|^s\Big)^{1/s} \cdot \Big(\sum_{k=1}^{\infty} \|R_kx\|^p\Big)^{1/p} \le A_pB_p^{-1}K_p\Big(\sum_{k=1}^{\infty} |c_k|^s\Big)^{1/s}\|x\|.\end{aligned}$$

The proof of the left-hand inequality in (2.20) needs more care. According to (2.22) there is a linear projection P of $L^p(0,1)$ onto $[S_kBx_k : k \in \mathbf{N}]$. We know that the restriction of $P(\sum_{k=1}^{\infty} c_kS_k)$ defines a compact operator $[Bx_k : k \in \mathbf{N}] \to [S_kBx_k : k \in \mathbf{N}]$ for $(c_k) \in \ell^s$. To circumvent the minor inconvenience that the restriction of $P(\sum_{k=1}^{\infty} c_kS_k)$ to $[Bx_k : k \in \mathbf{N}]$ need not be a block-diagonal operator (the off-diagonal terms S_kBx_j are not known for $k \neq j$), we first apply the unconditional blocking principle (Fact 2.5) to $P(\sum_{k=1}^{\infty} c_kS_k)$ with respect to the unconditional bases $(Bx_k) \approx \ell^p$ and $(S_kBx_k) \approx \ell^2$. For this purpose, let $\Delta\colon [Bx_k : k \in \mathbf{N}] \to [S_kBx_k :$

$k \in \mathbf{N}]$ denote the resulting diagonal operator: $\Delta(\sum_{j=1}^{\infty} a_j Bx_j) = \sum_{j=1}^{\infty} c_j a_j S_j Bx_j$ for $\sum_{j=1}^{\infty} a_j Bx_j \in [Bx_k : k \in \mathbf{N}]$. Thus

$$\|\Delta\| \le K \cdot \Big\| P\Big(\sum_{k=1}^{\infty} c_k S_k\Big) : [Bx_k : k \in \mathbf{N}] \to [S_k Bx_k : k \in \mathbf{N}]\Big\| \tag{2.24}$$

for some uniform constant $K > 0$. Fix constants $d_1, d_2 > 0$ so that

$$\Big\|\sum_{k=1}^{\infty} a_k Bx_k\Big\| \le d_1 \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p}, \quad (a_k) \in \ell^p, \tag{2.25}$$

$$d_2 \Big(\sum_{k=1}^{\infty} |b_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^{\infty} b_k S_k Bx_k\Big\|, \quad (b_k) \in \ell^2. \tag{2.26}$$

Suppose that $(a_j) \in \ell^p$ satisfies $\|(a_j)\|_p \le 1/d_1$. The estimates (2.24)–(2.26) yield that

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} c_k S_k\Big\| &\ge \|P\|^{-1} \Big\| P\Big(\sum_{k=1}^{\infty} c_k S_k\Big) : [Bx_k : k \in \mathbf{N}] \to [S_k Bx_k : k \in \mathbf{N}]\Big\| \\ &\ge \|P\|^{-1} K^{-1} \|\Delta\| \ge \|P\|^{-1} K^{-1} \Big\| \Delta\Big(\sum_{j=1}^{\infty} a_j Bx_j\Big)\Big\| \\ &= \|P\|^{-1} K^{-1} \Big\|\sum_{k=1}^{\infty} c_k a_k S_k Bx_k\Big\| \ge d_2 \|P\|^{-1} K^{-1} \Big(\sum_{k=1}^{\infty} |c_k|^2 |a_k|^2\Big)^{1/2}.\end{aligned}$$

By taking the supremum of the right-hand side over $(a_j) \in \frac{1}{d_1} B_{\ell^p}$ we get that

$$\Big\|\sum_{k=1}^{\infty} c_k S_k\Big\| \ge d_1^{-1} \|P\|^{-1} K^{-1} d_2 \|(c_k)\|_s,$$

where s satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$. The above inequality is seen from a standard duality argument and Hölder's inequality. This completes the proof of Claim 1. ■

To resume the proof of the implication (ii) $\Rightarrow$ (i) recall that (following our initial work) $L_A R_B$ is an isomorphism $[S_k : k \in \mathbf{N}] \to [AS_k B : k \in \mathbf{N}]$, where we may ensure that $\lim_k \|AS_k B - U_k\| = 0$ as quickly as we wish for some block-diagonal sequence $(U_k) \subset K\big(L^p(0,1)\big)$. We have

$$\Big\|\sum_{k=1}^{\infty} c_k AS_k B\Big\| \ge c' \Big(\sum_{k=1}^{\infty} |c_k|^s\Big)^{1/s}, \quad (c_k) \in \ell^s,$$

by Claim 1, where $c' = cC_p > 0$. Fix a sequence (R_k) of disjoint basis projections in $L^p(0,1)$ onto the supports (with respect to (h_n)) of the block-diagonal sequence

(U_k), so that $U_kR_k = U_k$ for $k \in \mathbf{N}$. For technical reasons we actually need a lower estimate of $\|\sum_{k=1}^{\infty} c_kAS_kBR_k\|$. We may ensure above that $(\sum_{k=1}^{\infty} \|U_k - AS_kB\|^{s'})^{1/s'} < c'/6K_p$, where s' is the dual exponent of s. Write

$$\sum_{k=1}^{\infty} c_kAS_kBR_k = \sum_{k=1}^{\infty} c_kAS_kB + \sum_{k=1}^{\infty} c_k(AS_kB - U_k)R_k + \sum_{k=1}^{\infty} c_k(U_k - AS_kB)$$

and use the Hölder inequality to get that

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} c_kAS_kBR_k\Big\| &\geq c'\Big(\sum_{k=1}^{\infty}|c_k|^s\Big)^{1/s} - 2K_p\Big(\sum_{k=1}^{\infty}\|U_k - AS_kB\|^{s'}\Big)^{1/s'}\Big(\sum_{k=1}^{\infty}|c_k|^s\Big)^{1/s}\\ &\geq \frac{2c'}{3}\Big(\sum_{k=1}^{\infty}|c_k|^s\Big)^{1/s},\end{aligned}$$

for $(c_k) \in \ell^s$. This estimate yields that

$$\Big\|\sum_{k\in J} S_kBR_k\Big\| \geq \frac{2c'}{3}\|A\|^{-1}\cdot \operatorname{card}(J)^{1/s} \tag{2.27}$$

for all finite subsets $J \subset \mathbf{N}$.

The strategy of the rest of the argument is to derive a contradiction from the strict singularity of B on $L^p(0,1)$ together with the following technical consequence of (2.27).

Claim 2 *There is $(m_k) \subset \mathbf{N}$ and a normalized block sequence $(x_k) \subset L^p(0,1)$ so that*

$$m_{k+1} - m_k > k - 1, \tag{2.28}$$

$$\|T_kx_k\| \geq \frac{c'}{2K_p\|A\|}\cdot k^{1/s} \tag{2.29}$$

for $T_k = \sum_{j=m_k}^{m_k+k-1} S_jBR_j$ and $k \in \mathbf{N}$.

