5

Superfield formalism

We saw in Chapter 3 how the Wess–Zumino model could be formulated in terms of the fields $\mathcal{S}$, ψ_L, and the auxiliary field $\mathcal{F}$, which transform into each other under a supersymmetry transformation. Here, we simply "pulled a Lagrangian out of a hat", and verified by brute force that (at least the free part of) this Lagrangian led to a supersymmetric action. While this example was instructive, it provided no guidance as to how to write down other more complicated supersymmetric theories. We alluded, however, to the fact that we could think of the fields, $\mathcal{S}$, ψ_L, and $\mathcal{F}$ as the components of a single entity, a chiral superfield.[1]

The superfield formalism provides a convenient way to formulate general rules for the construction of supersymmetric Lagrangians, even for theories with non-Abelian gauge symmetry that are the foundation of modern particle physics. The superfield calculus that we develop in this and succeeding chapters will provide us with a constructive procedure for writing down theories that are guaranteed to be supersymmetric. This procedure will ultimately be used to write down the simplest supersymmetric extension of the Standard Model. This theory, augmented with suitable soft supersymmetry breaking terms, is known as the Minimal Supersymmetric Standard Model, or MSSM.

5.1 Superfields

To begin, we would like to somehow combine the fields $\mathcal{S}$, ψ_L, and $\mathcal{F}$ into a single "superfield", in much the same way that the neutron and proton fields are combined into a single "nucleon" field in the isospin formalism. The fields $\mathcal{S}$ and ψ transform differently under Lorentz transformations so that it is by no means obvious how to combine these fields into a single entity called a superfield in which the component fields all enter on the same footing, i.e. we do not combine the scalar bilinear in ψ

[1] A. Salam and J. Strathdee, *Nucl. Phys.* **B76**, 477 (1974).

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

with the scalars $\mathcal{S}$ and $\mathcal{F}$. We are thus led to introduce a new Majorana spinor θ, with components θ_1, θ_2, θ_3, and θ_4, which can be combined with ψ to make a Lorentz scalar that can then be "added" to $\mathcal{S}$. Furthermore, since the components of ψ obey anticommutation relations, the components of θ will be taken to be anticommuting *Grassmann* numbers, so that

$$\{\theta_a, \theta_b\} = 0. \tag{5.1}$$

We will further assume that

$$\{\theta_a, \psi_b\} = 0. \tag{5.2}$$

Note that Eq. (5.1) implies that $\theta_a \theta_a = 0$ (no sum on a).

The spinor θ is determined by the four independent quantities θ_a that we have introduced. We emphasize that these are not complex numbers, but are a new type of object, a Grassmann number. Although these do not commute, we should be clear that they are *not* operators, but anticommuting numbers, in the same sense that usual complex numbers are commuting numbers. These Grassmann numbers (sometimes also referred to as a-numbers in analogy with commuting c-numbers) also anticommute with fermionic operators, but commute with bosonic operators.

The Majorana condition, $\bar{\theta} = \theta^T C$ means that the components of the conjugate spinor $\bar{\theta}$ are completely determined in terms of the four independent θ_as. Thus a product of any chain of larger than a total of four θ or $\bar{\theta}$s is identically zero. Alternatively, it will sometimes be convenient to think of two components of θ and two components of $\bar{\theta}$ as independent, or that each of the two components of θ_{L} and θ_{R} are independent.

A superfield is a function of x^μ and θ. The spinor θ (together with the coordinate vector x^μ) is a superfield label in exactly the same way that the coordinate vector x^μ is a label in the conventional formulation of field theory. We will denote superfields by carets and let $\hat{\Phi}(x, \theta)$ stand for a general superfield. The field $\hat{\Phi}$ thus depends on four (commuting) spacetime co-ordinates x^μ and on four anticommuting co-ordinates, θ_a. The extension of four-dimensional spacetime to include the four anticommuting dimensions is usually referred to as superspace. Whether the anticommuting variables have a physical significance, or whether they serve only as bookkeeping devices is something we will not dwell upon.

An important property of functions of Grassmann variables follows from the fact that any power series expansion in terms of the anticommuting co-ordinates always terminates because the square of any Grassmann variable is zero. For instance, if η is a Grassmann variable, and we have a function $f(\eta)$, then $f(\eta) = A + B\eta$, where A and B are just ordinary (real or complex) numbers. The power series expansion terminates with the first term in η. A function $g(x, \eta)$ would have a similar expansion, except that the coefficients A and B would now be (real or complex)

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

functions of x. We can similarly write the superfield $\hat{\Phi}$ in terms of independent products of the four θ_a variables, with coefficients that are functions of spacetime co-ordinates x^μ.

Exercise *Verify that from the four Grassmann variables θ_a, $a = 1, 2, 3, 4$, one can make exactly 16 independent products of* $0, 1 \ldots 4$ *θs. The most obvious choice is 1, θ_a (4 terms), $\theta_a\theta_b$ (6 terms, because of the anticommutativity of the θ_as), $\theta_a\theta_b\theta_c$ (4 terms, as any one θ_a from the unique quartic term in the θs can be omitted) and finally one quartic term, $\theta_1\theta_2\theta_3\theta_4$.*

