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A FORMULA FOR EIGENPAIRS OF CERTAIN SYMMETRIC TRIDIAGONAL MATRICES

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A closed form expression is given for the eigenvalues and eigenvectors of a symmetric tridiagonal matrix of odd order whose diagonal elements are all equal and whose superdiagonal elements alternate between the values c and d. An implicit formula is given for the even order case.

1. INTRODUCTION

Let $A = (a_{ij})$ be a symmetric tridiagonal matrix of order $2M$ or $2M+1$ where, for all i,

$$a_{ij} = b, \quad a_{2i-1\,2i} = c, \quad a_{2i\,2i+1} = d \tag{1.1}$$

with $cd \neq 0$. When $c = d$, the matrix A is exactly a tridiagonal Toeplitz matrix whose eigenvalues and eigenvectors are well known. Furthermore the eigenvalues for a general Toeplitz matrix have been investigated widely. (For example, see [**1**, **2**, **4**, **5**].) In this paper we derive a relation for the eigenpairs of the matrix A defined in (1.1). Such a matrix A occurs in a cubic collocation method designed for the numerical solution of a partial differential equation (see [**3**]). For the case of order $2M+1$, we give an explicit formula using a Fibonacci-type sequence. In the case of order $2M$, we give an implicit formula for the eigenpairs.

2. EIGENPAIRS

In this section using a Fibonacci-type sequence we present formulae for the eigenvalues and eigenvectors for the matrix A defined in (1.1). For convenience, define $T : \mathbb{C} \times \mathbb{C} \times \mathbb{N} \to \mathbb{C}$ by

$$T(\alpha, \beta, n) = \alpha^n + \alpha^{n-1}\beta + \cdots + \alpha\beta^{n-1} + \beta^n \quad \text{for } \alpha, \beta \in \mathbb{C} \text{ and } n \in \mathbb{N}, \tag{2.1}$$

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where $\mathbb{C}$ is the set of all complex numbers and $\mathbb{N}$ is the set of all natural numbers. Multiplying both sides in (2.1) by $(\alpha-\beta)$, we have

$$(\alpha-\beta)T(\alpha,\beta,n)=\alpha^{n+1}-\beta^{n+1}. \tag{2.2}$$

Lemma 1. *Let α and β be nonzero complex numbers. Then the Fibonacci-type sequence $\{a_n\}_{n=1}^{\infty}$ defined by*

$$a_{n+2}-(\alpha+\beta)a_{n+1}+\alpha\beta a_n=0, \qquad n=1,2,\cdots, \tag{2.3}$$

can be represented in terms of a_1 and a_2 by

$$a_n=T(\alpha,\beta,n-2)a_2-\alpha\beta T(\alpha,\beta,n-3)a_1, \qquad n=1,2,\cdots. \tag{2.4}$$

Proof: From (2.3), we have, for $n=1,2,\cdots$,

$$a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n), \quad a_{n+2}-\beta a_{n+1}=\alpha(a_{n+1}-\beta a_n).$$

Hence, inductively,

$$a_{n+1}-\alpha a_n=\beta^{n-1}(a_2-\alpha a_1), \quad a_{n+1}-\beta a_n=\alpha^{n-1}(a_2-\beta a_1). \tag{2.5}$$

If $\alpha=\beta$, (2.5) yields for $n=1,2,\cdots$,

$$a_n=(n-1)\alpha^{n-2}a_2-(n-2)\alpha^{n-1}a_1=T(\alpha,\beta,n-2)a_2-\alpha\beta T(\alpha,\beta,n-3)a_1. \tag{2.6}$$

If $\alpha\neq\beta$, from (2.5) we have

$$(\alpha-\beta)a_n=\left(\alpha^{n-1}-\beta^{n-1}\right)a_2-\alpha\beta\left(\alpha^{n-2}-\beta^{n-2}\right)a_1 \quad n=1,2,\cdots.$$

Hence for $n=1,2,\cdots$, we have

$$a_n=T(\alpha,\beta,n-2)a_2-\alpha\beta T(\alpha,\beta,n-3)a_1. \tag{2.7}$$

□

We now give the result for the odd order case. In this case we have an explicit formula for the eigenpairs for the matrix A defined in (1.1).

