Compositio Mathematica **133:** 49–63, 2002.

On the Rank of Picard Groups of Modular Varieties Attached to Orthogonal Groups

JAN HENDRIK BRUINIER
University of Wisconsin-Madison, Department of Mathematics, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706-1388, U.S.A. e-mail: bruinier@math.wisc.edu

(Received: 21 January 2001)

Abstract. We derive lower bounds for the rank of Picard groups of modular varieties associated with natural congruence subgroups of the orthogonal group of an even lattice of signature $(2, l)$. As an example we consider the Siegel modular group of genus 2. The analytic part of this paper also leads to certain class number identities.

Mathematics Subject Classifications (2000). 11F55, 14C22.

Key words. Picard group, orthogonal group, class number.

1. Introduction

Let L be an even lattice of signature $(2, l)$ with $l \geqslant 3$. Write $q(\cdot)$ for the quadratic form on L and $\mathcal{L}$ for the (finite) discriminant group of L.

Let $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$ be the spinor kernel of the real orthogonal group of L and denote the corresponding Hermitean symmetric domain by $\mathcal{H}_l$. We write $\mathrm{O}'(L)$ for the intersection of the integral orthogonal group of L with $\mathrm{O}'(L \otimes_{\mathbb{Z}} \mathbb{R})$. We consider the discriminant kernel $\Delta(L)$ of the group $\mathrm{O}'(L)$, that is the subgroup of those elements that act trivially on $\mathcal{L}$.

There is a natural notion of principal congruence groups for the group $\Delta(L)$: For any non-zero integer N we have the rescaled lattice $L(N)$, given by L as a $\mathbb{Z}$-module, but equipped with the quadratic form $Nq(\cdot)$. The discriminant kernel of $L(N)$ is a subgroup of $\Delta(L)$, defined by congruence conditions modulo N. We call it the principal congruence subgroup of level N and denote it by $\Gamma(N)$.

We consider the arithmetic quotient $X(N) = \mathcal{H}_l/\Gamma(N)$. By the theory of Baily and Borel, it carries the structure of a quasiprojective algebraic variety. A fundamental geometric invariant is its algebraic Picard group $\mathrm{Pic}(X(N))$. Our assumption on l implies that this group is finitely generated. In the present paper we shall derive a nontrivial lower bound for the rank of $\mathrm{Pic}(X(N))$. In particular we are interested in the asymptotic behavior of the numbers $\mathrm{rank}(\mathrm{Pic}(X(N)))$ as $N \to \infty$. Although this problem seems very natural, to the best of our knowledge, just partial results can be found in the literature. (See for instance [LW1,LW2] or [GN].) Certainly one would expect that the rank of $\mathrm{Pic}(X(N))$ tends to infinity as $N \to \infty$, reflecting the fact that the geometry of $X(N)$ gets more complicated as the level rises. However, even a result of this type seems not to be known in general.

Put $X = X(1)$. It is a consequence of the work of Borcherds [Bo1,Bo2] and the refinement given in [Br1] that there exists a homomorphism

$$S_{\kappa,L} \longrightarrow (\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}) \big/ \mathbb{C}[E_{\mathrm{Hodge}}] \tag{1}$$

from a certain space $S_{\kappa,L}$ of $\mathbb{C}[\mathcal{L}]$-valued cusp forms of weight $\kappa = 1 + l/2$ to the quotient of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ modulo the span of the class of the Hodge line bundle E_{Hodge}.

If L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, then the main result of [Br1] says that this map is injective (see also [Br2] or [BF] for related results). Hence in this case we can obtain a lower bound for rank(Pic(X)) by estimating the dimension of $S_{\kappa,L}$. By means of the Riemann–Roch theorem or the Selberg trace formula, the dimension of $S_{\kappa,L}$ can be computed. Thereby the original problem is reduced to estimating the different contributions in the dimension formula. Some of these are 'strange' invariants of the discriminant group $\mathcal{L}$ and the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on it induced by q. They are studied in Section 2, the technical heart of this paper.

Let us now assume that L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, i.e. has the special shape $L = L_0 \perp H \perp H$, where L_0 is an even negative definite lattice. Then the above argument can be used to find a bound for the rank of Pic(X). Unfortunately, it cannot be applied directly to get a bound for Pic($X(N)$), since $L(N)$ does not split two hyperbolic planes over $\mathbb{Z}$.

Therefore we first consider the lattice

$$L[N] = L_0(N) \perp H \perp H$$

and its discriminant kernel $\Gamma[N] = \Delta(L[N])$. We write $X[N]$ for the quotient $\mathcal{H}_l/\Gamma[N]$. The group $\Gamma[N]$ can be viewed as a subgroup of the rational orthogonal group of L with the property that $\Gamma(N) \subset \Gamma[N]$. In the O(2, 3)-case of the Siegel modular group of genus 2 it is isomorphic to the paramodular group of level N. Using the injectivity of the map (1) and the estimate of Section 2 for the dimension of $S_{\kappa,L}$, we obtain a bound for rank(Pic($X[N]$)) (see Theorem 8). In particular we find that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (which can be easily determined) such that

$$\mathrm{rank}(\mathrm{Pic}(X[N])) \geqslant \frac{l|\mathcal{L}|N^{l-2}}{48} - \begin{cases} C_\varepsilon N^{1/2+\varepsilon}, & \text{if } l = 3, \\ C_\varepsilon N^{l-3+\varepsilon}, & \text{if } l > 3, \end{cases} \tag{2}$$

for all $N \in \mathbb{N}$ (Corollary 9).

The projection $X(N) \to X[N]$ induces an injective homomorphism

$$\mathrm{Pic}(X[N]) \longrightarrow \mathrm{Pic}(X(N)).$$

Hence all bounds for the rank of Pic($X[N]$) give us also bounds for the rank of Pic($X(N)$). There are some reasons to believe that our estimate (2) actually describes the true asymptotic growth of rank(Pic($X[N]$)), whereas the resulting bound for Pic($X(N)$) seems rather poor (see Questions 1 and 2). Better results for $X(N)$ could be obtained by studying the injectivity properties of the map (1) more carefully for lattices which do not split two hyperbolic planes over $\mathbb{Z}$.