We could thus expand $\hat{\Phi}(x,\theta) = A + B\theta_1 + \cdots + P\theta_1\theta_2\theta_3\theta_4$. It is, however, more convenient to expand the superfield in terms of

$$\text{1 term independent of } \theta \;\; ; \;\; \mathbf{1}, \tag{5.3a}$$

$$\text{4 terms linear in } \theta \;\; ; \;\; \text{choose } \bar{\theta}\gamma_5, \tag{5.3b}$$

$$\text{6 terms bilinear in } \theta \;\; ; \;\; \text{choose } \bar{\theta}\theta,\; \bar{\theta}\gamma_5\theta,\; \bar{\theta}\gamma_\mu\gamma_5\theta, \tag{5.3c}$$

$$\text{4 terms trilinear in } \theta \;\; ; \;\; \text{choose } \bar{\theta}\gamma_5\theta\cdot\bar{\theta}, \tag{5.3d}$$

$$\text{1 term quartic in } \theta \;\; ; \;\; \text{choose } (\bar{\theta}\gamma_5\theta)^2, \tag{5.3e}$$

since this manifestly displays the Lorentz properties of the "expansion coefficients" which will ultimately be the usual fields in the theory. Terms such as $\bar{\theta}\gamma_\mu\theta$ and $\bar{\theta}\sigma_{\mu\nu}\theta$ are identically zero due to Eqs. (3.8c) and (3.8e). We can thus write a general superfield as,[2]

$$\begin{aligned}\hat{\Phi}(x,\theta) = {}& \mathcal{S} - \mathrm{i}\sqrt{2}\bar{\theta}\gamma_5\psi - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)\mathcal{M} + \frac{1}{2}(\bar{\theta}\theta)\mathcal{N} + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V^\mu \\ & + \mathrm{i}\,(\bar{\theta}\gamma_5\theta)[\bar{\theta}(\lambda + \frac{\mathrm{i}}{\sqrt{2}}\partial\!\!\!/\psi)] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2[\mathcal{D} - \frac{1}{2}\Box\mathcal{S}].\end{aligned} \tag{5.4}$$

Thus, the coefficients in the above expansion are the sixteen component fields

$$\mathcal{S},\; \psi,\; \mathcal{M},\; \mathcal{N},\; V^\mu,\; \lambda,\; \text{and } \mathcal{D}. \tag{5.5}$$

Here, V^μ is a vector field and ψ and λ are spinor fields. In general, the bosonic fields are complex, while ψ and λ are Dirac fields. The peculiar form of the coefficients of trilinear and quartic terms in θ in this expansion as well as the factors of half and $\sqrt{2}$ is chosen for future convenience. It should be obvious to the reader that although any scalar superfield can be written as in Eq. (5.4), this form is not unique. We will

[2] Actually, this is not the most general superfield since we have assumed that the θ independent term in the expansion is a Lorentz scalar. It is possible, and indeed necessary as we will see when we consider supersymmetric gauge theories, to introduce superfields where this is not the case. Such superfields will carry an additional index which specifies the Lorentz transformation property of their leading, i.e. θ-independent component. We will refer to the superfield in Eq. (5.4) as a *scalar* superfield.

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

regard (5.4) as the canonical form. Any other expansion can be straightforwardly reduced to this canonical form using identities amongst the Grassmann variables introduced later in this chapter.

Let us compute the Hermitian conjugate superfield $\hat{\Phi}^\dagger$. We will need the identities,

$$(\bar{\psi}\chi)^\dagger = \bar{\chi}\psi = \overline{\psi^c}\chi^c, \tag{5.6a}$$
$$(\bar{\psi}\gamma_5\chi)^\dagger = -\bar{\chi}\gamma_5\psi = -\overline{\psi^c}\gamma_5\chi^c, \quad \text{and} \tag{5.6b}$$
$$(\bar{\psi}\partial\!\!\!/\chi)^\dagger = \partial_\mu\bar{\chi}\gamma_\mu\psi = -\overline{\psi^c}\partial\!\!\!/\chi^c, \tag{5.6c}$$

so that

$$(\bar{\theta}\theta)^\dagger = \bar{\theta}\theta, \tag{5.7a}$$
$$(\mathrm{i}\bar{\theta}\gamma_5\theta)^\dagger = \mathrm{i}\bar{\theta}\gamma_5\theta, \quad \text{and} \tag{5.7b}$$
$$(\bar{\theta}\gamma_5\gamma_\mu\theta)^\dagger = \bar{\theta}\gamma_5\gamma_\mu\theta. \tag{5.7c}$$

Then,

$$\begin{aligned}\hat{\Phi}^\dagger(x,\theta) = \mathcal{S}^\dagger - \mathrm{i}\sqrt{2}\bar{\theta}\gamma_5\psi^c - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)\mathcal{M}^\dagger + \frac{1}{2}(\bar{\theta}\theta)\mathcal{N}^\dagger + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V^{\mu\dagger} \\ + \mathrm{i}(\bar{\theta}\gamma_5\theta)[\bar{\theta}(\lambda^c + \frac{\mathrm{i}}{\sqrt{2}}\partial\!\!\!/\psi^c)] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2[\mathcal{D}^\dagger - \frac{1}{2}\Box\mathcal{S}^\dagger].\end{aligned} \tag{5.8}$$

We define the superfield $\hat{\Phi}$ to be real if $\hat{\Phi} = \hat{\Phi}^\dagger$. In this case, we see that the bosonic fields are real and the fermionic fields are Majorana ($\psi = \psi^c$ and $\lambda = \lambda^c$). It was for this reason that we inserted the factors of i in our superfield expansion in Eq. (5.4). In general, however, $\hat{\Phi}$ need not be real.