Proof of Claim 2 The induction is again based on Lemma 2.3. Suppose that we have found finite sums $T_1,\dots,T_k$, integers $m_1 < \cdots < m_k$ and block vectors $x_1,\dots,x_k \in L^p(0,1)$ satisfying (2.28) and (2.29). Fix $r \in \mathbf{N}$ so that $Q_rx_j = 0$ for $j = 1,\dots,k$. We have $\sum_{j=n}^{n+k} S_jBR_jP_r = \sum_{j=n}^{n+k} S_j(Q_{n-1}BP_r)R_j$, where $\|Q_{n-1}BP_r\| \to 0$ as $n \to \infty$ by Lemma 2.3 (note that $R_jP_r = P_rR_j$ for each j and r, since $R_j = P_{n_{j+1}} - P_{n_j}$ for a suitable sequence (n_j)). Hence

$$\Big\|\sum_{j=n}^{n+k} S_jBR_jP_r\Big\| \leq (k+1)K_p\|Q_{n-1}BP_r\| \to 0 \quad \text{as } n \to \infty.$$

Moreover, $\|\sum_{j=n}^{n+k} S_j B R_j Q_r\| \geq \frac{2c'}{3\|A\|} \cdot (k+1)^{1/s} - \|\sum_{j=n}^{n+k} S_j B R_j P_r\|$ by (2.27). Hence there is $n = m_{k+1} > m_k + k$ and, after truncation, a normalized block $x_{k+1} \in L^p(0,1)$ satisfying $Q_r x_{k+1} = x_{k+1}$ and $\|T_{k+1} x_{k+1}\| = \|\sum_{j=m_{k+1}}^{m_{k+1}+k} S_j B R_j x_{k+1}\| \geq \frac{c'}{2K_p\|A\|} \cdot (k+1)^{1/s}$. This completes the proof of Claim 2. ■

To continue the main argument we fix ε so that $0 < \varepsilon < \frac{c'}{4\|A\|} \cdot \frac{B_p}{K_p^2 A_p}$ (where $A_p, B_p > 0$ are as in (2.18) and (2.19)). We estimate the growth of $\|T_k x_k\|$ for $k \in \mathbf{N}$, where $T_k x_k = \sum_{j=m_k}^{m_k+k-1} S_j B R_j x_k$ are from Claim 2. Define $J_1(k)$ and $J_2(k)$ by

$$J_1(k) = \left\{ j \in \{m_k, \dots, m_k + k - 1\} : \|BR_j x_k\| > \varepsilon \|R_j x_k\| \right\},$$
$$J_2(k) = \left\{ j \in \{m_k, \dots, m_k + k - 1\} : \|BR_j x_k\| \leq \varepsilon \|R_j x_k\| \right\},$$

for $k \in \mathbf{N}$, so that $\operatorname{card}\big(J_2(k)\big) \leq k$. For each $k \in \mathbf{N}$ we have

$$\|T_k x_k\| \leq \Big\| \sum_{j \in J_1(k)} S_j B R_j x_k \Big\| + \Big\| \sum_{j \in J_2(k)} S_j B R_j x_k \Big\| \equiv \Sigma_1 + \Sigma_2,$$

where Σ_1 and Σ_2 will be estimated separately.

The term Σ_2 is handled by applying (2.18) and (2.19) to the unconditional block vector sums $\sum_{j \in J_2(k)} S_j B R_j x_k$ and $\sum_{j \in J_2(k)} R_j x_k$. We get from Hölder's inequality (with $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$), $\|S_j\| = 1 = \|x_k\|$ and the definition of $J_2(k)$ that

$$\begin{aligned}
(2.30)\quad \Sigma_2 &\leq A_p \Big(\sum_{j \in J_2(k)} \|S_j B R_j x_k\|^2 \Big)^{1/2} \leq A_p \Big(\sum_{j \in J_2(k)} \|B R_j x_k\|^2 \Big)^{1/2} \\
&\leq A_p \varepsilon \Big(\sum_{j \in J_2(k)} \|R_j x_k\|^2 \Big)^{1/2} \leq A_p \varepsilon \Big(\sum_{j \in J_2(k)} \|R_j x_k\|^p \Big)^{1/p} \cdot \operatorname{card}\big(J_2(k)\big)^{1/s} \\
&\leq A_p \varepsilon \Big(\sum_{j=m_k}^{m_k+k-1} \|R_j x_k\|^p \Big)^{1/p} \cdot k^{1/s} \leq A_p B_p^{-1} \varepsilon \Big\| \sum_{j=m_k}^{m_k+k-1} R_j x_k \Big\|_p \cdot k^{1/s} \\
&\leq A_p B_p^{-1} K_p \varepsilon \|x_k\| \cdot k^{1/s} = A_p B_p^{-1} K_p \varepsilon \cdot k^{1/s}.
\end{aligned}$$

To handle Σ_1 we next formulate the specialized instance of the extraction of basic sequences in $L^p(0,1)$ that we will need here to complete the argument.

Lemma 2.10 *Let $2 < p < \infty$ and $U \in S\big(L^p(0,1)\big)$. Suppose that $(y_k) \subset L^p(0,1)$ is a normalized sequence such that*

(i) *$y_k \xrightarrow{w} 0$ as $k \to \infty$,*
(ii) *$\|U y_k\| \geq a > 0$ for $k \in \mathbf{N}$.*

Then there is a constant $d > 0$ (which is allowed to depend on $a > 0$, $U \in S\big(L^p(0,1)\big)$, and the sequence (y_k)), so that $\|y_k\|_{L^2(0,1)} \geq d$ for $k \in \mathbf{N}$.

We complete the main argument before indicating how to get Lemma 2.10 from [KP]. The term Σ_1 is estimated by applying the preceding lemma to the sequence (BR_jx_k), where $j \in \bigcup_{k\in\mathbf{N}} J_1(k)$. Here $\|B(\frac{R_jx_k}{\|R_jx_k\|})\| > \varepsilon$ for these j and k, where $B \in S\big(L^p(0,1)\big)$ and the sequence $(\frac{R_jx_k}{\|R_jx_k\|})$ determined by $\bigcup_{k\in\mathbf{N}} J_1(k)$ is weakly null. Thus Lemma 2.10 yields a constant $d > 0$ (which is allowed to depend on B, ε and $(\frac{R_jx_k}{\|R_jx_k\|})$), so that $\|R_jx_k\|_{L^2(0,1)} \geq d\|R_jx_k\|$ for all $j \in \bigcup_{k\in\mathbf{N}} J_1(k)$. In particular, $\|BR_jx_k\| \leq d^{-1}\|B\| \cdot \|R_jx_k\|_{L^2(0,1)}$ for these j and k. Thus we get from (2.18) and $\|S_j\| = 1 = \|x_k\|$ that

(2.31)

$$\begin{aligned}\Sigma_1 &\leq A_p\Big(\sum_{j\in J_1(k)} \|S_jBR_jx_k\|^2\Big)^{1/2} \leq A_pd^{-1}\|B\|\Big(\sum_{j\in J_1(k)} \|R_jx_k\|^2_{L^2(0,1)}\Big)^{1/2}\\ &\leq A_pd^{-1}\|B\|\Big(\sum_{j=m_k}^{m_k+k-1} \|R_jx_k\|^2_{L^2(0,1)}\Big)^{1/2} = A_pd^{-1}\|B\| \cdot \Big\|\sum_{j=m_k}^{m_k+k-1} R_jx_k\Big\|_{L^2(0,1)}\\ &\leq A_pd^{-1}\|B\| \cdot \Big\|\sum_{j=m_k}^{m_k+k-1} R_jx_k\Big\| \leq A_pK_pd^{-1}\|B\|.\end{aligned}$$

In (2.31) we also used the $L^2(0,1)$-orthogonality of the (Haar) block vectors (R_jx_k) for $j = m_k, \dots, m_k + k - 1$, as well as unconditionality. Finally, by combining (2.30) and (2.31) with Claim 2 we get the uniform estimates

$$\frac{c'}{2K_p\|A\|} \cdot k^{1/s} \leq \|T_kx_k\|_p \leq A_pK_pd^{-1}\|B\| + A_pB_p^{-1}K_p\varepsilon \cdot k^{1/s}, \quad k \in \mathbf{N}.$$

Since we have fixed $0 < \varepsilon < \frac{c'}{4\|A\|} \cdot \frac{B_p}{A_pK_p^2}$, the preceding inequalities are incompatible for all large enough k. This contradiction completes the proof of the implication (ii) $\Rightarrow$ (i), and hence of Theorem 2.9. ■

