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A TAUBERIAN THEOREM CONCERNING BOREL-TYPE AND RIESZ SUMMABILITY METHODS

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ABSTRACT. It is proved that the summability of a series by the Borel-type summability method (B, α, β) together with a certain Tauberian condition implies its summability by the Riesz method $\big(R, \log(n+1), p\big)$.

1. **Introduction.** Suppose throughout that $\alpha > 0$, $\alpha N + \beta > 0$ with N a non-negative integer, $p \geq 0$, and $s_n := a_0 + a_1 + \cdots + a_n$. The Borel-type summability method (B, α, β) and the Riesz method $\big(R, \log(n+1), p\big)$ are defined as follows:

$$s_n \longrightarrow s(B, \alpha, \beta) \text{ if } \alpha e^{-x} \sum_{n=N}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \longrightarrow s \text{ as } x \longrightarrow \infty;$$

$$s_n \longrightarrow s\big(R, log(n+1), p\big) \text{ if } \sum_{\log(n+1) < w} \left(1 - \frac{\log(n+1)}{w}\right)^p a_n \longrightarrow s \text{ as } w \longrightarrow \infty.$$

Both methods are regular, and $(B, 1, 1)$ is the standard Borel exponential method B.

Let

$$L_n := \sum_{r=0}^{n} \frac{1}{r+1},$$

$$t_n := t_n^{(1)} := \frac{1}{L_n} \sum_{r=0}^{n} \frac{s_r}{r+1},$$

and, for $k = 2, 3, \ldots$,

$$t_n^{(k)} := \frac{1}{L_n} \sum_{r=0}^{n} \frac{t_r^{(k-1)}}{r+1}.$$

The k-times iterated weighted mean method $\big(M, 1/(n+1), k\big)$ is defined by:

$$s_n \longrightarrow s\big(M, 1/(n+1), k\big) \text{ if } t_n^{(k)} \longrightarrow s \text{ as } n \longrightarrow \infty.$$

The object of this paper is to prove the following Tauberian theorem.

THEOREM. *Suppose that $s_n \longrightarrow s(B, \alpha, \beta)$ and*

(1) $$s_n = O\big((n^{1/2} \log n)^p\big),$$

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where p is a positive integer. Then $s_n \longrightarrow s\big(R, \log(n+1), p\big)$.

The case $\alpha = \beta = 1$ of the theorem was recently established by Kwee [8]. Our proof owes much to his. The present theorem is more general than Kwee's result since it is known ([1, Result (I)] and [2, Lemma 4]) that

if $s_n \longrightarrow s(B, \alpha, \beta)$ *and* $\alpha > \gamma > 0$, *then* $s_n \longrightarrow s(B, \gamma, \delta)$ *provided*

$$\sum_{n=N}^{\infty} s_n \frac{z^n}{\Gamma(\gamma n + \delta)}$$

is an entire function of z for N sufficiently large.

The proviso is certainly satisfied when (1) holds.

2. **Preliminary results.**

LEMMA 1 [1, RESULT (II)]. *If* $s_n \longrightarrow s(B, \alpha, \beta)$ *and* $\delta > \beta$, *then* $s_n \longrightarrow s(B, \alpha, \delta)$.

LEMMA 2 [4, THEOREM 1]. *If* $s_n \longrightarrow s(B, \alpha, \beta)$ *and* $s_n - s_{n-1} = O(n^{-1/2})$, *then* $s_n \longrightarrow s$.

This is a special case of a general Tauberian theorem [5, Theorem 1].

LEMMA 3. *For k a positive integer,* $s_n \longrightarrow s\big(M, 1/(n+1), k\big)$ *if and only if* $s_n \longrightarrow s\big(R, \log(n+1), k\big)$.

This is due to Kwee [8, Lemma 4] who deduced the equivalence from Kuttner's result [7] that the methods $\big(M, 1/(n+1), k\big)$ and (R, L_n, k) are equivalent.