As an important example we consider the special case of the Siegel modular group of genus 2 in somewhat more detail. We take $L = \mathbb{Z}(-2) \perp H \perp H$ and use the exceptional isomorphism from $\mathrm{Sp}(4, \mathbb{R})$ to $\mathrm{O}(2, 3)$. Due to the work of Weissauer [We1,We2] we know a lot about the Picard groups $\mathrm{Pic}(X(N))$ in this case. For instance the Tate conjecture for algebraic divisors is proved in [We1]. However, lower bounds for the rank of $\mathrm{Pic}(X[N])$ or $\mathrm{Pic}(X(N))$ seem not to be known in general.

The group $\Gamma[N]$ is isomorphic to the paramodular group of level N (cf. [GrNi]). The quotient $X[N]$ is the moduli space of Abelian surfaces with a $(1, N)$-polarization. Our result implies that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\mathrm{rank}(\mathrm{Pic}(X[N])) \geqslant N/8 - C_\varepsilon N^{1/2+\varepsilon}$$

for all $N \in \mathbb{N}$ (Corollary 10). The same estimate holds for the Siegel principal congruence subgroup of level N.

In the Appendix we apply some ideas of Section 2 to derive certain class number identities. Together with the lemmas in Section 2 they can be used to evaluate the formula for the dimension of $S_{\kappa,L}$ explicitly when L has the special shape $L = \mathbb{Z}(-2t_1) \perp \cdots \perp \mathbb{Z}(-2t_r)$ with nonzero integers $t_1, \ldots, t_r$. Moreover, these identities might be of independent interest.

2. The Dimension Formula

Let L be an even lattice of signature (b^+, b^-). We denote the bilinear form on L by $(\cdot, \cdot)$ and the associated quadratic form by $q(x) = \frac{1}{2}(x, x)$. We write L' for the dual lattice of L and $\mathcal{L} = L'/L$ for the (finite) discriminant group. Moreover, let $d = |\mathcal{L}/\{\pm 1\}|$, $r = b^+ + b^-$ be the rank of L, and denote by

$$D = \min\{n \in \mathbb{N}; \quad nq(\gamma) \in \mathbb{Z} \text{ for all } \gamma \in L'\} \tag{3}$$

the level of L.

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic 2-fold cover of $\mathrm{SL}_2(\mathbb{R})$ and denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. Recall that the elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ denotes a holomorphic function on the upper complex half plane $\mathcal{H}$ with $\phi(\tau)^2 = c\tau + d$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \qquad \text{and} \qquad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

There is a particular unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[\mathcal{L}]$ of $\mathcal{L}$. If we denote the standard basis of $\mathbb{C}[\mathcal{L}]$ by $(\mathbf{e}_\gamma)_{\gamma\in\mathcal{L}}$, then ρ_L can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows (see also [Bo1], [Bo2], where the dual of ρ_L is used):

$$\rho_L(T)\mathbf{e}_\gamma = e(-q(\gamma))\mathbf{e}_\gamma, \tag{4}$$

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

$$\rho_L(S)\mathbf{e}_\gamma = \frac{\sqrt{i}^{b^+-b^-}}{\sqrt{|\mathcal{L}|}}\sum_{\delta\in\mathcal{L}} e((\gamma,\delta))\mathbf{e}_\delta. \tag{5}$$

Here and throughout we abbreviate $e(z) = \mathrm{e}^{2\pi iz}$ for $z \in \mathbb{C}$. This representation is essentially the Weil representation attached to the quadratic module $(\mathcal{L}, q)$ (see [No]).

Let $k \in \frac{1}{2}\mathbb{Z}$. We denote by $M_{k,L}$ the vector space of $\mathbb{C}[\mathcal{L}]$-valued modular forms of weight k with representation ρ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$. The subspace of cusp forms is denoted by $S_{k,L}$. (See also [BF] or [Bo1].) It is easily seen that $M_{k,L} = 0$, if $2k \not\equiv b^- - b^+ \pmod 2$.

Since ρ_L factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$, it is clear that the dimension of $M_{k,L}$ is finite. It can be computed using the Riemann-Roch theorem or the Selberg trace formula in a standard way. This is carried out in [Fi] in a more general situation. In our special case the following formula holds (see [Bo3], [Bo2] p. 228): Assume that $2k \equiv b^- - b^+ \pmod 4$ (we will only be interested in this case). Then the d-dimensional subspace $W = \operatorname{span}\{\mathbf{e}_\gamma + \mathbf{e}_{-\gamma};\quad \gamma \in \mathcal{L}\}$ of $\mathbb{C}[\mathcal{L}]$ is invariant under ρ_L, more precisely ρ_L acts by multiplication with $e(-k/2)$ on it. We denote by ρ the restriction of ρ_L to W. If M is a unitary matrix of size d with eigenvalues $e(v_j)$ and $0 \leqslant v_j < 1$ (for $j = 1, \ldots, d$), then we define

$$\alpha(M) = \sum_{j=1}^{d} v_j.$$

The dimension of $M_{k,L}$ is given by

$$\dim_{\mathbb{C}}(M_{k,L}) = d + dk/12 - \alpha\big(\mathrm{e}^{\pi ik/2}\rho(S)\big) - \alpha\Big(\big(\mathrm{e}^{\pi ik/3}\rho(ST)\big)^{-1}\Big) - \alpha(\rho(T)). \tag{6}$$

Furthermore, using Eisenstein series, it can be easily shown that the codimension of $S_{k,L}$ in $M_{k,L}$ is equal to the number of elements of the set

$$\big\{\gamma \in \mathcal{L}/\{\pm 1\};\quad q(\gamma) \in \mathbb{Z}\big\} \tag{7}$$

(see also [Br1] chapter 1.2.3).