Proof of Lemma 2.10 Suppose to the contrary that $\liminf_{k\to\infty} \|y_k\|_{L^2(0,1)} = 0$, and pick a subsequence (y_{k_r}) so that $\|y_{k_r}\|_{L^2(0,1)} < 2^{-r}$ for $r \in \mathbf{N}$. Put

$$M(p,\delta) = \Big\{x \in L^p(0,1) : \big|\{t \in [0,1] : |x(t)| \geq \delta\|x\|_p\}\big| \geq \delta\Big\}$$

for $\delta > 0$. Since $\|y_{k_r}\|_{L^2(0,1)} \leq 2^{-r}\|y_{k_r}\|_p$ for each r, it follows from [KP, Theorem 1.1d] that $y_{k_r} \notin M(p,(2^{-r})^{2/3})$ for $r \in \mathbf{N}$. Hence the argument in [KP, Theorem 2] (see also [Wo, p. 327]) yields that (y_{k_r}) contains a basic subsequence, still denoted by (y_{k_r}), so that (y_{k_r}) is equivalent to the unit vector basis in ℓ^p. Denote this by $(y_{k_r}) \approx \ell^p$. Since (Uy_{k_r}) is weakly null by (i) and $\|Uy_{k_r}\|_p \geq a$ for $r \in \mathbf{N}$ by (ii), we may assume that (Uy_{k_r}) is a basic sequence. By the Kadec–Pełczyński dichotomy we may further assume that either $(Uy_{k_r}) \approx \ell^2$ or $(Uy_{k_r}) \approx \ell^p$. Since $U \in S\big(L^p(0,1)\big)$, it is then easy to check that both alternatives are impossible (see the argument following (2.22)). ■

Remarks (i) One may add to Theorem 2.9 the equivalent condition that $L_A R_B$ is strictly singular $K\big(L^p(0,1)\big) \to K\big(L^p(0,1)\big)$. This is seen by applying Fact 2.1(ii) to the restriction $L_A R_B$ on $K\big(L^p(0,1)\big)$, using that

$$(L_A R_B|_{K(L^p(0,1))})^{**} = L_A R_B \colon L(L^p(0,1)) \to L(L^p(0,1))$$

in trace-duality (see Section 4) and the facts that $U \in S(E)$ (respectively, $U \in P(E)$) whenever $U^* \in P(E^*)$ (respectively, $U^* \in S(E^*)$).

(ii) The maximal condition from Theorem 2.9 remains valid for multiplications $L_A R_B$ on $L(X)$, where X is a complemented subspace of $L^p(0,1)$ (*cf.* the argument for Example 4.5). This applies, *e.g.*, to $X = \ell^p \oplus \ell^2$ and $X = (\bigoplus_{\mathbf{N}} \ell^2)_{\ell^p}$. Apart from the case $X = \ell^p \oplus \ell^q$ in Section 4 we have not pursued the question to which classical spaces X the ideas of Theorem 2.9 might be extended.

Recall that $S\big(L^p(0,1)\big) = P\big(L^p(0,1)\big)$ for $1 < p < \infty$ and $p \neq 2$, see [W, Theorem]. The strictly singular result, combined with trace-duality, does not by itself yield a strictly cosingular version of Theorem 2.9, because of a general lack of duality between strict singularity and cosingularity. However, we conjecture that the answers to the following problems are in the affirmative.

Problem *Let $1 < p < \infty$, $p \neq 2$, and suppose that $A, B \in S(L^p(0,1)) = P(L^p(0,1))$.*

(i) *Is $L_A R_B$ strictly cosingular $L(L^p(0,1)) \to L(L^p(0,1))$?*

(ii) *Is $L_A R_B$ strictly cosingular $K\big(L^p(0,1)\big) \to K\big(L^p(0,1)\big)$?*

3 The Case of $\mathcal{L}^1$- and $\mathcal{L}^\infty$-Spaces

In this section we study Problems 1.1 and 1.2 for multiplications $L_A R_B$ on $L(X)$, where X belongs to the class of $\mathcal{L}^1$- or $\mathcal{L}^\infty$-spaces (our results will apply to classical non-reflexive spaces such as $L^1(0,1)$, $C(0,1)$ and $\ell^\infty \approx L^\infty(0,1)$). Many facts motivate this study. Firstly, there are characterizations of the non-strictly (co)singular operators on X for many $\mathcal{L}^1$- or $\mathcal{L}^\infty$-spaces X, which suggest that the strictly (co)singular multipliers on $L(X)$ could also be identified explicitly. For instance, $U \notin W(L^1(0,1))$ if and only if there is $M \subset L^1(0,1)$ so that $M \approx \ell^1$, U defines an isomorphism $M \to UM$, and both M and UM are complemented in $L^1(0,1)$ (see [P, Theorem II.1] or [Wo, III.C.12]). Here the strictly (co)singular multipliers on $L(\ell^1)$ are known from Example 2.2. Secondly, the weakly compact multiplications on $L(X)$ are known for $\mathcal{L}^1$- or $\mathcal{L}^\infty$-spaces.

Fact 3.1 ([R, Proposition 2], [ST1, 2.11]) *Let X be a $\mathcal{L}^1$- or $\mathcal{L}^\infty$-space, and $A, B \in L(X)$ be non-zero operators. Then $L_A R_B$ is weakly compact on $L(X)$ if and only if $A, B \in W(X)$.*

Our results in the case of $\mathcal{L}^1$- or $\mathcal{L}^\infty$-spaces can be considered as applications of Fact 3.1 and a result of Bourgain [B2] about the Dunford-Pettis property of certain spaces of bounded operators. Recall that the Banach space X has the *Dunford–Pettis property* (DPP) if for any Banach space Y and any weakly compact $S \in W(X,Y)$ one has $\|Sx_n\| \to 0$ as $n \to \infty$ for all weak-null sequences $(x_n) \subset X$. We refer to

[LT1, Chapter II.5] or [B1, Chapter 1] for the definitions and the basic properties of $\mathcal{L}^p$-spaces. Any $\mathcal{L}^1$- or $\mathcal{L}^\infty$-space has the DPP, see [LT1, II.4.30 and II.5.7].

We first characterize the strictly singular and cosingular multiplications on $L(X)$, where X is a $\mathcal{L}^1$-space. The result below applies, *e.g.*, to $L^1(0,1)$, $C(0,1)^*$ and $(\ell^\infty)^*$.

Theorem 3.2 *Let X be a $\mathcal{L}^1$-space and $A, B \in L(X)$ be non-zero operators. Then the following conditions are equivalent.*

(i) *$L_A R_B$ is strictly singular on $L(X)$,*
(ii) *$L_A R_B$ is strictly cosingular on $L(X)$,*
(iii) *$L_A R_B$ is weakly compact on $L(X)$,*
(iv) *$A, B \in S(X) = P(X) = W(X)$,*
(v) *$A \in S(X)$ and $B^* \in S(X^*)$,*
(vi) *$A \in P(X)$ and $B^* \in P(X^*)$.*

Proof The implications (i) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (vi) follow from Fact 2.1(ii). Moreover, the implications (v) $\Rightarrow$ (iv) and (vi) $\Rightarrow$ (iv) are easy consequences of duality and the equalities $W(X) = S(X) = P(X)$ for $\mathcal{L}^1$-spaces X. To recall these equalities note first that $W(X) \subset S(X) \cap P(X)$, since $\mathcal{L}^1$-spaces have the DPP. Moreover, any $\mathcal{L}^1$-space X is weakly sequentially complete, see, *e.g.*, [B1, 1.29]. It is then a known consequence (see, *e.g.*, [B1, 1.6]) of Rosenthal's ℓ^1-theorem that $U \notin S(X) \cup P(X)$ whenever $U \notin W(X)$.