Theorem 1. *For the matrix A of order $2M+1$ defined in (1.1), the eigenvalues λ are given by*

$$\lambda=b, \tag{2.8a}$$

$$\lambda=b\pm\sqrt{c^2+d^2+2cd\,\cos\left(\frac{k\pi}{M+1}\right)}, \qquad k=1,2,\cdots,M \tag{2.8b}$$

and the corresponding eigenvectors $x = (x_1, \cdots, x_{2M+1})^t$ *of* A *are given by*

$$(2.9)\quad x_{2n-1} = \left(-\frac{c}{d}\right)^{n-1}, \quad n = 1, 2, \cdots, M+1, \qquad x_{2n} = 0, \quad n = 1, 2, \cdots, M$$

and

$$(2.10a)\quad x_{2n-1} = \frac{d}{c}\sin\left(\frac{(n-1)k\pi}{M+1}\right) + \sin\left(\frac{nk\pi}{M+1}\right), \quad n = 1, 2, \cdots, M+1,$$

$$(2.10b)\quad x_{2n} = \frac{(\lambda - b)}{c}\sin\left(\frac{nk\pi}{M+1}\right), \quad n = 1, 2, \cdots, M,$$

respectively.

PROOF: Let λ be an eigenvalue of A and let $x = (x_1, x_2, \cdots, x_{2M+1})^t$ be the corresponding eigenvector of A. Then we have

$$(2.11a)\quad bx_1 + cx_2 = \lambda x_1,$$

$$(2.11b)\quad dx_{2M} + bx_{2M+1} = \lambda x_{2M+1},$$

$$(2.11c)\quad cx_{2n-1} + bx_{2n} + dx_{2n+1} = \lambda x_{2n},$$

$$(2.11d)\quad dx_{2n} + bx_{2n+1} + cx_{2n+2} = \lambda x_{2n+1}.$$

First, by taking $\lambda = b$ in (2.11), we see that b is an eigenvalue of A and the corresponding eigenvector x is given by

$$x_{2n+1} = \left(-\frac{c}{d}\right)^n, \quad n = 0, 1, \cdots, M \quad \text{and} \quad x_{2n} = 0, \quad n = 1, 2, \cdots, M.$$

Hence we have (2.8a) and (2.9).

Now suppose that $\lambda \neq b$. From (2.11c, d), we have

(2.12)

$$x_{2n+3} - \left(\frac{(\lambda - b)^2 - c^2 - d^2}{cd}\right)x_{2n+1} + x_{2n-1} = 0, \quad n = 1, 2, \cdots, M-1$$

and

(2.13)

$$x_{2n+2} - \left(\frac{(\lambda - b)^2 - c^2 - d^2}{cd}\right)x_{2n} + x_{2n-2} = 0, \quad n = 3, 4, \cdots, M-1.$$

Choose α and β such that

$$(2.14)\quad \alpha + \beta = \frac{(\lambda - b)^2 - c^2 - d^2}{cd} \quad \text{and} \quad \alpha\beta = 1.$$

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Then (2.14) implies

$$\lambda = b \pm \sqrt{c^2 + d^2 + cd\,(\alpha + \beta)}. \tag{2.15}$$

By (2.11a) we have

$$x_2 = \left[\frac{\lambda - b}{c}\right] x_1. \tag{2.16}$$

By (2.11c), (2.14) and (2.16) we have

$$x_3 = \left[\frac{d + c\,(\alpha + \beta)}{c}\right] x_1. \tag{2.17}$$

Now, from (2.11d), (2.13), (2.16) and (2.17) we have

$$x_4 = \left[\frac{(\lambda - b)\,(\alpha + \beta)}{c}\right] x_1. \tag{2.18}$$

By applying Lemma 1 to (2.12) and (2.13) respectively and using that $\alpha\beta = 1$, we have

$$x_{2n-1} = T(\alpha, \beta, n-2)x_3 - T(\alpha, \beta, n-3)x_1, \quad n = 1, 2, \cdots, M+1 \tag{2.19}$$

and

$$x_{2n} = T(\alpha, \beta, n-2)x_4 - T(\alpha, \beta, n-3)x_2, \quad n = 1, 2, \cdots, M. \tag{2.20}$$

Then by substituting (2.16)-(2.18) into (2.19) and (2.20), we have

$$x_{2n-1} = \left[\frac{d}{c}\, T(\alpha, \beta, n-2) + T(\alpha, \beta, n-1)\right] x_1, \quad n = 1, 2, \cdots, M+1 \tag{2.21}$$

and

$$x_{2n} = \frac{\lambda - b}{c}\, T(\alpha, \beta, n-1)x_1, \quad n = 1, 2, \cdots, M. \tag{2.22}$$