As already pointed out in the introduction, we need to find a lower bound for the dimension of $S_{k,L}$. In view of (6) and (7) we have to estimate the quantities

$$\begin{aligned}
\alpha_1 &:= \alpha\big(\mathrm{e}^{\pi ik/2}\rho(S)\big),\\
\alpha_2 &:= \alpha\Big(\big(\mathrm{e}^{\pi ik/3}\rho(ST)\big)^{-1}\Big),\\
\alpha_3 &:= \alpha(\rho(T)),\\
\alpha_4 &:= \Big|\big\{\gamma \in \mathcal{L}/\{\pm 1\};\quad q(\gamma) \in \mathbb{Z}\big\}\Big|.
\end{aligned}$$

This can easily be done for α_1, α_2, and α_4. However, for α_3 this problems turns out to be more difficult. In the appendix we will see that α_3 sometimes is related to class numbers of imaginary quadratic fields.

For the estimates we first need some facts on Gauss sums attached to L. Let $n \in \mathbb{Z}$. We define the Gauss sum $G(n, L)$ by

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

$$G(n, L) = \sum_{\gamma \in \mathcal{L}} e(nq(\gamma)). \tag{8}$$

Two basic but important properties of $G(n, L)$ are

$$G(-n, L) = \overline{G(n, L)}, \tag{9}$$

$$G(n + D, L) = G(n, L). \tag{10}$$

If n is an integer, we define

$$\mathcal{L}^n = \{\gamma \in \mathcal{L}; \quad n\gamma = 0\}.$$

Observe that $|\mathcal{L}^2| = 2d - |\mathcal{L}|$. In general it follows from the theorem of elementary divisors that

$$|\mathcal{L}^n| \leqslant (D, n)^r, \tag{11}$$

where (D, n) denotes the greatest common divisor of D and n.

LEMMA 1. *Let n be a positive integer*. (i) *If* $D|n$, *then* $G(n, L) = |\mathcal{L}|$. (ii) *The absolute value of* $G(n, L)$ *is given by*

$$|G(n, L)| = \sqrt{|\mathcal{L}|}\sqrt{|\mathcal{L}^n|}.$$

In particular $|G(n, L)| = \sqrt{|\mathcal{L}|}$, *if* $(n, D) = 1$.

The proof is left to the reader.

LEMMA 2. *The quantities* α_1 *and* α_2 *can be expressed in terms of Gauss sums as follows:*

$$\alpha_1 = \frac{d}{4} - \frac{1}{4\sqrt{|\mathcal{L}|}} e((2k + b^+ - b^-)/8)\Re(G(2, L)), \tag{12}$$

$$\alpha_2 = \frac{d}{3} + \frac{1}{3\sqrt{3|\mathcal{L}|}} \Re(e((4k + 3b^+ - 3b^- - 10)/24)(G(1, L) + G(-3, L))). \tag{13}$$

Proof. The idea of the proof was communicated to us by R. E. Borcherds. Let us first consider (12). In $\mathrm{Mp}_2(\mathbb{Z})$ we have the relation $S^2 = Z$. Since Z acts on $W \subset \mathbb{C}[\mathcal{L}]$ by multiplication with $e(-k/2)$, the identity

$$(e(k/4)\rho(S))^2 = e(k/2)\rho(Z) = \mathrm{id}$$

holds. Hence, all eigenvalues of $e(k/4)\rho(S)$ equal ± 1. If b denotes the number of eigenvalues equal to -1, then

$$\mathrm{tr}_W(e(k/4)\rho(S)) = -b + (d - b) = d - 2b.$$

Thus

$$\alpha_1 = b/2 = \frac{d}{4} - \frac{1}{4}\mathrm{tr}_W(e(k/4)\rho(S)).$$

Note that $\operatorname{tr}_W(\rho(S)) = \frac{1}{2}\operatorname{tr}_{\mathbb{C}[\mathcal{L}]}(\rho(S) + \rho(S)X)$, where X denotes the map $\mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ given by $\mathbf{e}_\gamma \mapsto \mathbf{e}_{-\gamma}$. Hence it follows from (5) that

$$\operatorname{tr}_W(e(k/4)\rho(S)) = \frac{1}{\sqrt{|\mathcal{L}|}} e\big((2k + b^+ - b^-)/8\big)\Re(G(2, L)).$$

This implies the assertion.

Equality (13) can be proved in the same way. Using the relation $(ST)^3 = Z$ we find

$$\alpha_2 = \frac{d}{3} + \frac{2}{3\sqrt{3}}\Re\big(e(-5/12 + k/6)\operatorname{tr}_W(\rho(ST))\big).$$

Furthermore, by (5) and (4) we have

$$\operatorname{tr}_W(\rho(ST)) = \frac{1}{2\sqrt{|\mathcal{L}|}} e((b^+ - b^-)/8)(G(1, L) + G(-3, L)).$$

□

From Lemma 2 we obtain the following corollary.

COROLLARY 3. *The quantities α_1 and α_2 satisfy the estimates*

$$|\alpha_1 - d/4| \leqslant \frac{1}{4}\sqrt{|\mathcal{L}^2|}, \tag{14}$$

$$|\alpha_2 - d/3| \leqslant \frac{1}{3\sqrt{3}}\left(1 + \sqrt{|\mathcal{L}^3|}\right). \tag{15}$$

We now derive an estimate for α_4. If n is a positive integer, we define the divisor sum $\sigma_t(n) = \sum_{a|n} a^t$.