LEMMA 4 [3, LEMMA 2]. *Let*

$$c_n(x) := \alpha e^{-x} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},$$

and let $h_n := n - \frac{x}{\alpha}$, $\frac{1}{2} < \xi < \frac{2}{3}$, *and* $0 < \eta < 2\xi - 1$. *Then, as* $x \longrightarrow \infty$,

(i) $\sum_{n=N}^{\infty} c_n(x) \longrightarrow 1$;

(ii) $\sum_{|h_n| > x^{\xi}} c_n(x) = O(e^{-x^{\eta}})$;

(iii) $c_n(x) = \dfrac{\alpha}{\sqrt{2\pi x}} \exp\left(-\dfrac{\alpha^2 h_n^2}{2x}\right) \{1 + O(x^{3\xi - 2})\}$ *when* $|h_n| \leq x^{\xi}$.

LEMMA 5. *Suppose that k is a positive integer, and that* $s_n \longrightarrow s(B, \alpha, \beta)$. *Then* $t_n^{(k)} \longrightarrow s(B, \alpha, \beta)$.

PROOF. Since $\{1/L_n\}$ is totally monotone there is a non-decreasing function χ on [0, 1] [6, Theorem 207] such that

$$\frac{1}{L_n} = \int_0^1 t^n \, d\chi(t); \tag{2}$$

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moreover, since $1/L_n \longrightarrow 0$, we must have $\chi(1) = \chi(1-)$.

Suppose as we may without loss of generality that $s = 0$ and, in view of Lemma 1, that $\beta \geq \max(1, \alpha)$. Let $x > 0$ and

$$\psi(x) := \sum_{n=0}^{\infty} s_n \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

Then

$$\psi(x^\alpha) = o(x^{1-\beta} e^x) \text{ as } x \longrightarrow \infty. \tag{3}$$

We first prove that

$$B(x) := e^{-x} \sum_{n=0}^{\infty} t_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \longrightarrow 0 \text{ as } x \longrightarrow \infty. \tag{4}$$

We have

$$\phi(x) := \sum_{n=0}^{\infty} \frac{s_n}{n+1} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} = x^{\beta - \alpha - 1} \int_0^{x^\alpha} \psi(t)\, dt = \alpha x^{\beta - \alpha - 1} \int_0^x \psi(t^\alpha) t^{\alpha - 1}\, dt.$$

Hence, by (3),

$$\begin{aligned} \phi(x) &= O(x^{\beta-\alpha-1}) + O(x^{\beta-\alpha-1}) \int_1^{x/2} t^{\alpha-\beta} e^t\, dt + x^{\beta-\alpha-1} \int_{x/2}^x o(t^{\alpha-\beta} e^t)\, dt \\ &= O(x^{\beta-\alpha-1}) + O(x^{\beta-\alpha-1} e^{x/2}) + o(x^{\beta-\alpha-1}(x/2)^{\alpha-\beta} e^x) \qquad (5) \\ &= o\left(\frac{e^x}{x}\right) \text{ as } x \longrightarrow \infty. \end{aligned}$$

Next, by (2),

$$\begin{aligned} B(x) &= e^{-x} \sum_{n=0}^{\infty} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \int_0^1 t^{\alpha n}\, d\chi(t^{\alpha n}) \sum_{r=0}^{n} \frac{s_{n-r}}{n - r + 1} \\ &= x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha) \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} \frac{(xt)^{\alpha n}}{\Gamma(\alpha n + \beta)} \frac{s_{n-r}}{n - r + 1} \\ &= x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha) \sum_{r=0}^{\infty} (xt)^{1-\beta} \sum_{n=0}^{\infty} \frac{(xt)^{\alpha n + \alpha r + \beta - 1}}{\Gamma(\alpha n + \alpha r + \beta)} \frac{s_n}{n+1}, \end{aligned}$$

the inversions in the order of operations, here and subsequently, being justified by absolute convergence. Since

$$\frac{(xt)^{\alpha n + \alpha r + \beta - 1}}{\Gamma(\alpha n + \alpha r + \beta)} = \frac{1}{\Gamma(\alpha r)\Gamma(\alpha n + \beta)} \int_0^{xt} (xt - u)^{\alpha r - 1} u^{\alpha n + \beta - 1}\, du$$

