NODAL NON-COMMUTATIVE JORDAN ALGEBRAS

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1. Introduction. A finite dimensional power-associative algebra $\mathfrak{A}$ with a unity element 1 over a field $\mathfrak{F}$ is called a nodal algebra by Schafer **(7)** if every element of $\mathfrak{A}$ has the form $\alpha 1 + z$ where α is in $\mathfrak{F}$, z is nilpotent, and if $\mathfrak{A}$ does not have the form $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ with $\mathfrak{N}$ a nil subalgebra of $\mathfrak{A}$. An algebra $\mathfrak{A}$ is called a non-commutative Jordan algebra if $\mathfrak{A}$ is flexible and $\mathfrak{A}^+$ is a Jordan algebra. Some examples of nodal non-commutative Jordan algebras were given in **(5)** and it was proved in **(6)** that if $\mathfrak{A}$ is a simple nodal non-commutative Jordan algebra of characteristic not 2, then $\mathfrak{A}^+$ is associative. In this paper we describe all simple nodal non-commutative Jordan algebras of characteristic not 2. Any such algebra has the form $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ with $\mathfrak{N}^+ = \mathfrak{F}[x_1, \ldots, x_n]$ for some n where the generators are all nilpotent of index p. The x_i can be selected so that $x_i x_j = \alpha_{ij} 1 + w_{ij}$ for w_{ij} in $\mathfrak{N}$ and α_{ij} in $\mathfrak{F}$ such that, for each i, some $\alpha_{ij} \neq 0$. Moreover, the multiplication table of $\mathfrak{A}$ is given by

$$f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) = f \cdot g + \tfrac{1}{2} \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j] \tag{1}$$

where the dot product $a \cdot b = \frac{1}{2}(ab + ba)$ is the product of $\mathfrak{A}^+$ and $[x_i, x_j] = x_i x_j - x_j x_i$.

The author would like to express his great indebtedness to R. D. Schafer for finding errors in the original manuscript and for showing how they could be corrected.

2. Properties of $\mathfrak{A}^+$. If $\mathfrak{D}$ is the derivation algebra of an algebra $\mathfrak{B}$, then Albert in **(1)** calls $\mathfrak{B}$ $\mathfrak{D}$-simple if there exists no ideal $\mathfrak{M}$, other than $\mathfrak{B}$ or 0, such that mD is in $\mathfrak{M}$ for every m in $\mathfrak{M}$ and D in $\mathfrak{D}$. We use a result of Harper **(2)** which for our purposes may be stated as follows.

THEOREM 1. (*Harper*) *Let $\mathfrak{B}$ be a commutative associative algebra with a unity quantity* 1 *over a field $\mathfrak{F}$ and let $\mathfrak{B}$ have the form $\mathfrak{B} = \mathfrak{F}1 + \mathfrak{N}$ with $\mathfrak{N}$ the radical of $\mathfrak{B}$. Also let $\mathfrak{B}$ be $\mathfrak{D}$-simple where $\mathfrak{D}$ is any set of derivations on $\mathfrak{B}$. Then $\mathfrak{N} = \mathfrak{F}[x_1, \ldots, x_n]$ for some n where the generators x_i have index p, p the characteristic of $\mathfrak{F}$.*

We remark that it is known that a $\mathfrak{D}$-simple algebra cannot have characteristic zero and Schafer has shown in **(7)** that a nodal non-commutative

Received February 17, 1958; in revised form August 31, 1959. Presented to the American Mathematical Society with the title *Nodal flexible associative-admissible algebras* on November 29, 1957.

https://doi.org/10.4153/CJM-1960-043-2 Published online by Cambridge University Press

Jordan algebra cannot have characteristic zero. He also uses a theorem of Jacobson **(4)** to prove that $\mathfrak{N}^+$ is a subalgebra of $\mathfrak{A}^+$ for any nodal non-commutative Jordan algebra.