LEMMA 4. *We have*

$$|\alpha_4| \leqslant \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2}\sigma_{r/2-1}(D).$$

Proof. We write α_4 as

$$\alpha_4 = \frac{1}{2}\sum_{\substack{\gamma\in\mathcal{L}^2\\ q(\gamma)\in\mathbb{Z}}} 1 + \frac{1}{2}\sum_{\substack{\gamma\in\mathcal{L}\\ q(\gamma)\in\mathbb{Z}}} 1.$$

The second term on the right hand side is equal to

$$\frac{1}{2D}\sum_{\gamma\in\mathcal{L}}\sum_{v(D)} e(q(\gamma)v) = \frac{1}{2D}\sum_{v(D)} G(v, L).$$

Thus, using Lemma 1, we obtain

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

$$\begin{aligned}|\alpha_4| &\leqslant \frac{|\mathcal{L}^2|}{2} + \frac{1}{2D}\sum_{v(D)}\sqrt{|\mathcal{L}|}\sqrt{|\mathcal{L}^v|}\\
&\leqslant \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D}\sum_{v(D)}(v, D)^{r/2}\\
&\leqslant \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D}\sum_{a|D}\sum_{\substack{\mu=1\\(\mu,D/a)=1}}^{D/a} a^{r/2}\\
&\leqslant \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2D}\sum_{a|D}\frac{D}{a}a^{r/2}\\
&\leqslant \frac{|\mathcal{L}^2|}{2} + \frac{\sqrt{|\mathcal{L}|}}{2}\sigma_{r/2-1}(D).\end{aligned}$$

□

Before we consider α_3 we introduce some more notation. If $x \in \mathbb{R}$, then we write $[x]$ for the greatest-integer function $\max\{n \in \mathbb{Z};\ n \leqslant x\}$. Moreover, we define

$$\mathbb{B}(x) = x - \tfrac{1}{2}([x] - [-x]). \tag{16}$$

Thus $\mathbb{B}(x)$ is the 1-periodic function on $\mathbb{R}$ with $\mathbb{B}(x) = 0$ for $x = 0, 1$ and $\mathbb{B}(x) = x - 1/2$ for $0 < x < 1$. By definition

$$\alpha_3 = \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} (-q(\gamma) - [-q(\gamma)]).$$

Using $\mathbb{B}(x)$ and α_4 we may rewrite this in the form

$$\alpha_3 = \frac{d}{2} - \frac{\alpha_4}{2} - \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} \mathbb{B}(q(\gamma)).$$

Hence, to obtain information on α_3, it suffices to consider the invariants

$$\alpha_5 = \sum_{\gamma \in \mathcal{L}/\{\pm 1\}} \mathbb{B}(q(\gamma)), \tag{17}$$

$$\alpha_5' = \sum_{\gamma \in \mathcal{L}} \mathbb{B}(q(\gamma)) \tag{18}$$

of L. Obviously the relation

$$\alpha_5 = \frac{1}{2}\sum_{\gamma \in \mathcal{L}^2} \mathbb{B}(q(\gamma)) + \frac{\alpha_5'}{2}$$

holds. For $\gamma \in \mathcal{L}^2$, we have $q(\gamma) \in \frac{1}{4}\mathbb{Z}$ and thereby $|\mathbb{B}(q(\gamma))| \leqslant 1/4$. Hence

$$|\alpha_5| \leqslant |\mathcal{L}^2|/8 + |\alpha_5'|/2 \quad \text{and} \quad |\alpha_3 - d/2 + \alpha_4/2| \leqslant |\mathcal{L}^2|/8 + |\alpha_5'|/2. \tag{19}$$

The main result of this section is the following estimate for α_5'.

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

LEMMA 5. *The invariant* α_5' *satisfies*

$$|\alpha_5'| \leqslant \frac{\sqrt{|\mathcal{L}|}}{\pi}(3/2+\ln(D))(\sigma_{r/2-1}(D) - D^{r/2-1}).$$

Proof. The 1-periodic function $\mathbb{B}(x)$ has the pointwise convergent Fourier expansion

$$\mathbb{B}(x) = -\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}-\{0\}}\frac{e(nx)}{n}. \tag{20}$$

Inserting this into the definition of α_5' we find

$$\begin{aligned}
\alpha_5' &= -\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{n}\sum_{\gamma\in\mathcal{L}} e(nq(\gamma))\\
&= -\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{n}G(n,L)\\
&= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Im(G(n,L)).
\end{aligned}$$

We use (9) and (10) and the fact $\Im(G(Dv,L))=0$ to rewrite this as follows:

$$\begin{aligned}
\alpha_5' &= -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{v=1}^{D-1}\left(\frac{\Im\big(G(Dn+v,L)\big)}{Dn+v} + \frac{\Im\big(G(D(n+1)-v,L)\big)}{D(n+1)-v}\right)\\
&= -\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{v=1}^{D-1}\left(\frac{1}{Dn+v} - \frac{1}{D(n+1)-v}\right)\Im(G(v,L))\\
&= -\frac{1}{\pi}\sum_{v=1}^{D-1}\frac{1}{v}\Im(G(v,L)) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{v=1}^{D-1}\frac{D-2v}{D^2n(n+1)+Dv-v^2}\Im(G(v,L)).
\end{aligned}$$

By means of Lemma 1 we obtain

$$\begin{aligned}
|\alpha_5'| &\leqslant \frac{1}{\pi}\sum_{v=1}^{D-1}\frac{1}{v}|G(v,L)| + \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{v=1}^{D-1}\frac{D-2}{D^2n(n+1)}|(G(v,L))|\\
&\leqslant \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{v=1}^{D-1}\frac{1}{v}\sqrt{|\mathcal{L}^v|} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{v=1}^{D-1}\sum_{n=1}^{\infty}\sqrt{|\mathcal{L}^v|}\frac{1}{n(n+1)}.
\end{aligned}$$

The latter sum over n equals 1. We apply (11) and rewrite the sum over v. We get

$$\begin{aligned}
|\alpha_5'| &\leqslant \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{\substack{a|D\\a\neq D}}\ \sum_{\substack{\mu=1\\(\mu,D/a)=1}}^{D/a}\frac{1}{a\mu}a^{r/2} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{\substack{a|D\\a\neq D}}\ \sum_{\substack{\mu=1\\(\mu,D/a)=1}}^{D/a} a^{r/2}\\
&\leqslant \frac{\sqrt{|\mathcal{L}|}}{\pi}\sum_{\substack{a|D\\a\neq D}}(1+\ln(D/a))a^{r/2-1} + \frac{\sqrt{|\mathcal{L}|}}{2\pi D}\sum_{\substack{a|D\\a\neq D}}\frac{D}{a}a^{r/2}
\end{aligned}$$

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

$$\leqslant \frac{\sqrt{|\mathcal{L}|}}{\pi}(3/2+\ln(D))(\sigma_{r/2-1}(D)-D^{r/2-1}).$$

Here we have also used the estimate $\sum_{\nu=1}^{n}\frac{1}{\nu}\leqslant 1+\ln(n)$. □

If we put the above lemmas together we finally obtain the desired estimate for the dimension of $S_{k,L}$.