when $r > 0$, it follows that

$$\begin{aligned} B(x) &= x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha)(xt)^{1-\beta} \left(\phi(xt) + \int_0^{xt} \phi(u)\, du \sum_{r=1}^{\infty} \frac{(xt-u)^{\alpha r - 1}}{\Gamma(\alpha r)} \right) \\ &= x^{\beta-1} e^{-x} \int_0^1 d\chi(t^\alpha)(xt)^{1-\beta} \left(\phi(xt) + \int_0^{xt} E(xt - u)\phi(u)\, du \right), \end{aligned}$$

where

$$E(x) := \sum_{r=1}^{\infty} \frac{x^{\alpha r-1}}{\Gamma(\alpha r)} \sim \frac{e^x}{\alpha} \text{ as } x \longrightarrow \infty, \tag{6}$$

by Lemma 4(i). Hence

$$\begin{aligned} B(x) &= x^{\beta-1}e^{-x}\int_0^1 d\chi(t^\alpha)(xt)^{1-\beta}\Big(\phi(xt) + x\int_0^t E(xt-xu)\phi(xu)\,du\Big) \\ &= x^{\beta-1}e^{-x}\int_0^1 (xt)^{1-\beta}\phi(xt)\,d\chi(t^\alpha) \\ &\quad + x^\beta e^{-x}\int_0^1 (xt)^{1-\beta}\phi(xt)\,dt\int_t^1 (u/t)^{1-\beta}E(xu-xt)\,d\chi(u^\alpha). \end{aligned} \tag{7}$$

Now let

$$F(x) := \sum_{n=0}^{\infty} \frac{1}{n+1}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)},$$

so that $F(x)$ is value of $\phi(x)$ when $s_n \equiv 1$. Then

$$F(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{\alpha n+\beta}{n+1}\frac{x^{\alpha n+\beta}}{\Gamma(\alpha n+\beta+1)} \sim \frac{e^x}{x} \text{ as } x \longrightarrow \infty,$$

by the regularity of $(B, \alpha, \beta+1)$. Hence, by (5),

$$\phi(x) = o\big(F(x)\big) \text{ as } x \longrightarrow \infty$$

and so, given $\epsilon > 0$, there is an $x_0 > 0$ such that

$$|\phi(x)| \le \epsilon F(x) \text{ for } x \ge x_0.$$

Further, replacing $\phi(xt)$ by $F(xt)$ in (7) yields a $B(x)$ which tends to $1/\alpha$ as $x \longrightarrow \infty$ and, since $\beta \ge 1$, to 0 as $x \longrightarrow 0+$, and hence this $B(x)$ is dominated by a constant M for all $x > 0$. Thus the contribution to (7) of the parts of the integrals over the range $x_0/x \le t \le 1$ is in modulus less than ϵM for all $x > 0$. Since ϵ can be taken arbitrarily small, in order to establish (4) it suffices to show that the contribution to (7) of the parts of the integrals over the range $0 < t < x_0/x$ tends to 0 as $x \longrightarrow \infty$.

Since $v^{1-\beta}\phi(v)$ is bounded for $0 < v \le x_0$, it follows that

$$x^{\beta-1}e^{-x}\int_0^{x_0/x}(xt)^{1-\beta}\phi(xt)\,d\chi(t^\alpha) = O\left(x^{\beta-1}e^{-x}\int_0^1 d\chi(t^\alpha)\right) = o(1) \text{ as } x \longrightarrow \infty.$$