Exercise *Verify the relations in (5.6a), (5.6b), and (5.6c). Notice that these hold regardless of whether the spinors are Dirac or Majorana.*

5.2 Representations of symmetry generators: a recap

In quantum field theory, symmetry transformations act on field operators which are the dynamical variables. We focus on symmetries which are linear transformations of the field operators. A symmetry operation, with a set of parameters α_a, due to the action of the set of generators Q_a can thus be written as,

$$\mathrm{e}^{\mathrm{i}\alpha_a Q_a}\phi_m\mathrm{e}^{-\mathrm{i}\alpha_b Q_b} = \left(\mathrm{e}^{-\mathrm{i}\alpha_a t_a}\right)_{mn}\phi_n. \tag{5.9a}$$

It is important to understand that $\left(\mathrm{e}^{-\mathrm{i}\alpha_a t_a}\right)_{mn}$ are simply numerical coefficients. There are, of course, as many parameters α_a as there are generators Q_a, and for

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

each Q_a we have a matrix coefficient (t_a). For an infinitesimal transformation, this becomes,

$$\delta\phi_m = \mathrm{i}\alpha_a[Q_a, \phi_m] = -\mathrm{i}(\alpha_a t_a)_{mn}\phi_n. \tag{5.9b}$$

By considering the action of successive symmetry transformations and using the Jacobi identity,

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

in the last step, it is straightforward to show that

$$\delta_2\delta_1\phi_m = [\mathrm{i}\alpha_{2b}Q_b, \delta_1\phi_m] = -[\mathrm{i}\alpha_{1a}Q_a, [\phi_m, \mathrm{i}\alpha_{2b}Q_b]] - [\phi_m, [\mathrm{i}\alpha_{2b}Q_b, \mathrm{i}\alpha_{1a}Q_a]],$$

which then yields,

$$(\delta_2\delta_1 - \delta_1\delta_2)\phi_m = [[\mathrm{i}\alpha_{2b}Q_b, \mathrm{i}\alpha_{1a}Q_a], \phi_m]. \tag{5.10a}$$

The result of the successive transformations can also be written in terms of the numerical coefficients t_{mn} introduced above as,

$$\delta_2\delta_1\phi_m = -\alpha_{1a}(t_a)_{mn}\alpha_{2b}(t_b)_{np}\phi_p,$$

so that the right-hand side of Eq. (5.10a) can also be written as,

$$(\delta_2\delta_1 - \delta_1\delta_2)\phi_m = -\alpha_{1a}\alpha_{2b}[t_a, t_b]_{mp}\phi_p, \tag{5.10b}$$

with the usual matrix multiplication rule for the product of the matrices t_a and t_b appearing on the right-hand side.

The set of generators Q_a satisfies algebraic commutation relations that are determined by the symmetry in question. If these are the generators of spacetime symmetries, these are the commutation relations of the Poincaré algebra. If these are generators of internal symmetry transformations, they satisfy the commutation relations of the corresponding symmetry algebra. In both these cases (and many others that we encounter), the algebra is a Lie algebra, so that the commutation rules can be written as,

$$[Q_a, Q_b] = i f_{abc} Q_c,$$

where the coefficients f_{abc} are the structure constants of the algebra. Requiring the right-hand sides of (5.10a) and (5.10b) to be the same, we see that the set of coefficient matrices t_a must satisfy,

$$[t_a, t_b] = -i f_{bac} t_c = i f_{abc} t_c.$$

In other words, these coefficient matrices obey the same commutation relations as the abstract generators Q. We say that these furnish a representation of the symmetry algebra.

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

Exercise *We implicitly assumed that the parameters α_a are commuting numbers when we showed that the matrices t_a obey the same commutation relations as the generators Q_a. If instead the parameters are anticommuting numbers, and the generators Q_a and Q_b obey an anticommutation relation, show that corresponding matrices t_a and t_b obey these same relations, and so, furnish a representation of this graded algebra. In this case, of course, the exponential in Eq. (5.9a) becomes a polynomial.*

The familiar Pauli matrices or the Gell-Mann matrices are examples of matrix representations of the generators of internal symmetry groups $SU(2)$ and $SU(3)$. But what does all this have to do with the representation of spacetime symmetry generators by differential operators that we have seen in Chapter 4? The underlying idea is the same. For instance, the momentum, defined as the generator of translations, satisfies

$$\phi \to \phi' = \phi(x+a) = \mathrm{e}^{\mathrm{i}a^\mu P_\mu}\phi\mathrm{e}^{-\mathrm{i}a^\mu P_\mu} \simeq \phi(x) + a^\mu \frac{\partial\phi}{\partial x^\mu} + \cdots \tag{5.11}$$

For an infinitesimal translation we find,

$$\phi' = \phi + \delta\phi = (1 + \mathrm{i}a^\mu P_\mu)\phi(1 - \mathrm{i}a^\mu P_\mu) = \phi + a^\mu \frac{\partial\phi}{\partial x^\mu}, \tag{5.12}$$

or

$$[P_\mu, \phi] = -\mathrm{i}\partial_\mu\phi. \tag{5.13}$$

Using Eq. (5.9b), we see that the translation generator P_μ can be represented by $\delta(x - x') \times \mathrm{i}\partial_\mu$ (where the indices m and n are the continuous spacetime indices x and x'). It is customary to omit the "identity matrix" $\delta(x - x')$ when writing this, and we frequently say that P_μ is represented by the differential operator $\mathrm{i}\partial_\mu$. The other generators of the Poincaré algebra can be similarly represented by differential operators. It is then straightforward to check that the differential operators furnish a representation of the Poincaré algebra, i.e. they obey the same commutation relations as the generators.

5.3 Representation of SUSY generators as differential operators

We have just seen that the generators of the Poincaré algebra can be represented by differential operators, where the derivative is with respect to the spacetime co-ordinate. We now want to realize the spinorial generator of supersymmetry transformations Q as a differential operator in superspace acting on the superfield $\hat{\Phi}(x, \theta)$.

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

This requires us to first explain what is meant by derivatives with respect to Grassmann numbers θ_a. First, since the four θ_as (or, the four $\bar{\theta}_a$s) are independent we *define*,

$$\frac{\partial \theta_a}{\partial \theta_b} = \delta_{ab} \quad \text{and} \quad \frac{\partial \bar{\theta}_a}{\partial \bar{\theta}_b} = \delta_{ab}. \tag{5.14}$$

Then, since $\theta_a = C_{ab}\bar{\theta}_b$, we have

$$\frac{\partial \theta_a}{\partial \bar{\theta}_b} = C_{ab}. \tag{5.15}$$

If we have a product of θs, we must bring $\partial/\partial\theta_a$ next to the θ we wish to differentiate, e.g.

$$\frac{\partial}{\partial \theta_c}(\theta_a\theta_b) = \frac{\partial \theta_a}{\partial \theta_c}\theta_b - \theta_a\frac{\partial \theta_b}{\partial \theta_c} = \delta_{ac}\theta_b - \theta_a\delta_{bc}, \tag{5.16}$$

where the "$-$" sign arises because the θs anticommute. Differentiation of a product of $\bar{\theta}$s or a combination of θs and $\bar{\theta}$s is analogously defined.