Conversely, if $B \in W(X)$, then $B^* \in W(X^*) \subset S(X^*)$ by the DPP of the $\mathcal{L}^\infty$-space X^*. The implication (iv) $\Rightarrow$ (vi) is checked in a similar manner. The conditions (iii) and (iv) are equivalent by Fact 3.1. It remains to prove that (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). Assume that $A, B \in W(X)$. Thus $A^{**}, B^{**} \in W(X^{**})$, where the $\mathcal{L}^1$-space X^{**} has the DPP, so that $L_{A^{**}} R_{B^{**}} \colon L(X^{**}) \to L(X^{**})$ is weakly compact according to Fact 3.1. It follows that $L_{A^{**}} R_{B^{**}}$ is also completely continuous, since $L(X^{**})$ itself has the DPP (this fact is verified separately in Lemma 3.3(i) below). This means that $L_{A^{**}} R_{B^{**}}$ maps weak-null sequences of $L(X^{**})$ to norm-null sequences. Suppose that $(T_j) \subset L(X)$ is a weak-null sequence, so that $T_j^{**} \xrightarrow{w} 0$ in $L(X^{**})$ as $j \to \infty$ (as $S \mapsto S^{**}$ is $w - w$ continuous). The complete continuity of $L_{A^{**}} R_{B^{**}}$ implies that $\|AT_j B\| = \|A^{**} T_j^{**} B^{**}\| \to 0$ as $j \to \infty$. Hence $L_A R_B$ is completely continuous on $L(X)$. Since $L_A R_B$ is also weakly compact on $L(X)$ by Fact 3.1, it follows that $L_A R_B$ is strictly singular $L(X) \to L(X)$. The DPP of X yields further that $L_A R_B(S) = ASB \in K(X)$ for $S \in L(X)$, so that $L_A R_B$ is also weakly compact considered as an operator $L(X) \to K(X)$. Since $K(X)$ has the DPP by Lemma 3.3(ii) below, we get that $L_A R_B$ is strictly cosingular $L(X) \to K(X)$ (as well as $L(X) \to L(X)$) by [P1, Proposition I.4b]. The proof will thus be complete once we have established Lemma 3.3 below. ∎

We next formulate the precise versions of the DPP-results, which are essential for the arguments of Theorems 3.2 and 3.4. We are not aware of references for these consequences of [B2], though we presume that they might be known to some specialists. Hence we are obliged to include quite careful arguments.

Let K be a compact Hausdorff space and (Ω, Σ, μ) a measure space. Here $L^1(\mu, C(K))$ will be the vector-valued function space consisting of (equivalence classes of) Bochner μ-integrable functions $\Omega \to C(K)$, and $C\big(K, L^1(\mu)\big)$ the space of continuous functions $K \to L^1(\mu)$. Let $E\hat{\bigotimes}_\pi F$ be the projective and $E\hat{\bigotimes}_\epsilon F$ the injective tensor product of the Banach spaces E and F. We refer to, *e.g.*, [DF] for the definitions and the general properties of these tensor products. Recall that $(E\hat{\bigotimes}_\pi F)^* = L(E, F^*)$. The identifications $L^1(\mu, C(K)) = L^1(\mu)\hat{\bigotimes}_\pi C(K)$ and $C(K, L^1(\mu)) = C(K)\hat{\bigotimes}_\epsilon L^1(\mu)$ used below are explained in [DF, 3.3 and 4.2.(2)].

Lemma 3.3 *Let E be a $\mathcal{L}^1$- or $\mathcal{L}^\infty$-space. Then*

(i) *$L(E^{**})^*$ and $L(E^{**})$ have the DPP.*

(ii) *$K(E)$ has the DPP.*

Proof (i) Suppose first that E is a $\mathcal{L}^1$-space, so that E^* is a $\mathcal{L}^\infty$-space by [LT1, II.5.8.(ii)]. Then E^{**} is isomorphic to a complemented subspace of $L^1(\mu)$ for some measure space (Ω, Σ, μ), and E^* is isomorphic to a complemented subspace of $C(K)$ for some compact space K (see, *e.g.*, [B1, 1.23]). It follows that $E^{**}\hat{\bigotimes}_\pi E^*$ is isomorphic to a complemented subspace of $L^1(\mu)\hat{\bigotimes}_\pi C(K)$, so that there are operators

$$j\colon E^{**}\hat{\bigotimes}_\pi E^* \to L^1(\mu)\hat{\bigotimes}_\pi C(K), \quad p\colon L^1(\mu)\hat{\bigotimes}_\pi C(K) \to E^{**}\hat{\bigotimes}_\pi E^*,$$

for which $p \circ j = I_{E^{**}\hat{\otimes}_\pi E^*}$. Hence $(E^{**}\hat{\bigotimes}_\pi E^*)^{**}$ is isomorphic to a complemented subspace of $(L^1(\mu)\hat{\bigotimes}_\pi C(K))^{**}$, since $p^{**} \circ j^{**} = I_{(E^{**}\hat{\otimes}_\pi E^*)^{**}}$. Bourgain showed [B2, Corollary 7] (*cf.* [D, pp. 47–51]) that the bidual

$$\big(L^1(\mu)\hat{\bigotimes}_\pi C(K)\big)^{**} = L^1\big(\mu, C(K)\big)^{**}$$

has the DPP. Hence $(E^{**}\hat{\bigotimes}_\pi E^*)^{**} = L(E^{**})^*$ has the DPP (recall here that $(E^{**}\hat{\bigotimes}_\pi E^*)^* = L(E^{**})$). Finally, the Dunford–Pettis property is inherited by the predual $L(E^{**})$. A similar argument applies to the $\mathcal{L}^\infty$-space E, since $E^{**}\hat{\bigotimes}_\pi E^* \approx E^*\hat{\bigotimes}_\pi E^{**}$, where E^* is a $\mathcal{L}^1$- and E^{**} is a $\mathcal{L}^\infty$-space by [LT1, II.5.8.(ii)].

(ii) Emmanuele [E, p. 475] pointed out (without including the details) that $E\hat{\bigotimes}_\epsilon F$ has the DPP whenever E is a $\mathcal{L}^\infty$-space and F is a $\mathcal{L}^1$-space. This general fact implies that $K(E) = E^*\hat{\bigotimes}_\epsilon E$ has the DPP whenever E is a $\mathcal{L}^1$-space (recall that E has the approximation property, see [LT1, II.5.7]). In a similar manner one gets that $K(E) = E^*\hat{\bigotimes}_\epsilon E \approx E\hat{\bigotimes}_\epsilon E^*$ has the DPP whenever E is a $\mathcal{L}^\infty$-space. We sketch here for completeness how to deduce Emmanuele's remark in [E, p. 475] from [B2, Corollary 7] by modifying some ideas from [E, Theorem 2] and [Ci, Theorem 1]. Let $S\colon E\hat{\bigotimes}_\epsilon F \to Z$ be any weakly compact operator, where Z is a Banach space, so that $S^{**} \in W\big((E\hat{\bigotimes}_\epsilon F)^{**}, Z\big)$. Here $E\hat{\bigotimes}_\epsilon F \subset E^{**}\hat{\bigotimes}_\epsilon F \subset (E\hat{\bigotimes}_\epsilon F)^{**}$ as closed subspaces (*cf.* [E, Lemma 1] for the latter isometry), so that the restriction $T = S^{**}|_{E^{**}\hat{\otimes}_\epsilon F}$ is weakly compact $E^{**}\hat{\bigotimes}_\epsilon F \to Z$. Since E^{**} is isomorphic to a complemented subspace of $C(K)$ for some compact set K, there are operators $J_1\colon E^{**} \to C(K)$ and $P\colon C(K) \to E^{**}$ satisfying $P \circ J_1 = I_{E^{**}}$. Let $P \otimes I_F\colon C(K)\hat{\bigotimes}_\epsilon F \to E^{**}\hat{\bigotimes}_\epsilon F$ be the

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corresponding tensored operator. Moreover, F^{**} is isomorphic to a complemented subspace of $L^1(\mu)$ for some measure space (Ω, Σ, μ), so there are $J_2 \colon F^{**} \to L^1(\mu)$ and $Q \colon L^1(\mu) \to F^{**}$ satisfying $Q \circ J_2 = I_{F^{**}}$. By repeating the preceding argument for $U = T \circ (P \otimes I_F) \in W\big(C(K)\hat{\bigotimes}_\epsilon F, Z\big)$ we get a weakly compact operator

$$V = (U^{**}|_{C(K)\hat{\bigotimes}_\epsilon F^{**}}) \circ (I_{C(K)} \otimes Q) \colon C(K)\hat{\bigotimes}_\epsilon L^1(\mu) \to Z.$$

Above $I_{C(K)} \otimes Q$ is a projection $C(K)\hat{\bigotimes}_\epsilon L^1(\mu) \to C(K)\hat{\bigotimes}_\epsilon J_2(F^{**})$, where $C(K)\hat{\bigotimes}_\epsilon L^1(\mu) = C\big(K, L^1(\mu)\big)$ has the DPP by [B2, Corollary 7]. It follows that $\|Tv_n\| \to 0$ as $n \to \infty$ whenever $(v_n) \subset E\hat{\bigotimes}_\epsilon F$ is weakly null, since T is (up to a linear isomorphism) a restriction of V. ■

There are analogues of Theorem 3.2 for quite large subclasses of $\mathcal{L}^\infty$-spaces, but it will be necessary to split the consideration into two parts (*cf.* Example 3.6 below).