Further, by (6) and because $\beta \ge 1$,

$$\begin{aligned} &x^\beta e^{-x}\int_0^{x_0/x}(xt)^{1-\beta}\phi(xt)\,dt\int_t^1 (u/t)^{1-\beta}E(xu-xt)\,d\chi(u^\alpha) \\ &= x^\beta e^{-x}O\left(\int_0^{x_0/x}dt\int_t^{x_0/x}e^{xu-xt}\,d\chi(u^\alpha) + \int_0^{x_0/x}dt\int_{x_0/x}^1 (u/t)^{1-\beta}e^{xu-xt}\,d\chi(u^\alpha)\right) \\ &= x^\beta e^{-x}O\left(e^{x_0}\int_0^{x_0/x}d\chi(u^\alpha)\int_0^u dt + \int_{x_0/x}^1 u^{1-\beta}e^{xu}\,d\chi(u^\alpha)\int_0^{x_0/x}t^{\beta-1}e^{-xt}\,dt\right) \\ &= O(x^{\beta-1}e^{-x}) + O\left(e^{-x}\int_{x_0/x}^1 u^{1-\beta}e^{xu}\,d\chi(u^\alpha)\int_0^{x_0}t^{\beta-1}e^{-t}\,dt\right) \\ &= o(1) + O\left(\int_{x_0/x}^{1/2}u^{1-\beta}e^{xu-x}\,d\chi(u^\alpha) + \int_{1/2}^1 u^{1-\beta}e^{xu-x}\,d\chi(u^\alpha)\right) \\ &= o(1) + O(x^{\beta-1}e^{-x/2}) + o(1) = o(1) \text{ as } x \longrightarrow \infty, \end{aligned}$$

the final integral tending to 0 by the Lebesgue-Stieltjes theorem on dominated convergence since, for $1/2 \leq u < 1$, $u^{1-\beta} > u^{1-\beta}e^{xu-x} \longrightarrow 0$ as $x \longrightarrow \infty$, and $\chi(u^\alpha) \longrightarrow \chi(1)$ as $u \longrightarrow 1-$.

This establishes the case $k = 1$ of the lemma. Applying this case $k-1$ times, we obtain the required result. ∎

LEMMA 6. *Suppose that $s_n \longrightarrow s(B, \alpha, \beta)$ and that (1) holds with p a positive integer. Then*

$$t_n^{(k)} = O\big((n^{1/2}\log n)^{p-k}\big) \text{ for } k = 1, 2, \ldots, p.$$

PROOF. Assume again that $s = 0$. Let $x > 0$, $\frac{1}{2} < \xi < \frac{2}{3}$, $0 < \eta < 2\xi - 1$,

$$h_n =: n - \frac{x}{\alpha}, \quad m := \left[\frac{x}{\alpha}\right], \text{ and } B_n := \sum_{r=0}^{n} \frac{s_r}{r+1}.$$

Then

$$L_n \sim \log n \text{ and } B_n = O\big((n^{1/2}\log n)^p \log n\big) \tag{8}$$

and, by Lemma 5,

$$T(x) := e^{-x}\sum_{n=0}^{\infty} t_n \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = o(1) \text{ as } x \longrightarrow \infty. \tag{9}$$

Write

$$\begin{aligned} T(x) &= e^{-x}\sum_{n=0}^{\infty}(B_n - B_m)\frac{x^{\alpha n+\beta-1}}{L_n\Gamma(\alpha n+\beta)} + e^{-x}B_m\sum_{n=0}^{\infty}\frac{x^{\alpha n+\beta-1}}{L_n\Gamma(\alpha n+\beta)} \\ &=: T_1(x) + T_2(x), \end{aligned} \tag{10}$$

$$T_1(x) = e^{-x}\left(\sum_{h_n<-x^\xi} + \sum_{|h_n|\leq x^\xi} + \sum_{h_n>x^\xi}\right) =: S_1(x) + S_2(x) + S_3(x). \tag{11}$$

By (8) and Lemma 4(ii), as $x \longrightarrow \infty$,

$$S_1(x) = O\left(m^{p/2}(\log m)^{p+1}e^{-x}\sum_{h_n<-x^\xi}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right) = O(e^{-x^\eta}), \tag{12}$$

and

$$\begin{aligned} S_3(x) &= O\left(e^{-x}\sum_{h_n>x^\xi}(n^{1/2}\log n)^p\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right) \\ &= O\left(e^{-x}\sum_{h_n>x^\xi}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta-p)}\right) = O(e^{-x^\eta}). \end{aligned} \tag{13}$$