Since Q is a spinor operator, its action on a superfield $\hat{\Phi}$ correspondingly changes its Lorentz transformation properties by either taking away or adding a θ to each term. Since, as we have just seen, differentiation with respect to θ_a removes a θ, we are led to try,

$$[Q_m, \hat{\Phi}] = \left(M_{mn}\frac{\partial}{\partial \bar{\theta}_n} + N_{mn}\theta_n\right)\hat{\Phi}(x, \theta), \tag{5.17}$$

where the matrices M_{mn} and N_{mn} (which may depend on x) have to be determined. The reader may wonder why we wrote the derivative with respect to $\bar{\theta}$ rather than θ. By Eq. (5.15), these are the same up to the numerical matrix C_{ab}. We will see shortly that by writing it as in Eq. (5.17), the matrix M becomes a multiple of the identity matrix, and we can write the representation of a SUSY transformation with the Majorana spinor parameter α as,

$$\left[\bar{\alpha}Q, \hat{\Phi}\right] = \left(\bar{\alpha}\frac{\partial}{\partial \bar{\theta}} + \bar{\alpha}N\theta\right)\hat{\Phi}. \tag{5.18}$$

We can work out what N must be by applying the Jacobi identity to two successive SUSY transformations by amounts α_1 and α_2. This gives,

$$\left[\left[\bar{\alpha}_1 Q, \bar{\alpha}_2 Q\right], \hat{\Phi}\right] = \left[\bar{\alpha}_1 Q, \left[\bar{\alpha}_2 Q, \hat{\Phi}\right]\right] - \left[\bar{\alpha}_2 Q, \left[\bar{\alpha}_1 Q, \hat{\Phi}\right]\right]. \tag{5.19}$$

We then write each term on the RHS as an action of successive SUSY transformations using (5.18) to obtain,

$$\left(\bar{\alpha}_1\frac{\partial}{\partial \bar{\theta}} + \bar{\alpha}_1 N\theta\right)\left(\bar{\alpha}_2\frac{\partial}{\partial \bar{\theta}} + \bar{\alpha}_2 N\theta\right)\hat{\Phi} - (2 \leftrightarrow 1).$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

A little manipulation of indices (and remembering that both θs and αs are anticommuting variables) shows that the terms involving no derivatives with respect to $\bar{\theta}$ give zero, as do the terms involving two such derivatives. We are then left only with two terms, each involving a θ derivative from one factor multiplying $N\theta$ from the other factor. We leave the following as an exercise for the reader.

Exercise *Verify that the RHS of (5.19) reduces to*
$\left[-\bar{\alpha}_{1a}\bar{\alpha}_{2b}\,(NC)_{ba} + \bar{\alpha}_{2b}\bar{\alpha}_{1a}\,(NC)_{ab}\right]\hat{\Phi}$.

On the other hand, the inner commutator of the LHS of (5.19) becomes

$$\bar{\alpha}_{2b}\bar{\alpha}_{1a}\{Q_a, Q_b\} = -2\bar{\alpha}_{2b}\bar{\alpha}_{1a}(\gamma_\mu C)_{ab}P_\mu$$

so that

$$\left[[\bar{\alpha}_1 Q, \bar{\alpha}_2 Q]\,, \hat{\Phi}\right] = 2\mathrm{i}\bar{\alpha}_{2b}\bar{\alpha}_{1a}(\gamma_\mu C)_{ab}\partial_\mu\hat{\Phi}.$$

We are thus led to require that the matrix N must satisfy,

$$(NC)_{ba} + (NC)_{ab} = 2\mathrm{i}\,(\partial\!\!\!/ C)_{ba}\,,$$

whose solution may be written as $N = \mathrm{i}\partial\!\!\!/$. Of course, because each term in the Jacobi identity is quadratic in Q, we cannot fix the overall factor in front of (5.18) from this. The choice of this factor is a convention. We will choose it to be i, which as we will see later is consistent with the SUSY transformations of chiral scalar superfields that we have already introduced in Chapter 3. We thus obtain the desired realization of the SUSY generator,

$$\left[\bar{\alpha}Q, \hat{\Phi}\right] = \mathrm{i}\left(\bar{\alpha}\frac{\partial}{\partial\bar{\theta}} + \mathrm{i}\bar{\alpha}\partial\!\!\!/\theta\right)\hat{\Phi}. \tag{5.20}$$

This expression for the supersymmetry generator is the analogue of (5.13) for the translation generator.

5.4 Useful θ identities

Before proceeding further, we have a short digression to establish a number of useful identities for Grassmann numbers θ that we have introduced into our formalism. These identities are especially useful when we do superfield manipulations. For instance, we may need to take a product of two (or more) superfields which, since it is just a function of x and θ coordinates, is itself a superfield, but not in the canonical form of Eq. (5.4). Indeed most manipulations will leave us with a superfield which is not in this canonical form. However, in order to read off the components of the resulting superfield, or simply to add superfields, we will need to be able to recast any superfield into canonical form. We have found the following identities to be very useful for this purpose, and we will use them repeatedly in our subsequent manipulations.