THEOREM 6. *Assume that* $2k\equiv b^- - b^+ \pmod 4$. *Then*

$$\left|\dim(S_{k,L})-\frac{(k-1)d}{12}\right|$$
$$\leqslant \frac{\sqrt{|\mathcal{L}^2|}}{4}+\frac{1+\sqrt{|\mathcal{L}^3|}}{3\sqrt{3}}+\frac{3}{8}|\mathcal{L}^2|+\frac{\sqrt{|\mathcal{L}|}}{4}\sigma_{r/2-1}(D)+$$
$$+\frac{\sqrt{|\mathcal{L}|}}{2\pi}(3/2+\ln(D))(\sigma_{r/2-1}(D)-D^{r/2-1}).$$

This estimate could be further improved by using the theorem of elementary divisors more carefully in the proof of Lemmas 4 and 5. However, since we are mainly interested in asymptotic questions, the above result suffices for our purposes. Recall that the quantities $|\mathcal{L}^\nu|$ are bounded by (11).

3. Picard Groups

For any lattice (L,q) and any nonzero integer N, we may consider the rescaled lattice $L(N)$. It is given by L as a $\mathbb{Z}$-module, but equipped with the rescaled quadratic form $Nq(\cdot)$. The dual is given by $L(N)'=(1/N)L'$.

From now on we suppose that L has signature $(2,l)$ with $l\geqslant 3$. The orthogonal group $\mathrm{O}(L)$ of L is a discrete subgroup of the real orthogonal group $\mathrm{O}(L\otimes_{\mathbb{Z}}\mathbb{R})\cong \mathrm{O}(2,l)$. Let $\mathrm{O}'(L\otimes_{\mathbb{Z}}\mathbb{R})$ be the spinor kernel of $\mathrm{O}(L\otimes_{\mathbb{Z}}\mathbb{R})$ and $\mathrm{O}'(L)=\mathrm{O}'(L\otimes_{\mathbb{Z}}\mathbb{R})\cap\mathrm{O}(L)$. We denote by $\Delta(L)$ the discriminant kernel of the group $\mathrm{O}'(L)$. By definition, this is the subgroup of those elements of $\mathrm{O}'(L)$, which act trivially on the discriminant group $\mathcal{L}$.

Let us briefly recall the construction of the Hermitean symmetric domain $\mathcal{H}_l$ associated to $\mathrm{O}'(L\otimes_{\mathbb{Z}}\mathbb{R})$. We extend the bilinear form $(\cdot,\cdot)$ on L to a $\mathbb{C}$-bilinear form on the complexification $L\otimes_{\mathbb{Z}}\mathbb{C}$ and consider the following chain of subsets of the associated projective space $P(L\otimes_{\mathbb{Z}}\mathbb{C})$:

$$\mathcal{H}_l\subset\mathcal{K}\subset\mathcal{N}\subset P(L\otimes_{\mathbb{Z}}\mathbb{C}).$$

Here $\mathcal{N}$ denotes the zero quadric, i.e. the subset of $P(L\otimes_{\mathbb{Z}}\mathbb{C})$ represented by vectors z of norm zero $(z,z)=0$. The open subset $\mathcal{K}$ is defined by the condition $(z,\bar{z})>0$. It has two connected components. We choose one of them and denote it by $\mathcal{H}_l$. The real orthogonal group of L acts on $L\otimes_{\mathbb{Z}}\mathbb{C}$, $P(L\otimes_{\mathbb{Z}}\mathbb{C})$, $\mathcal{N}$, and $\mathcal{K}$. The spinor kernel acts on $\mathcal{H}_l$.

Let $\Gamma = \Delta(L)$ and X be the quotient $\mathcal{H}_l/\Gamma$. By the theory of Baily and Borel, X is a quasi-projective algebraic variety.

If Γ acts freely on $\mathcal{H}_l$, then X is smooth. In this case we denote by $\mathrm{Pic}(X)$ the usual algebraic Picard group, i.e. the group of isomorphism classes of algebraic holomorphic line bundles on X. If Γ does not act freely, then we choose a normal subgroup Γ' of finite index which acts freely. We define the Picard group of X by

$$\mathrm{Pic}(X) = \mathrm{Pic}(\mathcal{H}_l/\Gamma')^{\Gamma/\Gamma'},$$

i.e. as the subgroup of $\mathrm{Pic}(\mathcal{H}_l/\Gamma')$, which is invariant under the action of the finite group Γ/Γ'. Our assumption on l implies that these Picard groups are finitely generated.

In the same way we define the divisor class group $\mathrm{Cl}(X)$ of X. (See also [Bo2] and [Br1].) Moreover, we write $\widetilde{\mathrm{Cl}}(X)$ for the quotient of $\mathrm{Cl}(X)$ modulo the subgroup $A(X)$ of divisor classes coming from meromorphic automorphic forms (of generally non-zero weight with a character of finite order) for the group Γ. There is the usual injective map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X),$$

which assigns to a divisor class its associated class of line bundles. (By our definition of Cl and Pic this map also makes sense if Γ does not act freely. Since X is quasi-projective, this map is in fact an isomorphism.) Thus the rank of $\mathrm{Pic}(X)$ is bounded by $\dim_{\mathbb{C}}(\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C})$. It follows from the Koecher boundedness principle (which holds since $l \geqslant 3$) that $\dim(A(X) \otimes_{\mathbb{Z}} \mathbb{C}) = 1$ and thereby

$$\mathrm{rank}(\mathrm{Pic}(X)) \geqslant 1 + \dim_{\mathbb{C}}(\widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}). \tag{21}$$

Put $\kappa = 1 + l/2$. It is a consequence of the existence of Borcherds' lifting from modular forms of negative weight $1 - l/2$ to automorphic products for the group Γ and Serre duality that there exists a homomorphism from the space of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of $S_{\kappa,L}$ to $\widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ (cf. [Bo1, Bo2]). By the refinement given in [Br1] chapter 5.1, we more precisely know that there is a homomorphism

$$\eta\colon\ S_{\kappa,L} \longrightarrow \widetilde{\mathrm{Cl}}(X) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{22}$$

We may infer the following fundamental proposition.