Theorem 3.4 *Let X be a $\mathcal{L}^\infty$-space, and $A, B \in L(X)$ non-zero operators.*

(a) *Assume that*

$$S(X) = W(X). \tag{3.1}$$

Then the following conditions are equivalent.

(i) *$L_A R_B$ is strictly singular on $L(X)$,*
(ii) *$L_A R_B$ is weakly compact on $L(X)$,*
(iii) *$A, B \in S(X) = W(X)$,*
(iv) *$A \in S(X)$ and $B^* \in S(X^*)$.*

(b) *Assume that*

$$P(X) = W(X). \tag{3.2}$$

Then the following conditions are equivalent.

(i) *$L_A R_B$ is strictly cosingular on $L(X)$,*
(ii) *$L_A R_B$ is weakly compact on $L(X)$,*
(iii) *$A, B \in P(X) = W(X)$,*
(iv) *$A \in P(X)$ and $B^* \in P(X^*)$.*

Proof The argument of Theorem 3.2 can be carried over almost verbatim to the case of $\mathcal{L}^\infty$-spaces. Recall first that $W(X^*) = S(X^*) = P(X^*)$, since X^* is a weakly sequentially complete $\mathcal{L}^1$-space. In this event (3.1) or (3.2) allow us to check that conditions (iii) and (iv) are equivalent in parts (a) and (b), respectively. The crucial implications (iii) $\Rightarrow$ (i) in parts (a) and (b) are proved as in Theorem 3.2, using the DPP of $L(X^{**})$ and $K(X)$ for $\mathcal{L}^\infty$-spaces X (see Lemma 3.3). ■

Corollary 3.5 *Let $X = C(K)$, where K is a compact metric space, or let $X = \ell^\infty = C(\beta\mathbf{N}) \approx L^\infty(0,1)$. Then* (3.1) *and* (3.2) *are both satisfied, so that the conditions in parts* (a) *and* (b) *of Theorem* 3.4 *are all equivalent.*

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Proof Recall that if K is a compact Hausdorff space, then $C(K)$ satisfies condition (3.1) by [P, Theorem I.1]. If K is a compact metric space, then $C(K)$ satisfies condition (3.2) by [P, I.2]. Moreover, $\ell^\infty = C(\beta\mathbf{N}) \approx L^\infty(0,1)$ satisfies (3.2) in view of [LT2, 2.f.4]. ■

There are $\mathcal{L}^\infty$-spaces having quite unexpected properties, see [B1, Chapter III] or [BP]. We note the following examples, which point out the limitations of Theorem 3.4.

Example 3.6 (i) Let X be the $\mathcal{L}^\infty$-space $c_0 \oplus \ell^\infty$. Then $W(X) = S(X) \subsetneq P(X)$, since the inclusion $J\colon c_0 \to \ell^\infty$ is strictly cosingular (see [P, p. 36] or [LT2, 2.f.4]).

(ii) Let Y be the separable $\mathcal{L}^\infty$-space constructed by Bourgain and Delbaen (see [B1, Chapter III]), so that Y has the Schur property. The separability of Y implies that Y does not have the Grothendieck property (that is, there is a w*-null sequence $(x_n^*) \subset Y^*$ without any weak-null subsequences). Hence $U \notin W(Y, c_0)$, where $Ux = (x_n^*(x))$ for $x \in Y$. It is easy to deduce that $U \in S(Y, c_0)$, since Y is ℓ^1-saturated (see, *e.g.*, [B1, Proposition I.1.3]), but c_0 is c_0-saturated by, *e.g.*, [LT2, 2.a.1 and 2.a.2]. Hence $W(X) \subsetneq S(X)$ for the $\mathcal{L}^\infty$-space $X = Y \oplus c_0$.

(iii) Both (3.1) and (3.2) fail to hold for the $\mathcal{L}^\infty$-space $X = Y \oplus c_0 \oplus \ell^\infty$.

4 Further Examples

This section contains examples that demonstrate the intrinsic dependence of Problem 1.1 on the space X, as well as the optimality of Fact 2.1(i). The main example (Theorem 4.1) identifies the strictly singular and cosingular multiplications on $L(\ell^p \oplus \ell^q)$ for $1 < p < q < \infty$. We write $S \in L(\ell^p \oplus \ell^q)$ as operator matrices $S = (S_{jk})$, where $S_{jk} = P_j S I_k$, and P_j and I_k are the natural projections and inclusions associated to $\ell^p \oplus \ell^q$ for $j, k \in \{1, 2\}$. Recall that

$$S(\ell^p \oplus \ell^q) = P(\ell^p \oplus \ell^q) = \begin{pmatrix} K(\ell^p) & K(\ell^q, \ell^p) \\ L(\ell^p, \ell^q) & K(\ell^q) \end{pmatrix} \tag{4.1}$$

by total incomparability (see [LT2, 2.a.3]) and Pitt's theorem. It follows from (4.1) that $U^* \in S(\ell^{p'} \oplus \ell^{q'})$ if and only if $U \in S(\ell^p \oplus \ell^q)$, and that $U^* \in P(\ell^{p'} \oplus \ell^{q'})$ if and only if $U \in P(\ell^p \oplus \ell^q)$. The following example should be contrasted with Example 4.5 for $X = \ell^p \oplus \ell^q \oplus \ell^r$, where $1 < p < q < r < \infty$.

Theorem 4.1 *Let $1 < p < q < \infty$. Then the following conditions are equivalent for non-zero operators $A, B \in L(\ell^p \oplus \ell^q)$:*

(i) *$L_A R_B$ is strictly singular $L(\ell^p \oplus \ell^q) \to L(\ell^p \oplus \ell^q)$,*
(ii) *$L_A R_B$ is strictly cosingular $L(\ell^p \oplus \ell^q) \to L(\ell^p \oplus \ell^q)$,*
(iii) $A, B \in \begin{pmatrix} K(\ell^p) & K(\ell^q, \ell^p) \\ L(\ell^p, \ell^q) & K(\ell^q) \end{pmatrix}$.

We will focus on the strictly cosingular case, which is the novel part. The basic strategy resembles that of Theorem 2.9, but applied here to certain spaces of nuclear operators. For the reader's convenience we will present the details, which are less involved than in the case of $X = L^p(0,1)$ (no versions of Lemmas 2.6 and 2.7 will be needed, and the concluding step is much simpler).

Let E and F be Banach spaces. The operator $S\colon E \to F$ is *nuclear* if there are sequences $(u_j^*) \subset E^*$ and $(v_j) \subset F$, so that $S = \sum_{j=1}^{\infty} u_j^* \otimes v_j$ and $\sum_{j=1}^{\infty} \|u_j^*\| \cdot \|v_j\| < \infty$. The nuclear norm of S is

$$\|S\|_N = \inf\Big\{ \sum_{j=1}^{\infty} \|u_j^*\| \cdot \|v_j\| : S = \sum_{j=1}^{\infty} u_j^* \otimes v_j \Big\}.$$

Then $(N(E,F), \|\cdot\|_N)$ is a Banach space, where $N(E,F)$ is the nuclear operators $E \to F$. The composition operator $S \mapsto ASB$ is bounded in the nuclear setting for bounded operators A and B, since $\|ASB\|_N \le \|A\| \cdot \|B\| \cdot \|S\|_N$ for $S \in N(E,F)$ and compatible A, B.

The proof will again be split into smaller steps. We begin by verifying some results for composition operators between spaces of nuclear operators in the setting (and the notation) of Lemma 2.4. Clearly the nuclear analogue of Lemma 2.3 holds, since nuclear operators can be approximated in $\|\cdot\|_N$ by finite rank operators. The unconditional operator blocking principle (Fact 2.5) also has a nuclear version, since its proof [LT2, 1.c.8 and Remark 1, p. 21] is based on averaging.

