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A Characterization of Bergman Spaces on the Unit Ball of $\mathbb{C}^n$. II

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Abstract. It has been shown that a holomorphic function f in the unit ball $\mathbb{B}_n$ of $\mathbb{C}_n$ belongs to the weighted Bergman space A^p_α, $p > n + 1 + \alpha$, if and only if the function $|f(z) - f(w)|/|1 - \langle z, w\rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\beta \times dv_\beta)$, where $\beta = (p + \alpha - n - 1)/2$ and $dv_\beta(z) = (1 - |z|^2)^\beta\, dv(z)$. In this paper we consider the range $0 < p < n + 1 + \alpha$ and show that in this case, $f \in A^p_\alpha$ (i) if and only if the function $|f(z) - f(w)|/|1 - \langle z, w\rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$, (ii) if and only if the function $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$. We think the revealed difference in the weights for the double integrals between the cases $0 < p < n + 1 + \alpha$ and $p > n + 1 + \alpha$ is particularly interesting.

1 Introduction

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$ and dv be the normalized volume measure on $\mathbb{B}_n$. For any $\alpha > -1$ we introduce the weighted volume measure

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha\, dv(z),$$

where c_α is a positive constant such that $v_\alpha(\mathbb{B}_n) = 1$ again.

For $0 < p < \infty$ and $\alpha > -1$ we use the notation $A^p_\alpha = H(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, dv_\alpha)$ to denote the weighted Bergman spaces on $\mathbb{B}_n$, where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$.

It was shown in [2] that for $p > n + 1 + \alpha$, we have $f \in A^p_\alpha$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w\rangle|^p}\, dv_\beta(z)\, dv_\beta(w) < \infty,$$

where $\beta = (p + \alpha - n - 1)/2$. An example in [3] shows that this is no longer true in general when $p = n + 1 + \alpha$.

The purpose of this article is to examine the remaining case $0 < p < n + 1 + \alpha$. Our main result is the following.

Theorem 1.1 *Suppose $\alpha > -1$, $0 < p < \alpha + n + 1$, and f is holomorphic in $\mathbb{B}_n$. Then the following conditions are equivalent.*

(i) $f \in A^p_\alpha(\mathbb{B}_n)$.

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(ii) *The double integral*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(w)|^p}{|z-w|^p}\,dv_\alpha(z)\,dv_\alpha(w)$$

converges.

(iii) *The double integral*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^p}\,dv_\alpha(z)\,dv_\alpha(w)$$

converges.

An interesting special case is obtained when $n = 1$, $\alpha > 0$, and $p = 2$. In this case, the unit ball $\mathbb{B}_n$ becomes the unit disk $\mathbb{D}$, and membership in the Bergman spaces $A^2_\alpha(\mathbb{D})$ and $A^2_\alpha(\mathbb{D}^2)$ can be described by Taylor coefficients. As a consequence, we see that for $\alpha > 0$,

$$\sum_{i=1}^{k}\frac{1}{i^{\alpha+1}(k+1-i)^{\alpha+1}}\sim\frac{1}{k^{\alpha+1}}$$

as $k \to \infty$. This does not seem to be obvious. Once again, this is not true when $\alpha = 0$.

2 Preliminary Estimates

We begin with an identity concerning Möbius maps on the unit ball and derive some useful inequalities. This will play an important role in the proof of Theorem 1.1.

Lemma 2.1 *For all z and w in $\mathbb{B}_n$ we have*

$$|z-\varphi_z(w)|^2=\frac{(1-|z|^2)(|w|^2-|\langle z,w\rangle|^2)}{|1-\langle z,w\rangle|^2}. \tag{2.1}$$

Consequently,

$$|z-\varphi_z(w)|\geq\frac{|w|(1-|z|^2)}{|1-\langle z,w\rangle|}, \tag{2.2}$$

and

$$|z-\varphi_z(w)|^2\leq\frac{2(1-|z|^2)}{|1-\langle z,w\rangle|}. \tag{2.3}$$

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Proof To prove the identity in (2.1), we use [5, Lemmas 1.2 and 1.3] along with the fact that $z = \varphi_z(0)$. Let $S = |z - \varphi_z(w)|^2$. Then