One manipulation that we need repeatedly is regrouping θs and $\bar{\theta}$s into a common set of bilinears. For this purpose, it is very useful to note that,

$$\theta_a \bar{\theta}_b = -\frac{1}{4}\left\{\bar{\theta}\gamma_5\theta(\gamma_5)_{ab} + \bar{\theta}\theta\delta_{ab} - (\bar{\theta}\gamma^\mu\gamma_5\theta)(\gamma_\mu\gamma_5)_{ab}\right\}. \tag{5.21}$$

This is the basic formula that underlies the Fierz re-arrangement discussed in Chapter 3.

We list below various relations that we have found very useful for superfield manipulation. We outline how to establish these, and leave it to the reader to verify these in detail.

Bilinear Identities

$$\bar{\theta}\gamma_\mu\theta = 0, \tag{5.22a}$$
$$\bar{\theta}\sigma_{\mu\nu}\theta = 0, \tag{5.22b}$$
$$\bar{\theta}\gamma_\mu\gamma_\nu\theta = g_{\mu\nu}\bar{\theta}\theta, \tag{5.22c}$$
$$\bar{\theta}\gamma_5\gamma_\mu\gamma_\nu\theta = g_{\mu\nu}\bar{\theta}\gamma_5\theta, \tag{5.22d}$$
$$\bar{\theta}\gamma_\mu\theta_{\mathrm{L/R}} = -\bar{\theta}\gamma_\mu\theta_{\mathrm{R/L}}, \tag{5.22e}$$
$$\bar{\theta}\gamma_\mu\gamma_5\theta_{\mathrm{L/R}} = \bar{\theta}\gamma_\mu\gamma_5\theta_{\mathrm{R/L}}. \tag{5.22f}$$

The first two are the result of the Majorana character of θ and follow immediately from (3.8b) and (3.8c) of Chapter 3. To establish the next two, decompose $\gamma_\mu\gamma_\nu$ into its symmetric and antisymmetric parts, and use (3.8e) to see that the latter gives zero. Finally, the last two follow from the fact the vector bilinear identically vanishes.

Trilinear Identities

$$\bar{\theta}\theta \cdot \theta = -\bar{\theta}\gamma_5\theta \cdot (\gamma_5\theta), \tag{5.23a}$$
$$\bar{\theta}\theta \cdot \bar{\theta} = -\bar{\theta}\gamma_5\theta \cdot (\bar{\theta}\gamma_5), \tag{5.23b}$$
$$\bar{\theta}\gamma_5\gamma_\mu\theta \cdot \theta = -\bar{\theta}\gamma_5\theta \cdot (\gamma_\mu\theta), \tag{5.23c}$$
$$\bar{\theta}\gamma_5\gamma_\mu\theta \cdot \bar{\theta} = \bar{\theta}\gamma_5\theta \cdot (\bar{\theta}\gamma_\mu). \tag{5.23d}$$

To prove the first, we note that we can write the left-hand side in terms of θ alone (using $\bar{\theta} = \theta^T C$) as $\theta_{\mathrm{L}}^T C\theta_{\mathrm{L}}\theta_{\mathrm{R}} + \theta_{\mathrm{R}}^T C\theta_{\mathrm{R}}\theta_{\mathrm{L}}$. Here, we have used the fact that any product of three θ_{L}s or three θ_{R}s identically vanishes as only two of these are independent (and θs anticommute). The reader can similarly check that the right-hand side of (5.23a) reduces to this same quantity. Eq. (5.23b) can be proven in the same manner, or alternatively, by taking the Dirac conjugate of (5.23a).

To establish (5.23c) we first show using Eq. (5.21) that

$$(\overline{\theta_L}\gamma_5\gamma_\mu\theta_L)\theta_R = (\overline{\theta_R}\gamma_5\gamma_\mu\theta_R)\theta_R = -\frac{1}{2}(\bar{\theta}\gamma_5\theta)P_R\gamma_\mu\theta,$$
$$(\overline{\theta_L}\gamma_5\gamma_\mu\theta_L)\theta_L = (\overline{\theta_R}\gamma_5\gamma_\mu\theta_R)\theta_L = -\frac{1}{2}(\bar{\theta}\gamma_5\theta)P_L\gamma_\mu\theta.$$

Combining these appropriately immediately leads to (5.23c). Eq. (5.23d) may be obtained by taking the Dirac conjugate of (5.23c).

Quartic Identities

$$\bar{\theta}\gamma_5\theta \cdot \bar{\theta}\theta = 0, \tag{5.24a}$$
$$\bar{\theta}\gamma_5\theta \cdot \bar{\theta}\gamma_\mu\gamma_5\theta = 0, \tag{5.24b}$$
$$\bar{\theta}\theta \cdot \bar{\theta}\gamma_\mu\gamma_5\theta = 0, \tag{5.24c}$$
$$(\bar{\theta}\theta)^2 = -(\bar{\theta}\gamma_5\theta)^2, \tag{5.24d}$$
$$\bar{\theta}\gamma_5\gamma_\mu\theta \cdot \bar{\theta}\gamma_5\gamma_\nu\theta = -g_{\mu\nu}(\bar{\theta}\gamma_5\theta)^2. \tag{5.24e}$$

The first of these follows if we recognize that $\bar{\theta}\Gamma\theta = \theta_R^T C\theta_R \pm \theta_L^T C\theta_L$ where the upper (lower) sign corresponds to $\Gamma = I(\gamma_5)$, and use the fact that a product of three or more θ_Ls or θ_Rs identically vanishes. Writing the left-hand side of (5.24b) or (5.24c) in terms of its chiral components immediately shows that it vanishes. Multiplying Eq. (5.23a) on the left by $\bar{\theta}$ immediately leads to (5.24d). Finally, the last of these identities may be obtained from $\bar{\theta}\gamma_5\gamma_\mu\theta \cdot \bar{\theta}\gamma_5\gamma_\nu\theta = -\bar{\theta}\gamma_5\theta \cdot \bar{\theta}\gamma_5\gamma_\nu\gamma_\mu\theta$ which follows from Eq. (5.23c); then using (5.22d) immediately yields (5.24e).