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By (8) and Lemma 4(iii), as $x \longrightarrow \infty$,

$$
\begin{aligned}
S_2(x) &= O\left(e^{-x} \sum_{|h_n|\le x^\xi} (|h_n|+1)x^{(p-2)/2}(\log x)^p \frac{x^{\alpha n+\beta-1}}{L_n\Gamma(\alpha n+\beta)}\right)\\
&= O\left(e^{-x}x^{(p-2)/2}(\log x)^{p-1} \sum_{|h_n|\le x^\xi} (|h_n|+1)\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right)\\
&= O\left(x^{(p-2)/2}(\log x)^{p-1} \sum_{|h_n|\le x^\xi} (|h_n|+1)\frac{1}{\sqrt{2\pi x}}\exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right) \qquad (14)\\
&= O\left(x^{(p-3)/2}(\log x)^{p-1} \int_{-\infty}^{\infty} (|t|+1)\exp\left(-\frac{\alpha^2 t^2}{2x}\right)dt\right)\\
&= O\big(x^{(p-1)/2}(\log x)^{p-1}\big) + O\big(x^{(p-2)/2}(\log x)^{p-1}\big)\\
&= O\big((x^{1/2}\log x)^{p-1}\big).
\end{aligned}
$$

It follows from (10), (11), (12), (13) and (14) that

$$
T_1(x) = O\big((x^{1/2}\log x)^{p-1}\big) \text{ as } x \longrightarrow \infty. \qquad (15)
$$

Next,

$$
T_2(x) = e^{-x}B_m\left(\sum_{h_n<-x^\xi} + \sum_{|h_n|\le x^\xi} + \sum_{h_n>x^\xi}\right) =: V_1(x)+V_2(x)+V_3(x). \qquad (16)
$$

By (8) and Lemma 4(ii), as $x \longrightarrow \infty$,

$$
\begin{aligned}
V_1(x)+V_3(x) &= e^{-x}t_m O\left(\sum_{h_n<-x^\xi}\frac{x^{\alpha n+\beta-1}\log x}{\Gamma(\alpha n+\beta)} + \sum_{h_n>x^\xi}\frac{x^{\alpha n+\beta-1}\log x}{\Gamma(\alpha n+\beta)}\right) \qquad (17)\\
&= O(t_m e^{-x^\eta}).
\end{aligned}
$$

Finally, as $x \longrightarrow \infty$,

$$
V_2(x) = t_m e^{-x}\sum_{h_n>x^\xi}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\frac{L_m}{L_n} = t_m\left(\frac{1}{\alpha}+o(1)\right), \qquad (18)
$$

since L_m/L_n in the above sum lies between $L_m/L_{[x/\alpha+x^\xi]}$ and $L_m/L_{[x/\alpha-x^\xi]}$ each of which tends to 1 as $x \longrightarrow \infty$ and, by Lemma 4(i) and (ii),

$$
\lim_{x\to\infty} e^{-x}\sum_{h_n>x^\xi}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = \lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty}\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = \frac{1}{\alpha}.
$$

It follows from (16), (17) and (18) that

$$
T_2(x) = t_m\left(\frac{1}{\alpha}+o(1)\right) \text{ as } x \longrightarrow \infty,
$$

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and hence from (9) and (15) that

$$t_m = O\big((x^{1/2}\log x)^{p-1}\big) \text{ as } x \longrightarrow \infty.$$

Taking $x = \alpha n$, we get

$$t_n = t_n^{(1)} = O\big((n^{1/2}\log n)^{p-1}\big).$$

When $p \geq 2$ we can replace t_n by $t_n^{(2)}$ in (9) and argue as above to obtain

$$t_n^{(2)} = O\big((n^{1/2}\log n)^{p-2}\big).$$

The proof can now be completed by induction in the obvious way. ∎

3. **Proof of the theorem.** By Lemma 6,

$$t_n^{(p)} - t_{n-1}^{(p)} = \frac{t_n^{p-1}}{(n+1)L_{n-1}} - \frac{1}{(n+1)L_nL_{n-1}}\sum_{r=0}^{n}\frac{t_r^{(p-1)}}{r+1} = O(n^{-1/2}).$$

Hence, by Lemma 5 and Lemma 2,

$$t_n^{(p)} \longrightarrow s \text{ as } n \longrightarrow \infty,$$

and so, by Lemma 3,

$$s_n \longrightarrow s\big(R, \log(n+1), p\big).$$ ∎

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