$$\begin{aligned}
S &= |z|^2 + |\varphi_z(w)|^2 - \langle z, \varphi_z(w)\rangle - \langle \varphi_z(w), z\rangle \\
&= |z|^2 + |\varphi_z(w)|^2 - \Big(1 - \frac{1-|z|^2}{1-\langle z,w\rangle}\Big) - \Big(1 - \frac{1-|z|^2}{1-\langle w,z\rangle}\Big) \\
&= -(1-|z|^2) - (1-|\varphi_z(w)|^2) + \frac{1-|z|^2}{1-\langle z,w\rangle} + \frac{1-|z|^2}{1-\langle w,z\rangle} \\
&= \frac{1-|z|^2}{1-\langle z,w\rangle} + \frac{1-|z|^2}{1-\langle w,z\rangle} - (1-|z|^2) - \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \\
&= (1-|z|^2)\Big[\frac{1}{|1-\langle z,w\rangle|^2} - \Big|1 - \frac{1}{1-\langle z,w\rangle}\Big|^2 - \frac{1-|w|^2}{|1-\langle z,w\rangle|^2}\Big] \\
&= \frac{(1-|z|^2)(|w|^2 - |\langle z,w\rangle|^2)}{|1-\langle z,w\rangle|^2}.
\end{aligned}$$

This proves the identity in (2.1). The inequality in (2.2) follows from (2.1) and the fact that $|\langle z,w\rangle| \le |z||w|$. The inequality in (2.3) follows from (2.1) and the following estimates:

$$\begin{aligned}
|w|^2 - |\langle z,w\rangle|^2 &\le 1 - |\langle z,w\rangle|^2 \\
&= (1 + |\langle z,w\rangle|)(1 - |\langle z,w\rangle|) \\
&\le 2|1 - \langle z,w\rangle|.
\end{aligned}$$

This completes the proof of the lemma. ■

Note that Lemma 2.1 appears as Exercise 3.18 in [5]. We believe it never appeared elsewhere before, so a full proof is given here.

Lemma 2.2 *There exists a positive constant C, independent of f, such that*

$$\int_{\mathbb{B}_n} \frac{|f(w)|^p}{|w|^p}\, dv_\alpha(w) \le C \int_{\mathbb{B}_n} |f(w)|^p\, dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ with $f(0) = 0$.

Proof This is well known. See [4, Lemma 4.26] in the one-dimensional case. The higher dimensional case is proved in a similar way. ■

Lemma 2.3 *For any $r > 0$ and $z \in \mathbb{B}_n$ let $D(z,r)$ be the Bergman metric ball at z.*

(i) *There exists a positive constant C (dependent on r, p, and α, but not on f and z) such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p\, dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$.

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(ii) *There exists a positive constant C (dependent on r but not on z and w) such that*

$$C^{-1}(1-|w|^2) \le |1-\langle z,w\rangle| \le C(1-|z|^2)$$

for all $z \in \mathbb{B}_n$ and $w \in D(z,r)$.

Proof This is also well known. See [5, Lemma 2.24], for example. ■

Lemma 2.4 *Suppose $\alpha > -1$ and $t < 0$. Then there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} \frac{dv_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+t}} \le C$$

for all $z \in \mathbb{B}_n$.

Proof This is part of Theorem 1.12 in [5]. ■

3 Main Result

We now proceed to prove Theorem 1.1, the main result of the paper. This will be accomplished by the following three lemmas.

Lemma 3.1 *There exists a positive constant C such that $I(f) \le C J(f)$ for all $f \in H(\mathbb{B}_n)$, where*

$$I(f) = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|z-w|^p}\, dv_\alpha(z)\, dv_\alpha(w),$$

and

$$J(f) = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^p}\, dv_\alpha(z)\, dv_\alpha(w).$$

Proof By Fubini's theorem and a change of variables (see [5, Proposition 1.13]),

$$I(f) = \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z)-f\circ\varphi_z(w)|^p}{|z-\varphi_z(w)|^p}\, \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(w).$$

By (2.2), there exists a positive constant C such that $I(f)$ is less than or equal to

$$C\int_{\mathbb{B}_n} (1-|z|^2)^{n+1+\alpha-p}\, dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z)-f\circ\varphi_z(w)|^p}{|w|^p|1-\langle z,w\rangle|^{2(n+1+\alpha)-p}}\, dv_\alpha(w).$$

By Lemma 2.2, there exists another constant $C > 0$ such that $I(f)$ is less than or equal to

$$C\int_{\mathbb{B}_n} (1-|z|^2)^{n+1+\alpha-p}\, dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z)-f\circ\varphi_z(w)|^p}{|1-\langle z,w\rangle|^{2(n+1+\alpha)-p}}\, dv_\alpha(w).$$

Another change of variables then gives

$$I(f) \le C\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^p}\, dv_\alpha(z)\, dv_\alpha(w) = C J(f).$$

This proves the lemma. ■

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Lemma 3.2 *There exists a positive constant C such that*

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p \, dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_\alpha(z) \, dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$.