PROPOSITION 7. *Suppose that the map* η *is injective. Then*

$$\mathrm{rank}(\mathrm{Pic}(X)) \geqslant 1 + \dim_{\mathbb{C}}(S_{\kappa,L}).$$

Recall that a hyperbolic plane is a lattice H which is isomorphic to the lattice $\mathbb{Z}^2$ equipped with the quadratic form $q((a, b)) = ab$. For the rest of this section we assume that L splits two orthogonal hyperbolic planes over $\mathbb{Z}$, i.e. has the special shape $L = L_0 \perp H \perp H$, where L_0 is an even negative definite lattice of rank $l - 2$.

Let N be a positive integer. We consider the lattice

$$L[N] = L_0(N) \perp H \perp H,$$

https://doi.org/10.1023/A:1016357029843 Published online by Cambridge University Press

its discriminant kernel $\Gamma[N] = \Delta(L[N])$, and the associated modular variety $X[N] = \mathcal{H}_l/\Gamma[N]$. We may view $\Gamma[N]$ as a subgroup of $O(L \otimes_{\mathbb{Z}} \mathbb{Q})$ which is commensurable with $\Gamma = \Delta(L)$.

THEOREM 8. *Let L be a lattice as above and $\mathcal{L}$ its discriminant group. Let D be the level of L as defined in* (3). *Then*

$$\begin{aligned}\operatorname{rank}(\operatorname{Pic}(X[N])) \geqslant{} & \frac{l|\mathcal{L}|N^{l-2}}{48} + l/48 + 1 - 2^{l/2-3} - 3\cdot 2^{l-5} - 3^{-3/2} - 3^{l/2-5/2} - \\ & - \frac{\sqrt{|\mathcal{L}|}}{4} N^{l/2-1}\sigma_{l/2-2}(DN) - \\ & - \frac{\sqrt{|\mathcal{L}|}}{2\pi} N^{l/2-1}(3/2 + \ln(DN))(\sigma_{l/2-2}(DN) - (DN)^{l/2-2}).\end{aligned}$$

Proof. By construction the lattice $L[N]$ splits two hyperbolic planes over $\mathbb{Z}$. The main result of [Br1] chapter 5.2 says that the map (22) is injective in this case. By Proposition 7 we find

$$\operatorname{rank}(\operatorname{Pic}(X[N])) \geqslant 1 + \dim(S_{\kappa,L[N]}) = 1 + \dim(S_{\kappa,L_0(N)}).$$

We apply Theorem 6 to estimate the dimension of $S_{\kappa,L_0(N)}$. The rank of $L_0(N)$ is $l-2$, the level of $L_0(N)$ is DN, and

$$\begin{aligned}&\left|L_0(N)'/L_0(N)\right| = N^{l-2}|\mathcal{L}|,\\ &\left|(L_0(N)'/L_0(N))/\{\pm 1\}\right| \geqslant \frac{1}{2}(1 + N^{l-2}|\mathcal{L}|).\end{aligned}$$

If we also take into account (11) we obtain the assertion. □

COROLLARY 9. *Let $\varepsilon > 0$. Then there exist positive constants $C_1 = C_1(L, \varepsilon)$ and $C_2 = C_2(L)$ (which can be easily determined explicitly) such that*

$$\operatorname{rank}(\operatorname{Pic}(X[N])) \geqslant \frac{l|\mathcal{L}|N^{l-2}}{48} - C_2 - \begin{cases} C_1 N^{1/2+\varepsilon}, & \text{if } l = 3,\\ C_1 N^{l-3+\varepsilon}, & \text{if } l > 3,\end{cases}$$

for all $N \in \mathbb{N}$.

In the above situation the map (22) induces in fact an isomorphism from $S_{\kappa,L[N]}$ to the subspace of $\widetilde{\operatorname{Cl}}(X[N]) \otimes_{\mathbb{Z}} \mathbb{C}$, which is generated by algebraic divisors $\lambda^{\perp}$, where $\lambda \in L[N]'$ is a negative norm vector and the orthogonal complement is taken is $\mathcal{H}_l$. According to the Tate conjecture one should expect that the codimension of this subspace in $\widetilde{\operatorname{Cl}}(X[N]) \otimes_{\mathbb{Z}} \mathbb{C}$ is small. This leads us to the following

QUESTION 1. *Is it true that* $\operatorname{rank}(\operatorname{Pic}(X[N])) \sim l|\mathcal{L}|N^{l-2}/48$, $N \to \infty$?

Let N be a positive integer. It is natural to define the *principal congruence subgroup* of level N of $\Gamma = \Delta(L)$ by $\Gamma(N) = \Delta(L(N))$.

We now consider the Picard groups of the modular varieties $X(N) = \mathcal{H}_l/\Gamma(N)$. In the same way as in [Fr] (chapter 2.6 Hilfssatz 6.5) it can be proved that for $N \geqslant 3$ the group $\Gamma(N)$ acts freely on $\mathcal{H}_l$. Thus $X(N)$ is smooth in this case.

To obtain an estimate for the rank of $\mathrm{Pic}(X(N))$ we cannot argue as above. Since $L(N)$ does not split two hyperbolic planes over $\mathbb{Z}$, we do not have the result of [Br1] saying that the map η (22) is injective.