THEOREM 2. *Let $\mathfrak{A}$ be a simple nodal non-commutative Jordan algebra over a field $\mathfrak{F}$ whose characteristic is not 2. Let $\mathfrak{D}$ be the derivation algebra of $\mathfrak{A}$. Then $\mathfrak{A}^+$ is $\mathfrak{D}$-simple.*

Suppose $\mathfrak{A}^+$ is not $\mathfrak{D}$-simple. Then there is an ideal $\mathfrak{B}$ of $\mathfrak{A}^+$ such that $\mathfrak{B}\mathfrak{D} \subseteq \mathfrak{B}$. We shall show that $\mathfrak{B}$ is then an ideal of $\mathfrak{A}$, contradicting the fact that $\mathfrak{A}$ is simple. The mapping $bD = [b, c]$ where c is any element of $\mathfrak{A}$ and $[b, c] = bc - cb$ is a derivation of $\mathfrak{A}^+$. This is so because $(a \cdot b)D = aD \cdot b + a \cdot bD$ if and only if $[a \cdot b, c] = [a, c] \cdot b + a \cdot [b, c]$ and the last identity follows from $(ab)c + (cb)a = a(bc) + c(ba)$, the linearized form of the flexible law $(ab)a = a(ba)$. Now let b be in $\mathfrak{B}$ and a in $\mathfrak{A}$. Since $\mathfrak{B}$ is a $\mathfrak{D}$-ideal of $\mathfrak{A}^+$, $bD = [b, a]$ is in $\mathfrak{B}$. Also, since $\mathfrak{B}$ is an ideal of $\mathfrak{A}^+$, $a \cdot b$ is in $\mathfrak{B}$. Then $ba - ab$ and $ab + ba$ in $\mathfrak{B}$ imply ab and ba are in $\mathfrak{B}$. That is, $\mathfrak{B}$ is an ideal of $\mathfrak{A}$.

COROLLARY. *If $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ is a simple nodal non-commutative Jordan algebra over a field $\mathfrak{F}$ whose characteristic is not 2, then $\mathfrak{N}^+ = \mathfrak{F}[x_1, \ldots, x_n]$ for some n, where $x_i^p = 0$, $x_i^{p-1} \neq 0$. Thus, $\mathfrak{A}$ has order p^n.*

3. The multiplication table of $\mathfrak{A}$. Assume that $\mathfrak{A}$ is simple so that, by the corollary above, $\mathfrak{A}^+ = \mathfrak{F}[1, x_1, \ldots, x_n]$ with $x_i^p = 0$. In **(3)**, Jacobson has shown that if D is any derivation on $\mathfrak{A}^+$, then

$$fD = \sum_i \frac{\partial f}{\partial x_i} \cdot a_i$$

for any f in $\mathfrak{A}^+$ and for a_i in $\mathfrak{A}^+$. The a_i of course depend on the derivation D. If g is any element of $\mathfrak{A}^+$, we have seen that the mapping $fD = [f, g]$ is a derivation of $\mathfrak{A}^+$. Hence

$$fD = [f, g] = \sum_i \frac{\partial f}{\partial x_i} \cdot a_i(g).$$

To evaluate the $a_i(g)$, we note that $x_i D = [x_i, g] = a_i(g)$ and

$$[g, x_i] = \sum_j \frac{\partial g}{\partial x_j} \cdot a_j(x_i).$$

Since $[x_i, g] = -[g, x_i]$,

$$a_i(g) = -\sum_j \frac{\partial g}{\partial x_j} \cdot a_j(x_i)$$

and since $[x_j, x_i] = a_j(x_i)$, it follows that

$$[f, g] = \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j].$$

https://doi.org/10.4153/CJM-1960-043-2 Published online by Cambridge University Press

THEOREM 3. *If $\mathfrak{A}$ is a simple algebra, then for any f, g in $\mathfrak{A}$,*

$$fg = f\cdot g + \tfrac{1}{2}\sum_{i,j}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot[x_i, x_j].$$

This result follows from the above formula for $[f, g]$ and the fact that $fg = f\cdot g + \frac{1}{2}[f, g]$. The assumption that $\mathfrak{A}$ is nodal implies that at least one of the $[x_i, x_i]$ is not in $\mathfrak{N}$. This is equivalent to the statement that for some i, j, $x_i x_j$ is not in $\mathfrak{N}$.

