Canad. Math. Bull. Vol. **15** (2), 1972

AN ELEMENTARY RESULT ON EXPONENTIAL MEASURE SPACES

BY
C. Y. SHEN

A simple but useful result in the measure theory for product spaces can be stated as follows:

THEOREM A. *A necessary and sufficient condition that a measurable subset E of $X\times Y$ has measure zero is that almost every X-section (or almost every Y-section) has measure zero (see* [1, §36]).

We will show, in this short note, that a similar result also holds for the exponential of measure spaces. Before proceeding any further, we describe briefly here the exponential construction of a measure space.

Let (X, χ, ξ) be a σ-finite measure space. For each nonnegative integer n, $(X^{\cdot n}, \chi^{\cdot n}, \xi^{\cdot n})$ denotes the nth product space. When $n=0$, $X^{\cdot n}=\{0\}$ and $\xi^{\cdot 0}(\{0\})=1$. Let $X_e^{\cdot}=\bigcup_{n=0}^{\infty} X^{\cdot n}$. Then

$$\chi_e^{\cdot} = \{\textstyle\bigcup_{n=0}^{\infty} A_n\colon A_n \in \chi^{\cdot n} \text{ for each } n\}$$

is a σ-algebra of subsets of $X_e^{\cdot}$, and the set function $\xi_e^{\cdot}$ defined on $\chi_e^{\cdot}$ by

$$\xi_e^{\cdot}(E) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \xi^{\cdot n}(E \cap X^{\cdot n})$$

is a σ-finite measure. Two sequences $x, y \in X^{\cdot n}$ are equivalent if one is a rearrangement of the other. The set of equivalence classes of $X^{\cdot n}$ is denoted by X^n and the set $X_e=\bigcup_{n=0}^{\infty} X^n$ is called the exponential of the set X. The quotient space of $(X_e^{\cdot}, \chi_e^{\cdot}, \xi_e^{\cdot})$ under the natural projection $p\colon X_e^{\cdot} \to X_e$ is called the exponential space of (X, χ, ξ) and is denoted by (X_e, χ_e, ξ_e). Exponential spaces arise naturally as the underlying sample spaces in the general theory of counting processes. For more detailed discussion, we refer readers to [2], [3].

Each unordered sequence $x \in X_e$, $x\neq 0$, can be regarded as a formal product $t_1 \ldots t_n$ of elements in X where the order of the factors is irrelevant. On X_e, one can introduce a binary operation as follows: If $x=t_1\ldots t_m$, $y=t_1'\ldots t_n'$, then

$$xy = t_1\ldots t_m t_1' \ldots t_n'$$

with 0 as the identity element. For each $E\subset X_e$ and each $x \in X_e$, we define the "x-section" of E as

$$E_x = \{y\colon xy \in E\}.$$

Received by the editors March 8, 1971 and, in revised form, April 6, 1971.

THEOREM B. *Let $E \in \chi_e$. Then for all $x \in X_e$, E_x is measurable. If $\xi_e(E) = 0$, then for almost all $x \in X_e$, $\xi_e(E_x) = 0$.*

Proof. For each integer $n \geq 0$, let $E^n = E \cap X^n$. It is easy to show that

$$E_x = \bigcup_{n=0}^{\infty} (E^n)_x.$$

By taking each E^n separately, we see that the theorem reduces to the following: Let n, k be nonnegative integers, and suppose that $E \in \chi_e$ and $E \subset X^n$. Then for all $x \in X^k$, E_x is measurable. Moreover, if $\xi_e(E) = 0$ then $\xi_e(E_x) = 0$ for almost all x in X^k.

Case 1. $k > n$. Here $E_x = \varnothing$ and the result follows trivially.

Case 2. $k = 0$. Here $E_x = E$ and the result is again trivial.

Case 3. $k = n$. Here $E_x = \begin{cases} X^0 & \text{if } x \in E \\ \varnothing & \text{if } x \notin E. \end{cases}$

If $\xi_e(E) = 0$ then $\xi_e(E_x) = 0$ for almost all $x \in X^n$.

Case 4. $1 \leq k \leq n-1$. Consider the set $F = p^{-1}(E)$. Let $x = p(u)$, and F_u denote the section $\{v: (u, v) \in F\}$. It is easily shown that $F_u = p^{-1}(E_x)$. By [1, §34, Theorem A], F_u is measurable; hence E_x is measurable. If E has measure zero, then $\xi_e^{\cdot}(F) = \xi_e(E) = 0$ and by [1, §36, Theorem A] there is a null set $F' \subset X^{\cdot k}$ such that if $u \in X^{\cdot k} \sim F'$ then $\xi^{\cdot n-k}(F_u) = 0$. Take $E' = p(F')$. It follows that

$$\xi_e(E') = \xi^{\cdot k}(p^{-1}(p(F'))) = 0.$$

If $x \notin E'$ then $u \notin F'$, and $\xi_e(E_x) = \xi^{\cdot n-k}(F_u) = 0$.

A typical application of Theorem B can be given as follows: For each real-valued function φ on X_e and for each $x \in X_e$, we define φ_x on X_e by the formula

$$\varphi_x(y) = \varphi(xy).$$

COROLLARY C. *If $\varphi = \psi$ a.e., then for almost all $x \in X_e$, $\varphi_x = \psi_x$ a.e.*

REFERENCES

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N.J., 1950.

2. D. S. Carter and P. M. Prenter, *Exponential spaces and counting processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete (to appear).

3. D. S. Carter, *The exponential of a measure space*, Tech. Rep. No. 29, Dept. of Math., Oregon State Univ., Corvallis, 1966.

SIMON FRASER UNIVERSITY,
BURNABY, BRITISH COLUMBIA