Exercise *Convince yourself that the θ identities that we have listed are valid.*

The trilinear [quartic] identities show how various trilinear [quartic] terms in θ can be recast as $\bar{\theta}\gamma_5\theta \cdot \bar{\theta}$ [$(\bar{\theta}\gamma_5\theta)^2$] that appear in our canonical form of the superfield in (5.4). Quadratic terms can be similarly cast into the forms appearing there. We expect that it is now clear to the reader how any other form for the expansion of the superfield may be reduced to this canonical form.

5.5 SUSY transformations of superfields

We are now in a position to compute how a general superfield $\hat{\Phi}(x,\theta)$ changes under an infinitesimal SUSY transformation. Our starting point is the relation

$$\delta\hat{\Phi} = \mathrm{i}\left[\bar{\alpha}Q, \hat{\Phi}\right] = \left(-\bar{\alpha}\frac{\partial}{\partial\bar{\theta}} - \mathrm{i}\bar{\alpha}\partial\!\!\!/\theta\right)\hat{\Phi}. \tag{5.25}$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

To proceed, we must work out the action of $\partial/\partial\bar{\theta}$ on various terms in $\hat{\Phi}$. For instance, to work out $\frac{\partial}{\partial\bar{\theta}}(\bar{\theta}\theta)$, it helps again to keep track of spinor indices:

$$\frac{\partial}{\partial\bar{\theta}_a}(\bar{\theta}_b\theta_b) = \theta_a - \bar{\theta}_b C_{ba}.$$

But

$$\bar{\theta}_b C_{ba} = C^T_{ab}\bar{\theta}^T_b = -(C\bar{\theta}^T)_a = -\theta_a$$

so that

$$\frac{\partial}{\partial\bar{\theta}}(\bar{\theta}\theta) = 2\theta. \tag{5.26a}$$

In a similar fashion, we can show that,

$$\frac{\partial}{\partial\bar{\theta}}(\bar{\theta}\gamma_5\theta) = 2\gamma_5\theta, \tag{5.26b}$$

$$\frac{\partial}{\partial\bar{\theta}}(\bar{\theta}\gamma_\mu\gamma_5\theta) = 2\gamma_\mu\gamma_5\theta, \tag{5.26c}$$

$$\frac{\partial}{\partial\bar{\theta}_a}(\bar{\theta}\gamma_5\theta)\cdot\bar{\theta}_b = 2(\gamma_5\theta)_a\bar{\theta}_b + \bar{\theta}\gamma_5\theta\delta_{ab}, \tag{5.26d}$$

$$\frac{\partial}{\partial\bar{\theta}}(\bar{\theta}\gamma_5\theta)^2 = 4(\bar{\theta}\gamma_5\theta)\cdot(\gamma_5\theta). \tag{5.26e}$$

We can now evaluate the RHS of Eq. (5.25). First, using $\chi \equiv \lambda + \frac{\mathrm{i}}{\sqrt{2}}\not\partial\psi$, we find

$$\begin{aligned}-\bar{\alpha}\frac{\partial}{\partial\bar{\theta}}\hat{\Phi} = {} & \mathrm{i}\sqrt{2}\bar{\alpha}\gamma_5\psi + \mathrm{i}\bar{\theta}\gamma_5\alpha\mathcal{M} - \bar{\theta}\alpha\mathcal{N} - \bar{\theta}\gamma_5\gamma_\mu\alpha V^\mu + \frac{\mathrm{i}}{2}\bar{\theta}\theta\bar{\alpha}\gamma_5\chi \\ & -\frac{\mathrm{i}}{2}\bar{\theta}\gamma_5\theta\bar{\alpha}\chi + \frac{\mathrm{i}}{2}\bar{\theta}\gamma_\mu\gamma_5\theta\bar{\alpha}\gamma^\mu\chi + \bar{\theta}\gamma_5\theta\bar{\theta}\gamma_5\alpha\left[\mathcal{D} - \frac{1}{2}\Box\mathcal{S}\right].\end{aligned} \tag{5.27}$$

Next,

$$\begin{aligned}-\mathrm{i}\bar{\alpha}\not\partial[\theta\hat{\Phi}] = {} & -\mathrm{i}\bar{\alpha}\not\partial\mathcal{S}\theta - \sqrt{2}\bar{\alpha}\not\partial\theta\bar{\theta}\gamma_5\psi - \frac{1}{2}\bar{\theta}\gamma_5\theta\bar{\alpha}\not\partial\theta\mathcal{M} \\ & -\frac{\mathrm{i}}{2}\bar{\theta}\theta\bar{\alpha}\not\partial\theta\mathcal{N} - \frac{\mathrm{i}}{2}\bar{\theta}\gamma_5\gamma_\mu\theta\bar{\alpha}\not\partial\theta V^\mu + \bar{\theta}\gamma_5\theta\bar{\alpha}\not\partial\theta\bar{\theta}\chi.\end{aligned} \tag{5.28}$$