THEOREM 4. *The generators $x_1, \ldots, x_n$ can be selected so that $x_i x_j = \alpha_{ij}1 + w_{ij}$ with w_{ij} in $\mathfrak{N}$ and α_{ij} in $\mathfrak{F}$ such that, for each i, some $\alpha_{ij} \neq 0$.*

Let $\mathfrak{M}$ be the vector space with $x_1, \ldots, x_n$ as a basis. If we write $\alpha_{ij} = \alpha(x_i, x_j)$ then $x_j x_i = 2x_i\cdot x_j - x_i x_j = -\alpha_{ij} - w_{ij} + 2x_i\cdot x_j$ together with the fact that $x_i\cdot x_j$ is in $\mathfrak{N}$, implies that $\alpha(x_j, x_i) = -\alpha(x_i, x_j)$. Therefore $\alpha(x_i, x_j)$ is a skew-symmetric bilinear form on $\mathfrak{M}$. If the rank of the form is $2r$, there exists a basis $x_1', \ldots, x_n'$ such that we have the canonical form

$$\alpha(x_i', x_{i+r}') = 1 = -\alpha(x_{i+r}', x_i')$$

for $i \leqslant r$, $\alpha(x_i', x_j') = 0$ for all other pairs i, j. Next take $x_i'' = x_i'$ for $i \leqslant 2r$ and $x_i'' = x_i' + x_1'$ for $i > 2r$. Then, if $i \leqslant r$, $\alpha(x_i'', x_{i+r}'') = \alpha(x_i', x_{i+r}') = 1$; if $r < i \leqslant 2r$,

$$\alpha(x_i'', x_{i-r}'') = \alpha(x_i', x_{i-r}') = -\alpha(x_{i-r}', x_{(i-r)+r}') = -1;$$

and if $i > 2r$, $\alpha(x_i'', x_{r+1}'') = \alpha(x_i' + x_1', x_{r+1}') = \alpha(x_1', x_{r+1}') = 1$. The basis $x_1'', \ldots, x_n''$ of $\mathfrak{M}$ has the properties stated in Theorem 4.

4. Construction of algebras. Let $\mathfrak{F}$ be any field of characteristic $p \neq 2$. Define $\mathfrak{A}^+$ by $\mathfrak{A}^+ = \mathfrak{F}1 + \mathfrak{N}^+$ where $\mathfrak{N}^+ = \mathfrak{F}[x_1, \ldots, x_n]$ with $x_1, \ldots, x_n$ nilpotent generators of index p. That is, $\mathfrak{A}^+$ consists of elements $\alpha 1 + z$ where α is in F, 1 is the unity quantity of $\mathfrak{A}^+$, and z is a polynomial in $x_1, \ldots, x_n$. Define the algebra $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ to be the same vector space as $\mathfrak{A}^+$ and to have a product defined by $x_i x_j = \alpha_{ij}1 + w_{ij}$ for any $\alpha_{ij} = -\alpha_{ji}$ in $\mathfrak{F}$ and and $w_{ij} = 2x_i\cdot x_j - w_{ji}$ in $\mathfrak{N}$, $i < j$. Further define

$$fg = f\cdot g + \tfrac{1}{2}\sum_{i,j}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot[x_i, x_j]$$

for f, g any elements in $\mathfrak{A}$.

THEOREM 5. *If at least one $\alpha_{ij} \neq 0$, the algebra $\mathfrak{A}$ described above is a nodal non-commutative Jordan algebra.*

Linearization of the flexible law $(fg)f = f(gf)$ yields the identity $(fg)h + (hg)f = f(gh) + h(gf)$. Add $(gf)h + (gh)f$ to both sides of the equality to obtain

$$(f\cdot g)h + (g\cdot h)f = (gf)\cdot h + (gh)\cdot f. \tag{2}$$

Since $\mathfrak{A}$ has characteristic $\neq 2$, flexibility is equivalent to identity (2). The expression

$$gf\cdot h + gh\cdot f - (g\cdot h)f - (f\cdot g)h$$
$$= f\cdot g\cdot h + \tfrac{1}{2}\sum_{i,j}\frac{\partial g}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot[x_i, x_j]\cdot h + f\cdot g\cdot h$$
$$+ \tfrac{1}{2}\sum_{i,j}\frac{\partial g}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\cdot[x_i, x_j]\cdot f - f\cdot g\cdot h$$
$$- \tfrac{1}{2}\sum_{i,j}\frac{\partial (g\cdot h)}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot[x_i, x_j] - f\cdot g\cdot h$$
$$- \tfrac{1}{2}\sum_{i,j}\frac{\partial (f\cdot g)}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\cdot[x_i, x_j].$$