Proof We use $I(f)$ again to denote the integral on the right-hand side of the inequality to be proved. And again we make a change of variables to obtain

$$I(f) = \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f \circ \varphi_z(w)|^p}{|z - \varphi_z(w)|^p} \, \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} \, dv_\alpha(w).$$

By the inequality in (2.3) of Lemma 2.1, there exists a positive constant C such that $I(f)$ is greater than or equal to

$$C \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f \circ \varphi_z(w)|^p}{\frac{(1-|z|^2)^{p/2}}{|1-\langle z,w \rangle|^{p/2}}} \, \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} \, dv_\alpha(w).$$

If we let $D(z, r)$ denote the Bergman metric ball, then $I(f)$ is greater than or equal to

$$C \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f \circ \varphi_z(w)|^p \, \frac{(1 - |z|^2)^{n+1+\alpha-p/2}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)-p/2}} \, dv_\alpha(w).$$

This together with Lemma 2.3(ii) shows that there is another positive constant C, independent of f and z, such that

$$I(f) \geq C \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} \frac{|f(z) - f \circ \varphi_z(w)|^p}{(1 - |z|^2)^{n+1+\alpha}} \, dv_\alpha(w).$$

Combining this with Lemma 2.3(i), we obtain yet another positive constant C such that

$$I(f) \geq C \int_{\mathbb{B}_n} |f(z) - f \circ \varphi_z(z)|^p \, dv_\alpha(z),$$

or

$$I \geq C \int_{\mathbb{B}_n} |f(z) - f(0)|^p \, dv_\alpha(z).$$

This proves the lemma. ∎

Lemma 3.3 *If $0 < p < n + 1 + \alpha$ and $f \in A^p_\alpha(\mathbb{B}_n)$, then the double integral*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_\alpha(z) \, dv_\alpha(w)$$

converges.

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Proof Once again we use $I(f)$ to denote the double integral above. By the remark at the end of [3], there exists a positive continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \le d(z,w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n,$$

where

$$d(z,w) = \frac{|z-w|}{|1 - \langle z, w\rangle|}.$$

It follows that

$$I(f) \le \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \Big(\frac{g(z) + g(w)}{|1 - \langle z, w\rangle|}\Big)^p \, dv_\alpha(z)\, dv_\alpha(w).$$

Choose a positive constant C such that $(x + y)^p \le C(x^p + y^p)$ for all positive values x and y. Then

$$I(f) \le C \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{g(z)^p + g(w)^p}{|1 - \langle z, w\rangle|^p} \, dv_\alpha(z)\, dv_\alpha(w).$$

This along with Fubini's theorem gives

$$I(f) \le 2C \int_{\mathbb{B}_n} g(z)^p \, dv_\alpha(z) \int_{\mathbb{B}_n} \frac{dv_\alpha(w)}{|1 - \langle z, w\rangle|^p}.$$

Since $p < n+1+\alpha$, it follows from Lemma 2.4 that there is another positive constant C such that

$$I(f) \le C \int_{\mathbb{B}_n} g(z)^p \, dv_\alpha(z) < \infty.$$

This proves the lemma. ■

Note that Lemma 3.3 above was obtained in [3] in the one-dimensional case. Now Lemmas 3.2and 3.3 prove the equivalence of conditions (i) and (ii) in Theorem 1.1. Lemma 3.1 shows that condition (iii) implies condition (ii). That condition (i) implies condition (iii) follows from the elementary inequality $|f(z) - f(w)|^p \le C(|f(z)|^p + |f(w)|^p)$ and the proof of Lemma 3.3. This completes the proof of Theorem 1.1. We can slightly strengthen Theorem 1.1 as follows.

Corollary 3.4 *Fix $\alpha > -1$ and $0 < p < n + 1 + \alpha$. For any holomorphic function f in $\mathbb{B}_n$ let*

$$\|f\|_1^p = \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z),$$

$$\|f\|_2^p = |f(0)|^p + \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w\rangle|^p} \, dv_\alpha(z)\, dv_\alpha(w),$$

$$\|f\|_3^p = |f(0)|^p + \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_\alpha(z)\, dv_\alpha(w).$$

Then there exists a positive constant C such that

$$C^{-1}\|f\|_k \le \|f\|_l \le C\|f\|_k, \quad 1 \le l \le 3,\ 1 \le k \le 3,$$

for all $f \in H(\mathbb{B}_n)$.

Proof This follows from Theorem 1.1 and the open mapping theorem. ∎

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