However, we can still get an estimate for the rank of $\mathrm{Pic}(X(N))$ in the following way. There exists a lattice $\tilde{L}$, which is isomorphic to $L[N]$ and contains

$$L(N) = L_0(N) \perp H(N) \perp H(N)$$

as a sub-lattice. It is easily seen that

$$\Gamma(N) = \Delta(L(N)) \subset \Delta(\tilde{L}).$$

(In fact, taking the discriminant kernel of a lattice is functorial.) Therefore we may view $\Gamma(N)$ as a subgroup of $\Gamma[N]$. The natural projection $X(N) \to X[N]$ induces an injective map of Picard groups

$$\mathrm{Pic}(X[N]) \longrightarrow \mathrm{Pic}(X(N)).$$

Thus Theorem 8 gives us a lower bound for $\mathrm{rank}(\mathrm{Pic}(X(N)))$, too. The asymptotic bound of corollary 9 also holds.

It is clear that these bounds for the rank of $\mathrm{Pic}(X(N))$ are probably not optimal. Here it is natural to ask

QUESTION 2. *What is the asymptotic behavior of the numbers* $\mathrm{rank}(\mathrm{Pic}(X(N)))$ *for* $N \to \infty$?

3.1. THE SIEGEL MODULAR GROUP OF GENUS 2

If R is a subring of $\mathbb{C}$, then we denote by

$$\mathrm{Sp}(2, R) = \{M \in \mathrm{GL}(4, R); \quad M^t I M = I\}$$

the symplectic group of genus 2 with coefficients in R. Here I denotes the matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and E the 2×2 identity matrix. The group $\mathrm{Sp}(2, \mathbb{R})$ acts on the Siegel half plane $\mathbb{H}_2$. Let N be a positive integer. The paramodular group $\Gamma_S[N]$ of level N is the subgroup of $\mathrm{Sp}(2, \mathbb{Q})$ given by matrices of the form

$$\begin{pmatrix} * & N* & * & * \\ * & * & * & N^{-1}* \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix},$$

where the $*$ are all integral. The quotient $\mathbb{H}_2/\Gamma_S[N]$ is the moduli space of Abelian surfaces with a $(1, N)$-polarization.

Let L be the lattice $H \perp H \perp \mathbb{Z}(-2)$ of signature $(2, 3)$. It is well known that there exists an isomorphism $\mathrm{Sp}(2, \mathbb{R})/\{\pm 1\} \to \mathrm{O}'(L[N] \otimes \mathbb{R})/\{\pm 1\}$, which commutes with the action of $\mathrm{Sp}(2, \mathbb{R})$ on $\mathbb{H}_2$ and the action of $\mathrm{O}'(L \otimes \mathbb{R})$ on $\mathcal{H}_3$, and which induces an isomorphism

$$\Gamma_S[N]/\{\pm 1\} \longrightarrow \Gamma[N]/\{\pm 1\} = \Delta(L[N])/\{\pm 1\}$$

(see [GN]). Hence Corollary 9 implies

COROLLARY 10. *Let* $\varepsilon > 0$. *Then there exist positive constants* $C_1 = C_1(\varepsilon)$ *and* $C_2 < 0.6$ (*which can be easily determined*) *such that*

$$\mathrm{rank}(\mathrm{Pic}(\mathbb{H}_2/\Gamma_S[N])) \geqslant N/8 - C_2 - C_1 N^{1/2+\varepsilon}$$

for all $N \in \mathbb{N}$.

Note that $\dim(S_{\kappa, L[N]})$ can be computed explicitly in this case. By Lemma 2 the quantities α_1 and α_2 can be expressed in terms of standard Gauss sums $G(n, a) = \sum_{v(a)} e(nv^2/a)$. Moreover, α_4 is equal to $[1 + b/2]$, where b is the largest integer whose square divides N. Finally, using Theorem 11 of the appendix, α_5 can be written as a sum of class numbers. Therefore we could obtain a sharper estimate than in Theorem 8. However, in the asymptotic estimate Corollary 10 this would only improve the constants C_1 and C_2.

Let $\Gamma_S(N) \subset \mathrm{Sp}(2, \mathbb{Z})$ be the principal congruence subgroup of level N, i.e. the kernel of the reduction homomorphism $\mathrm{Sp}(2, \mathbb{Z}) \to \mathrm{Sp}(2, \mathbb{Z}/N\mathbb{Z})$. Since $\Gamma_S(N) \subset \Gamma_S[N]$, the above estimate also holds for the group $\Gamma_S(N)$. (To see this we could have also used the fact that the orthogonal principal congruence subgroup $\Gamma(N)$ is isomorphic to a group G with $\Gamma_S(2N) \subset G \subset \Gamma_S(N)$.)

Appendix

In Section 2 we saw that the quantities $\alpha_1, \alpha_2, \alpha_4$ can all be expressed in terms of Gauss sums. We now indicate, how the idea of the proof of Lemma 5 can sometimes be used to obtain a closed formula for α'_5 (and thereby for α_3) in terms of class numbers.

Let L be the negative definite lattice of rank r given by

$$L = \mathbb{Z}(-2N) \perp \cdots \perp \mathbb{Z}(-2N).$$

Define

$$A_r(N) = \sum_{v_1, \ldots, v_r\,(N)} \mathbb{B}\left(\frac{v_1^2}{N} + \cdots + \frac{v_r^2}{N}\right),$$

where $v_1, \ldots, v_r$ run through a set of representatives of $\mathbb{Z}/N\mathbb{Z}$. Then for our particular lattice L we have $\alpha'_5 = -\frac{1}{2}A_r(4N)$.

We denote by $H(a)$ for $a \neq -3, -4$ the class number of positive definite binary quadratic forms of discriminant a and put $H(-3) = 1/3$, $H(-4) = 1/2$. Then $H(a) = 0$, if $a > 0$ or $a \not\equiv 0, 1 \pmod 4$. Moreover, we write χ_a for the Dirichlet character defined by the Kronecker symbol $x \mapsto \left(\frac{a}{x}\right)$.