Using

$$\frac{\partial (a\cdot b)}{\partial x} = \frac{\partial a}{\partial x}\cdot b + a\cdot\frac{\partial b}{\partial x},$$

the above expression becomes

$$\tfrac{1}{2}\sum_{i,j}[x_i, x_j]\cdot\left(\frac{\partial g}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot h + \frac{\partial g}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\cdot f - \frac{\partial g}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot h - \frac{\partial h}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot g - \frac{\partial f}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\cdot g - \frac{\partial g}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\cdot f\right)$$
$$= \tfrac{1}{2}\sum_{i,j}[x_i, x_j]\cdot\left(-\frac{\partial h}{\partial x_i}\cdot\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial h}{\partial x_j}\right)\cdot g$$
$$= f\cdot g\cdot h - (hf)\cdot g + f\cdot g\cdot h - (fh)\cdot g = 0$$

as desried. The algebra is nodal since at least one α_{ij} is not zero.

The proof of Theorem 4 depends only on $\mathfrak{A}$ having the form as described at the beginning of this section and it is not necessary for $\mathfrak{A}$ to be simple in order to obtain the result of Theorem 4. Thus we may assume that the generators $x_1, \ldots, x_n$ have the properties of Theorem 4 and that we have the associated bilinear form of rank $2r$.

THEOREM 6. *If $n = 2r$, then $\mathfrak{A}$ is simple.*

Suppose $\mathfrak{B}$ is a proper ideal of $\mathfrak{A}$. Then there exists a polynomial $f = f(x_1, \ldots, x_n)$ in $\mathfrak{B}$ with least possible degree t in $x_1, \ldots, x_n$. Since $n = 2r$, $\alpha_{ij} = 0$ except for the following: $\alpha_{i,r+i} = 1$ for $i \leqslant r$; and $\alpha_{i,i-r} = -1$ for $r < i \leqslant 2r$. Then for each i there exists a k such that $\alpha_{ki} \neq 0$ but $\alpha_{kj} = 0$ for all $j \neq i$. Then for this i,

$$x_k f = \sum_j \alpha_{kj}\frac{\partial f}{\partial x_j} + \text{terms of degree} \geqslant t = \alpha_{ki}\frac{\partial f}{\partial x_i} + \text{terms of degree} \geqslant t.$$

https://doi.org/10.4153/CJM-1960-043-2 Published online by Cambridge University Press

Therefore, if any monomial of f of degree t has a power x_i as a factor, $x_k f$ is a polynomial of degree $t - 1$. The fact that f is in $\mathfrak{B}$ implies that $x_k f$ is in $\mathfrak{B}$ and this contradicts the assumption that f has minimal degree t.

If $n > 2r$, $\mathfrak{A}$ is not necessarily simple. For example, consider $x_1 - x_{2r+1}$ which has the property that $(x_1 - x_{2r+1})\mathfrak{A} \subseteq \mathfrak{N}$. Then $\mathfrak{B} = (x_1 - x_{2r+1})\cdot\mathfrak{A}$ is an ideal of $\mathfrak{A}$ if

$$
\begin{aligned}
[(x_1 - x_{2r+1})\cdot g]f &= (x_1 - x_{2r+1})\cdot g\cdot f + \tfrac{1}{2}\sum_{i,j} \frac{\partial[(x_1 - x_{2r+1})\cdot g]}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot[x_i, x_j] \\
&= (x_1 - x_{2r+1})\cdot g\cdot f + \tfrac{1}{2}\sum_{j} \frac{\partial f}{\partial x_j}\cdot g\cdot[x_1 - x_{2r+1}, x_j] \\
&\qquad + \tfrac{1}{2}\sum_{i,j} \frac{\partial g}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\cdot[x_i, x_j]\cdot(x_1 - x_{2r+1})
\end{aligned}
$$

is in $\mathfrak{B}$ for every g and f in $\mathfrak{A}$. This will be so if $[x_1 - x_{2r+1}, x_j]$ is in $\mathfrak{B}$ for every j. This can be accomplished by setting $x_1 x_j = x_j x_1 = x_1\cdot x_j$ and $x_{2r+1}x_j = x_j x_{2r+1} = x_{2r+1}\cdot x_j$. Then $[x_1 - x_{2r+1}, x_j] = 0$ is certainly in $\mathfrak{B}$ for every j.

It seems clear that whether or not $\mathfrak{A}$ is simple with $n > 2r$ depends on the nature of the nilpotent elements w_{ij}.

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