By applying (2.21) and (2.22) to (2.11b), we have $T(\alpha, \beta, M) = 0$. If $\alpha = \beta$, then we have

$$T(\alpha, \beta, M) = T(\alpha, \alpha, M) = (M+1)\alpha^M = 0,$$

which leads to a contradiction. Therefore $\alpha \neq \beta$. Hence on multipling both sides in $T(\alpha, \beta, M) = 0$ by $(\alpha - \beta)$, we have

$$\alpha^{M+1} - \beta^{M+1} = (\alpha - \beta)T(\alpha, \beta, M) = 0, \tag{2.23}$$

so that $\alpha^{M+1} = \beta^{M+1}$. This implies

$$|\alpha| = |\beta|. \tag{2.24}$$

Since $\alpha\beta = 1$ and $\alpha \neq \beta$, we can choose

$$\alpha = e^{i\theta}, \quad \beta = e^{-i\theta}, \quad 0 < \theta < \pi. \tag{2.25}$$

Then, from (2.23) and (2.25), we have

$$\alpha + \beta = 2\cos\theta \quad \text{and} \quad \sin{(M+1)\theta} = 0. \tag{2.26}$$

The second equation in (2.26) yields

$$\theta = \frac{k\pi}{M+1}, \qquad 1 \leqslant k \leqslant M. \tag{2.27}$$

Therefore, by substituting (2.25)-(2.27) in (2.15), (2.21) and (2.22), we have the conclusions (2.8b) and (2.10). □

By applying the ideas employed in Theorem 1, we can get an implicit formula for the eigenpairs of a matrix A of order $2M$.

THEOREM 2. *For the matrix A of order $2M$ defined in (1.1), the eigenvalues λ are given by*

$$\lambda = b \pm \sqrt{c^2 + d^2 + (\alpha + \alpha^{-1})\, cd}, \tag{2.28}$$

and the corresponding eigenvectors $x = (x_1, \cdots, x_{2M})^t$ of A are given by

(2.29a)

$$x_{2n-1} = \frac{d}{c}\, T(\alpha, \alpha^{-1}, n-2) + T(\alpha, \alpha^{-1}, n-1), \qquad n = 1, 2, \cdots, M,$$

(2.29b)

$$x_{2n} = \frac{\lambda - b}{c}\, T(\alpha, \alpha^{-1}, n-1), \qquad n = 1, 2, \cdots, M,$$

where α satisfies the equation:

$$c\, T(\alpha, \alpha^{-1}, M) + d\, T(\alpha, \alpha^{-1}, M-1) = 0. \tag{2.30}$$

PROOF: Let λ be an eigenvalue of A and let $x = (x_1, x_2, \cdots, x_{2M})^t$ be the corresponding eigenvector of A. Then we have

$$bx_1 + cx_2 = \lambda x_1, \tag{2.31a}$$

$$dx_{2M-1} + bx_{2M} = \lambda x_{2M}, \tag{2.31b}$$

$$cx_{2n-1} + bx_{2n} + dx_{2n+1} = \lambda x_{2n}, \tag{2.31c}$$

$$dx_{2n} + bx_{2n+1} + cx_{2n+2} = \lambda x_{2n+1}. \tag{2.31d}$$

First, if we take $\lambda = b$ in (2.31), then we can easily check that $x_n = 0$ for $n = 1, 2, \cdots, 2M$. Therefore there is no eigenvector for A, and hence $\lambda \neq b$.

Now suppose that $\lambda \neq b$. Similarly as in Theorem 1, we have the eigenvalues λ such that

$$(2.32) \qquad (\lambda - b)^2 = c^2 + d^2 + (\alpha + \alpha^{-1})cd$$

and the corresponding eigenvectors $x = (x_1, \cdots, x_{2M})^t$ are given by

$$(2.33) \qquad x_{2n-1} = \left[\frac{d}{c}\, T(\alpha, \alpha^{-1}, n-2) + T(\alpha, \alpha^{-1}, n-1)\right] x_1, \qquad n = 1, 2, \cdots, M$$

and

$$(2.34) \qquad x_{2n} = \frac{\lambda - b}{c}\, T(\alpha, \alpha^{-1}, n-1)\, x_1, \qquad n = 1, 2, \cdots, M.$$

Now applying (2.33) and (2.34) to (2.31b), we have

$$(2.35) \qquad c\, T(\alpha, \alpha^{-1}, M) + d\, T(\alpha, \alpha^{-1}, M-1) = 0.$$

Therefore (2.33) and (2.34) with (2.35) complete (2.29) and (2.30). $\square$

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