Let a, b, c denote sides of triangle ABC; h the projection of c on a; p, q, r, s the sides of the required quadrilateral.

Then $$q+p=c, \quad s-r=b \quad \text{(from } AB^2\text{)}$$
$$q-p=h-a, \quad s+r=h \quad \text{(from } BD^2\text{)}.$$

These relations determine p, q, r, s and provide simple constructions for drawing the quadrilateral. The second set (dotted lines) is got by changing the third equation into $p-q=h-a$, so that this set has the same sides as the previous set but in different order.

When $a=h$ we get $p=q$ and the two sets degenerate into one and give Perigal's dissection of Pythagoras' theorem (Mackay's Euclid, 1897, p. 93). When $h=b$ the quadrilaterals become triangles ($r=0$). When $h<b$ r becomes negative; the construction still holds if the sign convention is admitted.

The construction of the quadrilateral may also be approached trigonometrically. Taking θ for the angle between sides s, r we find

$$b\cos\theta+h\sin\theta=c$$

which determines two values of θ when $b^2+h^2>c^2$ and one when $b^2+h^2=c^2$. Circles could be drawn centrally in AB^2, BD^2 with radii $\frac{1}{2}b$, $\frac{1}{2}(h-a)$ and tangents to them making θ with AB, etc.

G. D. C. STOKES.

A new form for the sum of a trigonometric series.

1. The usual text book formula for $\Sigma \equiv \sum_{t=0}^{n-1}[\gamma^t \sin(\alpha+t\beta)]$ is

$$\Sigma=[\sin\alpha-\gamma\sin(\alpha-\beta)-\gamma^n\sin(\alpha+n\beta)+\gamma^{n+1}\sin(\alpha+\overline{n-1}\,\beta)]/R_0$$

where $R_0\equiv 1-2\gamma\cos\beta+\gamma^2=(1-\gamma\cos\beta)^2+\gamma^2\sin^2\beta$.

Now $$\sin\alpha-\gamma\sin(\alpha-\beta)=\sin\alpha\,(1-\gamma\cos\beta)+\cos\alpha\,(\gamma\sin\beta)$$
$$=\sqrt{R_0}\sin(\alpha+\theta)$$

where $\theta=\tan^{-1}[\gamma\sin\beta/(1-\gamma\cos\beta)]$.

Similarly $$\gamma^n\sin(\alpha+n\beta)-\gamma^{n+1}\sin(\alpha+\overline{n-1}\beta)$$
$$=\gamma^n\sqrt{R_0}\sin(\alpha+\theta+n\beta).$$

Hence $$\Sigma = [\sin(\alpha+\theta) - \gamma^n \sin(\alpha+\theta+n\beta)] / \sqrt{R_0} \quad\text{..........(1)}$$
$$= -\frac{1}{\sqrt{R_0}}[\gamma^t \sin(\alpha+\theta+t\beta)]_0^n \quad\text{..........(2)}$$
$$= \frac{1}{\sqrt{R_0}}[\gamma^t \sin(\pi+\alpha+\theta+t\beta)]_0^n \quad\text{..........(3)}$$

where R_0 and θ are independent of n and α.

By a similar transformation it may be shown that [ ] in (1) is equal to $\sqrt{R_n}\sin(\alpha+\theta-\phi)$ where $R_n \equiv 1 - 2\gamma^n \cos\beta + \gamma^{2n}$ and $\phi = \tan^{-1}[\gamma^n \sin n\beta / (1-\gamma^n \cos n\beta)]$.

Thus $\Sigma = \sqrt{R_n / R_0}\sin(\alpha+\theta-\phi)$..........(4)
a form which might be practically useful if we required a number of separate sums with varying initial angles, but each consisting of n terms; so that R_n and ϕ, as well as R_0 and θ, would be the same for each sum.

The positive values of the square roots are to be taken. The function $\tan^{-1}$ being many-valued, a value of θ must be selected giving $\sin\theta$ the sign of $\gamma\sin\beta$ and $\cos\theta$ the sign of $1-\gamma\cos\beta$; and similarly ϕ will take a value which gives $\sin\phi$ the sign of $\gamma^n \sin n\beta$ and $\cos\phi$ the sign of $1-\gamma^n\cos n\beta$. It is readily seen that for the sum of the corresponding cosine series we have merely to substitute *cos* for *sin* in the expressions given, the values of R_0, R_n, θ and ϕ remaining unchanged.

2. In what precedes we have merely transformed a known expression for Σ, but it is simpler to start *ab initio* and obtain Σ as the inverse of Δ.

Using the transformation of § 1, with $(\alpha+t\beta)$ for α and $-\beta$ for β, and therefore $-\theta$ for θ

$$\Delta_t[\gamma^t \sin(\alpha+t\beta)] = -\gamma^t[\sin(\alpha+t\beta) - \gamma\sin(\alpha+\overline{t+1}\beta)]$$
$$= -\sqrt{R_0}\,\gamma^t \sin(\alpha-\theta+t\beta) \quad\text{..........(5)}$$

where R_0 and θ have the values previously found.

Performing Σ on both sides

$$\gamma^t \sin(\alpha+t\beta) = -\sqrt{R_0}\,\Sigma\gamma^t \sin(\alpha-\theta+t\beta)$$

whence, putting $\alpha+\theta$ for α and dividing by $-\sqrt{R_0}$

$$-\sqrt{1/R_0}\,\gamma^t \sin(\alpha+\theta+t\beta) = \Sigma\gamma^t \sin(\alpha+t\beta) \quad\text{..........(6)}$$

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3. The integral $\int e^{\lambda x} \sin(\alpha + \beta x)\, dx$ is deducible in the form $e^{\lambda x} \sin(\alpha + \beta x - \theta) / \sqrt{\lambda^2 + \beta^2}$ by treating it as a limiting case of the summation cited in §2. Δ is replaced by δ, t by x, γ by e^{λ}, and the limits of $\sqrt{R} / \delta x$ and $\tan\theta$ are easily shown to be $\sqrt{\lambda^2 + \beta^2}$ and β / λ.

The method of obtaining this integral in the above form by reversing the derivative process is given by Edwards (*Int. Calc.* p. 46).

G. J. LIDSTONE.

To construct an ellipse whose focus F is given, which shall pass through three given points A, A′, A″ (Halley's Problem).

The following method is applicable to ellipses of small excentricity (such as the orbits of planets) and may be used as an alternative method to finding the directrix which in such cases is situated at a considerable distance.

With centre F describe a circle of any arbitrary and convenient radius r. If FA, FA', FA'' intersect this circle in a, a', a'', then corresponding chords such as AA', aa' intersect on a fixed straight line DE perpendicular to the focal axis of the ellipse, which is in fact the common chord of the two curves and distant r/e from the S-directrix. For if AA' intersect the directrix in Z, SZ is the external bisector of the angle AFA' and is therefore parallel to aa'. Also $Fa : FA = ZD : ZA =$ ratio of perpendiculars from D and A on directrix XZ. Hence the perpendicular from D on $XZ = r/e$.

Similarly by taking AA'', aa'' together we get another point E on the common chord which is now completely determined. The line through F perpendicular to DE is the focal axis.

Now let x be any other point on the circle and let xa' meet DE in G; then the intersection X of GA' and Fx is a point on the ellipse. It follows then that if the tangent at a' meets DE in T, TA' will be the tangent to the ellipse at A'. Hence the second

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