The superfield in Eq. (5.27) is already in the canonical form. We have used the (anti)symmetry properties of Majorana spinor bilinears as well as (5.21) to write it this way. We must similarly re-arrange the last expression so that we can combine it with (5.27) to obtain $\delta\hat{\Phi}$ (with components $\delta\mathcal{S}, \delta\psi, \ldots$) in the canonical form. By comparing "coefficients", we obtain the transformation laws for the components of a general scalar superfield:

$$\delta\mathcal{S} = \mathrm{i}\sqrt{2}\bar{\alpha}\gamma_5\psi, \tag{5.29a}$$

$$\delta\psi = -\frac{\alpha\mathcal{M}}{\sqrt{2}} - \mathrm{i}\frac{\gamma_5\alpha\mathcal{N}}{\sqrt{2}} - \mathrm{i}\frac{\gamma_\mu\alpha V^\mu}{\sqrt{2}} - \frac{\gamma_5\partial\!\!\!/\mathcal{S}\alpha}{\sqrt{2}}, \tag{5.29b}$$

$$\delta\mathcal{M} = \bar{\alpha}\left(\lambda + \mathrm{i}\sqrt{2}\partial\!\!\!/\psi\right), \tag{5.29c}$$

$$\delta\mathcal{N} = \mathrm{i}\bar{\alpha}\gamma_5\left(\lambda + \mathrm{i}\sqrt{2}\partial\!\!\!/\psi\right), \tag{5.29d}$$

$$\delta V^\mu = -\mathrm{i}\bar{\alpha}\gamma^\mu\lambda + \sqrt{2}\bar{\alpha}\partial^\mu\psi, \tag{5.29e}$$

$$\delta\lambda = -\mathrm{i}\gamma_5\alpha\mathcal{D} - \frac{1}{2}[\partial\!\!\!/, \gamma_\mu]V^\mu\alpha, \tag{5.29f}$$

$$\delta\mathcal{D} = \bar{\alpha}\partial\!\!\!/\gamma_5\lambda. \tag{5.29g}$$

Exercise *Perform the required algebra to obtain the transformation laws for the components of the scalar superfield.*

Equations (5.29a)–(5.29g) define a linear transformation of the component fields, and, as expected for a SUSY transformation, the variation of a bosonic (fermionic) field is proportional to a fermionic (bosonic) field.

5.6 Irreducible SUSY multiplets

We have just seen that the components of a general scalar superfield transform into one another under supersymmetry. This does not, however, mean that we require *all* the components to be simultaneously present. Of course, we cannot arbitrarily leave out any component since these would "be generated" by the transformation. For instance, if we said $\mathcal{S}$ was absent, we would see that it would be generated by the transformation as long as $\psi \neq 0$. It is, however, possible that there might be a smaller set of component fields which transform into just one another under SUSY. If we find such a set, we say the representation furnished by the original (larger) set is *reducible*. If this set cannot be reduced any further, we say that it furnishes an *irreducible* representation of supersymmetry.

Exercise *A familiar example of the concept of irreducibility is the representation (of the 3-D rotation transformation) furnished by the tensor $T^{ij} = x^i y^j$, where x^i and y^j are the components of two co-ordinate vectors. Under rotations, the nine components of T^{ij} clearly transform into one another. Show that the six components of $S^{ij} = x^i y^j + x^j y^i$ as well as the three components of $A^{ij} = x^i y^j - x^j y^i$*

separately transform into one another. Show further that while A^{ij} furnishes an irreducible representation, the representation furnished by S^{ij} can be reduced further into a traceless symmetric tensor $\bar{S}^{ij} = S^{ij} - \frac{1}{3}\mathrm{Trace}(S)\delta^{ij}$ whose five components transform among themselves, and the unit tensor δ^{ij}, which is inert under the transformations. This is, of course, the familiar statement that the combination of two angular momentum **1** *states gives states with angular momenta* **0**,**1**, *and* **2**.

5.6.1 Left-chiral scalar superfields

Our examination of the Wess–Zumino model in Chapter 3 showed us that there is a consistent supersymmetric model that can be written down in terms of just the $\mathcal{S}$, ψ_{L}, and $\mathcal{F}$ fields. Furthermore, Eq. (3.16a)–(3.16c) show that these three fields (which are contained in our general superfield) form a multiplet under SUSY transformations. It should, therefore, be possible to find a representation where several of the components of the general superfield $\hat{\Phi}$ are zero or unphysical. In other words, the representation furnished by the components of $\hat{\Phi}$ should be reducible. Since the Wess–Zumino multiplet (3.15) does not include any vector field, we naturally look for a representation where the field strength $(\partial_\mu V_\nu - \partial_\nu V_\mu)$ vanishes, i.e. $V_\mu = \partial_\mu \zeta$. We must, of course, require that this is not altered by the SUSY transformations (5.29a)–(5.29g). In order that δV^μ is also a pure gradient, we infer from (5.29e) $\lambda = 0$. Then, requiring $\delta\lambda = 0$, gives us

$$\delta\lambda = -\mathrm{i}\gamma_5\alpha\mathcal{D} - \frac{1}{2}[\gamma_\rho, \gamma_\mu]\partial^\rho\partial^\mu\zeta\alpha = 0,$$

which, in turn, implies $\mathcal{D} = 0$. We can thus consistently choose

$$\lambda = \mathcal{D} = 0, \quad V_\mu = \partial_\mu\zeta.$$

The set of SUSY transformations then reduces to,

$$\delta\mathcal{S} = \mathrm{i}\sqrt{2}\bar{\alpha}\gamma_5\psi, \tag{5.30a}$$

$$\delta\psi = -\frac{\alpha\mathcal{M}}{\sqrt{2}} - \mathrm{i}\frac{\gamma_5\alpha\mathcal{N}}{\sqrt{2}} - \mathrm{i}\frac{\gamma_\mu\alpha V^\mu}{\sqrt{2}} - \frac{\gamma_5\partial\!\!\!/\mathcal{S}\alpha}{\sqrt{2}}, \tag{5.30b}$$

$$\delta\mathcal{M} = \mathrm{i}\sqrt{2}\bar{\alpha}\partial\!\!\!/\psi, \tag{5.30c}$$

$$\delta\mathcal{N} = \sqrt{2}\bar{\alpha}\partial\!\!\!/\gamma_5\psi, \tag{5.30d}$$

$$\delta V^\mu = \sqrt{2}\bar{\alpha}\partial^\mu\psi. \tag{5.30e}$$

These can then be written as,

$$\delta\left[\frac{\partial^\mu \mathcal{S} \mp \mathrm{i}V^\mu}{\sqrt{2}}\right] = \mp 2\mathrm{i}\bar{\alpha}\partial^\mu \psi_{\substack{\mathrm{L}\\\mathrm{R}}}, \tag{5.31a}$$