Lemma 4.2 *Suppose that $E_1, \dots, E_4$ are reflexive Banach spaces having unconditional bases, and let $A \in L(E_3, E_4)$ and $B \in L(E_1, E_2)$ be fixed. Assume moreover that there is a normalized block-diagonal sequence $(S_k) \subset N(E_2, E_3)$, so that $\|AS_kB\|_N \ge c > 0$ for $k \in \mathbf{N}$. Then there is a subsequence (S_{k_j}) so that $\big(L_A R_B(S_{k_j})\big) = (AS_{k_j}B)$ is equivalent (and as close as we wish in $\|\cdot\|_N$) to a block-diagonal sequence $(T_j) \subset N(E_1, E_4)$.*

Proof Since $(S_k) \subset N(E_2, E_3)$ is a normalized block-diagonal sequence, it is not difficult to modify the argument of Lemma 2.4 (replacing the operator norm by $\|\cdot\|_N$). We leave the details to the reader. ■

The following technical lemma is needed for the main reduction step.

Lemma 4.3 *Let E_1 and E_2 be reflexive Banach spaces having unconditional bases with unconditional basis constants $d_1, d_2 \ge 1$. Let $(n_k) \subset \mathbf{N}$ be a strictly increasing sequence (where $n_0 = 1$). Assume that the sequence $(R_k) \subset N(E_1, E_2)$ satisfies*

(i) $\|R_k\|_N = 1$,
(ii) $\|P_{n_k}^{(2)} R_k P_{n_k}^{(1)} - R_k\|_N \le \frac{1}{8d_1d_2} 2^{-k}$,
(iii) $P_{n_{k-1}}^{(2)} R_k P_{n_{k-1}}^{(1)} = 0$,

for $k \in \mathbf{N}$. Then there is a constant $c_1 = c_1(d_1, d_2) > 0$ so that

$$\Big\| \sum_{k=1}^{\infty} a_k P_{n_k, n_{k-1}}^{(2)} R_k P_{n_k, n_{k-1}}^{(1)} \Big\|_N \le c_1 \Big\| \sum_{k=1}^{\infty} a_k R_k \Big\|_N$$

for all scalar sequences (a_k) such that $\sum_{k=1}^{\infty} a_k R_k$ converges in $N(E_1, E_2)$.

Proof Note first that

$$\Big\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\Big\|_N \leq 2\Big\|\sum_{k=1}^{\infty} a_k R_k\Big\|_N \tag{4.2}$$

whenever $\sum_{k=1}^{\infty} a_k R_k$ converges in $N(E_1, E_2)$. Indeed, $\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}$ converges in $N(E_1, E_2)$, and it follows from (iii) that

$$\begin{aligned} |a_j| \cdot \|P^{(2)}_{n_j} R_j P^{(1)}_{n_j}\|_N &= \Big\| P^{(2)}_{n_j}\Big[\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\Big] P^{(1)}_{n_j} \\ &\qquad - P^{(2)}_{n_{j-1}}\Big[\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\Big] P^{(1)}_{n_{j-1}}\Big\|_N \\ &\leq 2d_1 d_2 \Big\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\Big\|_N, \quad j \in \mathbf{N}. \end{aligned}$$

Hence $\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\|_N \leq \|\sum_{k=1}^{\infty} a_k R_k\|_N + \frac{2}{7}\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\|_N$ by assumption (ii), since $\|P^{(2)}_{n_j} R_j P^{(1)}_{n_j}\|_N \geq \frac{7}{8}$. The nuclear version of Fact 2.5, applied to the operator $S = \sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k} \in N(E_1, E_2)$, and (4.2) yield that

$$\Big\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k,n_{k-1}} R_k P^{(1)}_{n_k,n_{k-1}}\Big\|_N \leq c\Big\|\sum_{k=1}^{\infty} a_k P^{(2)}_{n_k} R_k P^{(1)}_{n_k}\Big\|_N \leq 2c\Big\|\sum_{k=1}^{\infty} a_k R_k\Big\|_N,$$

where $c = c(d_1, d_2) > 0$. Note that above we clearly have $\sum_{r=1}^{\infty} P^{(2)}_{n_r,n_{r-1}} S P^{(1)}_{n_r,n_{r-1}} = \sum_{k=1}^{\infty} a_k P^{(2)}_{n_k,n_{k-1}} R_k P^{(1)}_{n_k,n_{k-1}}$ by (iii). ■

The following result is a nuclear analogue of Proposition 2.8, and it contains the main reduction step of the argument.

Proposition 4.4 *Let $E_1, \dots, E_4$ be reflexive Banach spaces having unconditional bases. Assume that $A \in S(E_3, E_4)$ and $B^* \in S(E_2^*, E_1^*)$ are such that $L_A R_B$ is a non-strictly singular operator $N(E_2, E_3) \to N(E_1, E_4)$.*

Then there is a normalized block-diagonal sequence $(S_k) \subset N(E_2, E_3)$ so that

- *$L_A R_B$ is bounded below on $[S_k : k \in \mathbf{N}]$,*
- *$(L_A R_B(S_k)) = (AS_k B)$ is equivalent (and as close as we wish in $\|\cdot\|_N$) to a block-diagonal sequence $(T_k) \subset N(E_1, E_4)$.*

Proof By assumption there is an infinite-dimensional subspace $M \subset N(E_2, E_3)$ and $c > 0$ so that

$$\|L_A R_B(S)\|_N = \|ASB\|_N \geq c\|S\|_N, \quad S \in M.$$

We first construct a sequence $(R_k) \subset M$ and an increasing sequence $(n_k) \subset \mathbf{N}$ so that

$$\|R_k\|_N = 1, \tag{4.3}$$

$$\|P^{(3)}_{n_k} R_k P^{(2)}_{n_k} - R_k\|_N \le \frac{1}{16 d_2 d_3} \cdot 2^{-k}, \tag{4.4}$$

$$P^{(3)}_{n_{k-1}} R_k P^{(2)}_{n_{k-1}} = 0, \tag{4.5}$$

$$\|L_A R_B(R_k) - L_A R_B(P^{(3)}_{n_k,n_{k-1}} R_k P^{(2)}_{n_k,n_{k-1}})\|_N \le b \cdot 2^{-k} \tag{4.6}$$

for $k \in \mathbf{N}$. Here $d_2, d_3 \ge 1$ are the unconditional basis constants in E_2 and E_3, and b satisfies $0 < b < \min\{\frac{c}{2}, \frac{c_1}{8\|A\|\cdot\|B\| d_2 d_3}\}$, where $c_1 = c_1(d_2, d_3) > 0$ is the constant given by Lemma 4.3.

Suppose that we have chosen operators $R_1, \dots, R_{k-1}$ and $1 = n_0 < n_1 < \cdots < n_{k-1}$ satisfying (4.3)–(4.6). We proceed as in Proposition 2.8 and choose a normalized $R_k \in M$ so that $P^{(3)}_{n_{k-1}} R_k P^{(2)}_{n_{k-1}} = 0$ and

$$\|L_A R_B(R_k) - L_A R_B(Q^{(3)}_{n_{k-1}} R_k Q^{(2)}_{n_{k-1}})\|_N \le b \cdot 2^{-k-1}.$$

For this we need to note that Fact 2.1(i) remains valid for compositions $N(E_2, E_3) \to N(E_1, E_4)$ (see [LS, 2.1, 2.2 and 2.3]). Conditions (4.4) and (4.6) are then ensured by truncation in $\|\cdot\|_N$. Indeed, the nuclear version of Lemma 2.3 gives $n_k > n_{k-1}$, so that $\|P^{(3)}_{n_k} R_k P^{(2)}_{n_k} - R_k\|_N$ and $\|P^{(3)}_{n_k} Q^{(3)}_{n_{k-1}} R_k Q^{(2)}_{n_{k-1}} P^{(2)}_{n_k} - Q^{(3)}_{n_{k-1}} R_k Q^{(2)}_{n_{k-1}}\|_N$ are small enough.