THEOREM 11. *Suppose that r is odd. Then*

$$A_r(N) = -\chi_{-4}(r)N^{r-1}\sum_{\substack{a|N\\ a\equiv -1\,(4)}} a^{\frac{1-r}{2}}H(-a) - \chi_{-8}(r)\left(\sqrt{2}N\right)^{r-1}\sum_{\substack{a|N\\ a\equiv 0\,(4)}} a^{\frac{1-r}{2}}H(-a).$$

Here the sums run through the positive divisors of N satisfying the indicated conditions.

Proof. If $n \in \mathbb{Z}$ and $a \in \mathbb{N}$, then we denote by $G(n, a) = \sum_{\nu(a)} e(n\nu^2/a)$ the standard Gauss sum. By means of the Fourier expansion (20) of the function $\mathbb{B}$, we can rewrite $A_r(N)$ as a Dirichlet series:

$$A_r(N) = -\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{n}G(n, N)^r = -\frac{1}{\pi}\sum_{n\geqslant 1}\frac{1}{n}\Im(G(n, N)^r).$$

Using the fact $G(n, N) = aG(n/a, N/a)$ for $a|(n, N)$, we find

$$A_r(N) = -\frac{N^{r-1}}{\pi}\sum_{a|N}\sum_{\substack{m\geqslant 1\\ (m,a)=1}}\frac{1}{m}a^{1-r}\Im(G(m, a)^r).$$

If we insert the explicit formula for $G(m, a)$ (cf. [La] chapter 4.3), we obtain by a lengthy but straightforward calculation

$$A_r(N) = -\frac{N^{r-1}}{\pi}\sum_{a|N}a^{1-r/2}L(\chi_{-a}, 1)\cdot\begin{cases}0, & \text{if } a\equiv 1, 2 \pmod 4,\\ \chi_{-4}(r), & \text{if } a\equiv -1 \pmod 4,\\ 2^{(r-1)/2}\chi_{-8}(r), & \text{if } a\equiv 0 \pmod 4).\end{cases}$$

Here $L(\chi_a, s)$ denotes the Dirichlet series associated to the Dirichlet character χ_a. Since $L(\chi_{-a}, 1) = \pi H(-a)/\sqrt{a}$ (cf. [Za] §8), this implies the assertion. □

By virtue of the above argument, A_r can also be evaluated for even r. In this case, class numbers do not show up. For instance for $r \equiv 0 \pmod 4$ one finds that $A_r(N) = 0$. More generally α_5' can be computed for any lattice of the form $\mathbb{Z}(-2N_1) \perp \cdots \perp \mathbb{Z}(-2N_r)$ with $N_1, \ldots, N_r \in \mathbb{N}$. Note that for $r = 1$ the above formula is already contained in the book [EZ] in §10 (but with a different proof).

Acknowledgements

I would like to thank M. Bundschuh, E. Freitag, and R. Weissauer for several helpful conversations.

References

[Bo1] Borcherds, R. E.: Automorphic forms with singularities on Grassmannians, *Invent. Math.* **132** (1998), 491–562.
[Bo2] Borcherds, R. E.: The Gross–Kohnen–Zagier theorem in higher dimensions, *Duke Math. J.* **97** (1999), 219–233.
[Bo3] Borcherds, R. E.: Reflection groups of Lorentzian lattices, *Duke Math. J.* **104** (2000), 319–366.

[Br1] Bruinier, J. H.: *Borcherds Products on* O(2, *l*) *and Chern Classes of Heegner Divisors*, Lecture Notes in Math. 1780, Springer-Verlag, New York, 2002.

[Br2] Bruinier, J. H.: Borcherds products and Chern classes of Hirzebruch–Zagier divisors, *Invent. Math.* **138** (1999), 51–83.

[BF] Bruinier, J. H. and Freitag, E.: Local Borcherds products, *Ann. Inst. Fourier* **51** (2001), 1–27.

[EZ] Eichler, M. and Zagier, D.: *The Theory of Jacobi Forms*, Progr. in Math. 55, Birkhäuser, Basel, 1985.

[Fi] Fischer, J. *An Approach to the Selberg Trace Formula via the Selberg Zeta-Function*, Lecture Notes in Math. 1253, Springer-Verlag, New York, 1987.

[Fr] Freitag, E.: *Siegelsche Modulfunktionen*, Springer-Verlag, New York, 1983.

[GN] Van Geemen, B. and Nygaard, N. O.: On the geometry and arithmetic of some Siegel modular threefolds, *J. Number Theory* **53** (1995), 45–87.

[GrNi] Gritsenko, V. and Nikulin, V.: Automorphic forms and Lorentzian Kac–Moody algebras, Part II, *Internat. J. Math.* **9** (1998), 201–275.

[La] Lang, S.: *Algebraic Number Theory*, Addison-Wesley, Englewood Cliffs, 1970.

[LW1] Lee, R. and Weintraub, S. H.: Cohomology of $\mathrm{Sp}(4, \mathbb{Z})$ and related groups and spaces, *Topology* **24** (1985), 391–410.

[LW2] Lee, R. and Weintraub, S. H.: On certain Siegel modular varieties of genus two and levels above two, In: *Algebraic Topology and Transformation groups*, Lecture Notes in Math. 1361, Springer-Verlag, New York, 1988, pp. 29–52.

[No] Nobs, A.: Die irreduziblen Darstellungen der Gruppen $\mathrm{SL}_2(\mathbb{Z}_p)$, insbesondere $\mathrm{SL}_2(\mathbb{Z}_2)$. I. Teil, *Comment Math. Helv.* **51** (1976), 465–489.

[We1] Weissauer, R. Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades, *J. reine angew. Math.* **391** (1988), 100–156.

[We2] Weissauer, R.: The Picard group of Siegel modular threefolds, *J. reine angew. Math.* **430** (1992), 179–211.

[Za] Zagier, D.: *Zetafunktionen und quadratische Körper*, Springer-Verlag, Berlin, 1981.