Put $U_k = P^{(3)}_{n_k,n_{k-1}} R_k P^{(2)}_{n_k,n_{k-1}}$ for $k \in \mathbf{N}$. Then $\|U_k\|_N \ge \frac{c}{2\|A\|\cdot\|B\|}$ for $k \in \mathbf{N}$, since $\|L_A R_B(U_k)\|_N \ge \frac{c}{2}$ by (4.6) and $R_k \in M$. By arguing as in the proof of Lemma 4.3 we get

$$\frac{c}{2\|A\| \cdot \|B\|} \sup_{k \in \mathbf{N}} |a_k| \le 2 d_2 d_3 \Big\| \sum_{k=1}^{\infty} a_k U_k \Big\|_N \tag{4.7}$$

whenever $\sum_{k=1}^{\infty} a_k U_k$ converges in $N(E_2, E_3)$. Since $\sum_{k=1}^{\infty} a_k R_k \in M$, we get from Lemma 4.3 and (4.6) that

$$\Big\| \sum_{k=1}^{\infty} a_k U_k \Big\|_N \le c_1 \Big\| \sum_{k=1}^{\infty} a_k R_k \Big\|_N \le \frac{c_1}{c} \Big\| L_A R_B\Big(\sum_{k=1}^{\infty} a_k R_k \Big) \Big\|_N$$

$$\le \frac{c_1}{c} \Big\| L_A R_B\Big(\sum_{k=1}^{\infty} a_k U_k \Big) \Big\|_N + \frac{c_1}{c} \cdot b \cdot \sup |a_k|.$$

Lemma 4.3 can be applied here thanks to (4.3)–(4.6). Since $b < \frac{c_1}{8\|A\|\cdot\|B\| d_2 d_3}$, it follows from (4.7) that

$$\Big\| L_A R_B\Big(\sum_{k=1}^{\infty} a_k U_k \Big) \Big\|_N \ge \frac{c}{2c_1} \Big\| \sum_{k=1}^{\infty} a_k U_k \Big\|_N$$

whenever $\sum_{k=1}^{\infty} a_k U_k$ converges in $N(E_2, E_3)$. The proof of Proposition 4.4 is thus complete by using Lemma 4.2 to pass to a suitable subsequence (S_k) of (U_k). ■

Remark Proposition 4.4, as well as Lemmas 4.2 and 4.3, also hold for compositions on $K(E_1, E_2)$. It is not hard to modify the arguments.

If E and F are reflexive Banach spaces having Schauder bases, then there is an isometric identification $K(E,F)^* = N(F,E)$ given by $\langle S,T\rangle = \operatorname{tr}(TS) = \sum_{j=1}^{\infty} y_j^*(Tx_j)$ for $S = \sum_{j=1}^{\infty} y_j^* \otimes x_j \in N(F,E)$ and $T \in K(E,F)$ (see, *e.g.*, [FS, Theorem 1]). Similarly, $N(F,E)^* = L(E,F)$ and $K(E,F) \subset K(E,F)^{**} = L(E,F)$ is the natural inclusion. Let $A, B \in L(E,F)$. Then $L_A R_B$ maps $K(F,E) \to K(E,F)$ and $(L_A R_B)^* \colon K(E,F)^* \to K(F,E)^*$ is identified with $L_B R_A \colon N(F,E) \to N(E,F)$ in this trace-duality, since

$$\begin{aligned}\langle (L_A R_B)^*(y^* \otimes x), T\rangle &= \operatorname{tr}\big(ATB \circ (y^* \otimes x)\big) = y^*(ATBx)\\ &= \operatorname{tr}\big(T \circ (A^* y^* \otimes Bx)\big) = \langle L_B R_A(y^* \otimes x), T\rangle\end{aligned}$$

for $y^* \in F^*$, $x \in E$ and $T \in K(E,F)$. (There are alternative trace-duality identifications $K(E,F)^* = N(E^*,F^*)$ and $N(E^*,F^*)^* = L(E,F)$, for which $(L_A R_B)^* = L_{A^*} R_{B^*}$. The one described above avoids dual exponents here.)

Proof of Theorem 4.1 The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) follow from Fact 2.1 and the duality facts recorded after (4.1).

(iii) $\Rightarrow$ (ii): Suppose that $A, B \in \begin{pmatrix} K(\ell^p) & K(\ell^q,\ell^p) \\ L(\ell^p,\ell^q) & K(\ell^q)\end{pmatrix}$. Decompose

$$A = \begin{pmatrix} 0 & 0 \\ A_{21} & 0\end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{pmatrix} \equiv A_0 + K_1, \quad B = B_0 + K_2,$$

where $K_1, K_2 \in K(\ell^p \oplus \ell^q)$, so that $L_A R_B = L_{A_0} R_{B_0} + L_{A_0} R_{K_2} + L_{K_1} R_{B_0} + L_{K_1} R_{K_2}$. Fact 2.1(i) and (4.1) imply that $L_{A_0} R_{K_2} + L_{K_1} R_{B_0} + L_{K_1} R_{K_2}$ is strictly cosingular on $L(\ell^p \oplus \ell^q)$. We must verify that

$$S \mapsto L_{A_0} R_{B_0}(S) = \begin{pmatrix} 0 & 0 \\ A_{21} S_{21} B_{21} & 0\end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} \in L(\ell^p \oplus \ell^q),$$

is strictly cosingular on $L(\ell^p \oplus \ell^q)$. Hence, by using the natural projections on $L(\ell^p \oplus \ell^q)$ associated to the operator matrix $S = (S_{jk}) \in L(\ell^p \oplus \ell^q)$, it will be enough (after simplifying our notation) to verify that $L_A R_B$ is strictly cosingular $L(\ell^q,\ell^p) = K(\ell^q,\ell^p) \to K(\ell^p,\ell^q)$ for $A, B \in L(\ell^p,\ell^q)$. One has $(L_A R_B)^* = L_B R_A \colon N(\ell^q,\ell^p) \to N(\ell^p,\ell^q)$ in the trace-duality described above. It will then suffice, by easy duality, to verify the following

Claim 3 *$L_B R_A$ is strictly singular $N(\ell^q,\ell^p) \to N(\ell^p,\ell^q)$ for $A, B \in L(\ell^p,\ell^q)$.*

Proof of Claim 3 Suppose to the contrary that $L_B R_A$ is not strictly singular $N(\ell^q,\ell^p) \to N(\ell^p,\ell^q)$. Proposition 4.4 yields a normalized block-diagonal sequence $(S_k) \subset N(\ell^q,\ell^p)$, so that $L_B R_A$ defines an isomorphism $[S_k : k \in \mathbf{N}] \to [BS_kA : k \in \mathbf{N}]$, and (BS_kA) is equivalent to a block-diagonal sequence $(T_k) \subset N(\ell^p,\ell^q)$. We check that

$$\Big\| \sum_{k=1}^{\infty} c_k S_k \Big\|_N = \sum_{k=1}^{\infty} |c_k|, \quad (c_k) \in \ell^1. \tag{4.8}$$

Clearly $\|\sum_{k=1}^{\infty} c_k S_k\|_N \leq \sum_{k=1}^{\infty} |c_k|$ for $(c_k) \in \ell^1$. By finite-dimensional trace-duality there is a normalized block-diagonal sequence $(U_k) \subset K(\ell^p, \ell^q)$ so that $\langle S_j, U_k\rangle = \delta_{j,k}$ for $j, k \in \mathbf{N}$. Since $p < q$ it is not difficult to check that $\|\sum_{k=1}^{\infty} b_k U_k\| = \sup_{k\in\mathbf{N}} |b_k|$ in $K(\ell^p, \ell^q)$ for all $(b_k) \in c_0$. Hence

$$\Big\|\sum_{k=1}^{\infty} c_k S_k\Big\|_N \geq \sup\Big\{\Big|\Big\langle \sum_{k=1}^{\infty} c_k S_k, \sum_{j=1}^{\infty} b_j U_j\Big\rangle\Big| : \sup_{j\in\mathbf{N}} |b_j| \leq 1\Big\}$$
$$= \sup\Big\{\sum_{k=1}^{\infty} |b_k c_k| : \sup_{j\in\mathbf{N}} |b_j| \leq 1\Big\} = \sum_{k=1}^{\infty} |c_k|.$$

On the other hand, $[S_k : k \in \mathbf{N}] \approx [BS_kA : k \in \mathbf{N}] \subset N(\ell^p, \ell^q)$, where $N(\ell^p, \ell^q) = K(\ell^q, \ell^p)^*$ is reflexive by [K, Corollary 2], since $L(\ell^q, \ell^p) = K(\ell^q, \ell^p)$ for $p < q$. This clearly contradicts (4.8), which proves the Claim.

(iii) $\Rightarrow$ (i): We only sketch the idea, and leave the details to the reader. By a similar reduction as above it will be enough to verify that $L_A R_B$ is strictly singular $K(\ell^q, \ell^p) \to K(\ell^p, \ell^q)$ for $A, B \in L(\ell^p, \ell^q)$. If $L_A R_B$ is not strictly singular $K(\ell^q, \ell^p) \to K(\ell^p, \ell^q)$, then the version of Proposition 4.4 for spaces of compact operators yields a normalized block-diagonal sequence $(S_k) \subset K(\ell^q, \ell^p)$, so that $L_A R_B$ is an isomorphism $[S_k : k \in \mathbf{N}] \to [AS_kB : k \in \mathbf{N}]$, where (AS_kB) is equivalent to a semi-normalized block-diagonal sequence $(T_k) \subset K(\ell^p, \ell^q)$. One verifies that $[S_k : k \in \mathbf{N}] \approx [T_k : k \in \mathbf{N}] \approx c_0$, which contradicts the reflexivity of $L(\ell^q, \ell^p) = K(\ell^q, \ell^p)$ for $p < q$, [K, Corollary 2]. ∎

The following example shows that, contrary to Theorem 4.1, the maximal conditions for strict (co)singularity are not the correct ones for $X = \ell^p \oplus \ell^q \oplus \ell^r$, where $1 < p < q < r < \infty$, or for (certain) sums $X = L^p(0,1) \oplus L^q(0,1)$. Here the strict (co)singularity of A and B does not always imply the strict (co)singularity of $L_A R_B$.

Example 4.5 (i) Suppose that $X = \ell^p \oplus \ell^q \oplus \ell^r$, where $1 < p < q < r < \infty$. Let $j_2 \colon \ell^p \to \ell^q$, $j_1 \colon \ell^q \to \ell^r$ be the natural inclusions, and define $J_1, J_2 \in S(X) \cap P(X)$ by

$$J_1(x, y, z) = (0, 0, j_1 y), \;\; J_2(x, y, z) = (0, j_2 x, 0), \quad \text{for } (x, y, z) \in \ell^p \oplus \ell^q \oplus \ell^r.$$

Then $L_{J_1} R_{J_2}$ is neither strictly singular nor cosingular on $L(X)$.

(ii) Let $X = L^p(0,1) \oplus L^q(0,1)$, where $p, q \in (1, \infty) \setminus \{2\}$ and $p \neq q$. Then there are $A, B \in S(X) \cap P(X)$, so that $L_A R_B$ is neither strictly singular nor cosingular on $L(X)$.

Proof (i) $J_1, J_2 \in S(X) \cap P(X)$, since j_1 and j_2 are strictly singular and cosingular (recall that $L(\ell^u, \ell^v) = S(\ell^u, \ell^v) = P(\ell^u, \ell^v)$ for $1 < u < v < \infty$ by the total incomparability of ℓ^u and ℓ^v, see [LT2, 2.a.3], and reflexivity). For 3×3-operator matrices $S = (S_{jk}) \in L(X)$ we have

$$L_{J_1} R_{J_2}(S) = J_1 (S_{jk}) J_2 = \begin{pmatrix} 0 & 0 & 0 \\ j_1 S_{22} j_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It suffices to verify that $L_{j_1} R_{j_2}$ is neither strictly singular nor cosingular $L(\ell^q) \to L(\ell^p, \ell^r)$.

Clearly $\| \sum_{j=1}^{\infty} c_j e_j^* \otimes e_j \| = \sup_j |c_j|$ in $L(\ell^q)$ for $(c_j) \in \ell^\infty$, where $(e_n) \subset \ell^q$ and $(e_n^*) \subset \ell^{q'}$ are the unit vector bases, and q' is the conjugate exponent of q. Moreover, it is not difficult to check that

$$\Big\| L_{j_1} R_{j_2} \Big(\sum_{j=1}^{\infty} c_j e_j^* \otimes e_j \Big) \Big\| = \Big\| \sum_{j=1}^{\infty} c_j j_2^* e_j^* \otimes j_1 e_j \Big\| = \sup_{j \in \mathbf{N}} |c_j|, \quad (c_j) \in \ell^\infty$$

in $L(\ell^p, \ell^r)$. Thus $L_{j_1} R_{j_2}$ is a linear isometry $M \to L_{j_1} R_{j_2}(M)$, where $M = [e_n^* \otimes e_n : n \in \mathbf{N}]$ is isometric to ℓ^∞ in $L(\ell^q)$. The injectivity of ℓ^∞ implies that $L_{j_1} R_{j_2}(M)$ is complemented in $L(\ell^p, \ell^r)$. Let $N \subset L(\ell^p, \ell^r)$ be an infinite-dimensional closed subspace, so that $L(\ell^p, \ell^r) = L_{j_1} R_{j_2}(M) \oplus N$. Thus $Q_N \circ L_{j_1} R_{j_2}$ is surjective $L(\ell^q) \to L(\ell^p, \ell^r)/N$, so that $L_{j_1} R_{j_2} \notin P\big(L(\ell^q), L(\ell^p, \ell^r)\big)$.

(ii) $\ell^r \oplus \ell^2$ is isomorphic to a complemented subspace of $L^r(0,1)$ for $r \in (1, \infty)$, so that we may decompose $L^p(0,1) \oplus L^q(0,1) = M \oplus N$, where $M \approx \ell^p \oplus \ell^q \oplus \ell^2$. According to part (i) there are $A_0, B_0 \in S(M) \cap P(M)$ so that $L_{A_0} R_{B_0}$ is neither strictly singular nor cosingular $L(M) \to L(M)$. Let $A(x, y) = (A_0 x, 0)$ and $B(x, y) = (B_0 x, 0)$ for $(x, y) \in X = M \oplus N$. One checks as before that $L_A R_B$ is neither strictly singular nor cosingular on $L(X)$. ■

Remarks Strict singularity and cosingularity of $L_A R_B$ are, in general, unrelated. For instance, let $A \in L(\ell^1, \ell^2)$ be a linear surjection, and let $B \in K(\ell^1)$ be non-zero. Then Fact 2.1 yields that $L_A R_B$ is strictly singular $L(\ell^1) \to L(\ell^1, \ell^2)$, but not strictly cosingular. Moreover, let $A \colon \ell^2 \to C(0,1)$ be a linear embedding, and let $B \in K(\ell^2)$ be non-zero. Then $L_A R_B$ is strictly cosingular $L(\ell^2) \to L\big(\ell^2, C(0,1)\big)$, but not strictly singular. Here the fact that $A \in P\big(\ell^2, C(0,1)\big)$ follows, *e.g.*, from [P, Proposition I.4b]. These examples transfer to $X = \ell^1 \oplus \ell^2$ and $X = \ell^2 \oplus C(0,1)$, respectively.

Finally, there has been substantial parallel work on properties of tensor products of operators in the literature. We refer to [DF] for a systematic exposition, and to [DiF] and [R] for results closer to the topic of the present paper. Several of our results yield information about the strict (co)singularity of tensor products of operators between ϵ-tensor products of concrete Banach spaces. As a sample we restate Theorem 2.9.

Theorem 4.6 *Let $A \in L(L^{p'}(0,1))$ and $B \in L(L^p(0,1))$ be non-zero operators. Then $A \hat{\bigotimes}_\epsilon B$ is strictly singular on $L^{p'}(0,1) \hat{\bigotimes}_\epsilon L^p(0,1)$ if and only if $A \in S(L^{p'}(0,1))$ and $B \in S(L^p(0,1))$.*

Proof One may identify $K(L^p(0,1)) = L^{p'}(0,1) \hat{\bigotimes}_\epsilon L^p(0,1)$ and the tensor operator

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$A\hat{\bigotimes}_\epsilon B$ with $L_B R_{A^*}$, since $B(x^*\otimes y)A^* = Ax^*\otimes By$ for $x^* \in L^{p'}(0,1)$ and $y \in L^p(0,1)$. The claim now follows from Theorem 2.9 and [W, Corollary 2]. ■

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