$$\delta\psi_{\substack{\mathrm{L}\\\mathrm{R}}} = -\frac{\mathcal{M} \mp \mathrm{i}\mathcal{N}}{\sqrt{2}}\alpha_{\substack{\mathrm{L}\\\mathrm{R}}} \pm \frac{\partial^\mu \mathcal{S} \mp \mathrm{i}V^\mu}{\sqrt{2}}\gamma_\mu \alpha_{\substack{\mathrm{R}\\\mathrm{L}}}, \tag{5.31b}$$

$$\delta\left[\frac{\mathcal{M} \mp \mathrm{i}\mathcal{N}}{\sqrt{2}}\right] = 2\mathrm{i}\bar{\alpha}\not{\partial}\psi_{\substack{\mathrm{L}\\\mathrm{R}}}. \tag{5.31c}$$

We then see that the fields

$$\frac{(\partial^\mu \mathcal{S} - \mathrm{i}V^\mu)}{\sqrt{2}}, \quad \psi_\mathrm{L}, \quad \frac{\mathcal{M} - \mathrm{i}\mathcal{N}}{\sqrt{2}} \tag{5.32}$$

transform into one another, as does the set

$$\frac{(\partial^\mu \mathcal{S} + \mathrm{i}V^\mu)}{\sqrt{2}}, \quad \psi_\mathrm{R}, \quad \frac{\mathcal{M} + \mathrm{i}\mathcal{N}}{\sqrt{2}}. \tag{5.33}$$

Let us recapitulate what we have accomplished. Starting with a scalar multiplet, by choosing $\lambda = \mathcal{D} = 0$ and $V_\mu = \partial_\mu \zeta$, we have *reduced* the original multiplet into two multiplets such that the component fields of each multiplet transform only among themselves. If the superfield $\hat{\Phi}$ that we started with was real, then these two reduced multiplets are conjugates of one another. If, however, we had started with a complex field $\hat{\Phi}$, the two multiplets are unrelated. A superfield transforming as the set (5.32) is called a left-chiral superfield, while one transforming as the set (5.33) is called a right-chiral superfield. We trust that it is clear that our reduction procedure is conceptually identical to the example of reducing the second rank co-ordinate tensor into its scalar and the traceless symmetric and antisymmetric parts, discussed in the last exercise.

Finally, let us recover the field content of the Wess–Zumino model. We can reduce a complex superfield $\hat{\Phi}$ as described above, and set all the components in the set (5.33) to zero, consistent with SUSY transformations. In other words, we can choose $\psi_\mathrm{R} = 0$, $V^\mu = \mathrm{i}\partial^\mu \mathcal{S}$ and let $\mathcal{N} = \mathrm{i}\mathcal{M} \equiv \mathrm{i}\mathcal{F}$. Then, the field content of our model will be a complex spin zero field $\mathcal{S}$, ψ_L (or equivalently, a four-component Majorana spinor ψ whose right-handed components are chosen to make it Majorana) and a complex field $\mathcal{F}$. Making the appropriate substitutions in (5.4), we obtain the expansion of a left-chiral scalar superfield,

$$\begin{aligned}\hat{\mathcal{S}}_\mathrm{L} = \mathcal{S} + \mathrm{i}\sqrt{2}\bar{\theta}\psi_\mathrm{L} + \mathrm{i}\bar{\theta}\theta_\mathrm{L}\mathcal{F} + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S} \\ - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta \cdot \bar{\theta}\not{\partial}\psi_\mathrm{L} + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\mathcal{S}.\end{aligned} \tag{5.34}$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

The transformation laws for the component fields are then

$$\delta\mathcal{S} = -\mathrm{i}\sqrt{2}\bar{\alpha}\psi_{\mathrm{L}}, \tag{5.35a}$$
$$\delta\psi_{\mathrm{L}} = -\sqrt{2}\mathcal{F}\alpha_{\mathrm{L}} + \sqrt{2}\not{\partial}\mathcal{S}\alpha_{\mathrm{R}}, \tag{5.35b}$$
$$\delta\mathcal{F} = \mathrm{i}\sqrt{2}\bar{\alpha}\not{\partial}\psi_{\mathrm{L}}, \tag{5.35c}$$

which is exactly the same as in Eq. (3.16a)–(3.16c). Throughout the remainder of this book, we will reserve $\mathcal{S}$, ψ, and $\mathcal{F}$ to denote components of chiral superfields.

Exercise *Convince yourself that the components of the left-chiral superfield form an irreducible multiplet. In other words, show that it is not possible to set any of the components (or combinations thereof) to zero.*

Exercise *In our reduction of the general superfield to the left-chiral scalar superfield, we took $V_\mu = \partial_\mu\zeta$, and $\lambda = \mathcal{D} = 0$. Show that any attempt to reduce the system by setting $V^\mu = 0$ with $\mathrm{i}\gamma_\mu\lambda + \sqrt{2}\partial^\mu\psi = 0$, etc. collapses the system of equations.*

5.6.2 Right-chiral scalar superfields

In order to obtain a right-chiral scalar superfield, we set $\psi_{\mathrm{L}} = 0$, $V^\mu = -\mathrm{i}\partial^\mu\mathcal{S}$ and $\mathcal{N} = -\mathrm{i}\mathcal{M} \equiv \mathcal{F}$ in (5.4) so that

$$\begin{aligned}\hat{\mathcal{S}}_{\mathrm{R}} = {} & \mathcal{S} - \mathrm{i}\sqrt{2}\bar{\theta}\psi_{\mathrm{R}} - \mathrm{i}\bar{\theta}\theta_{\mathrm{R}}\mathcal{F} - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S} \\ & - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\not{\partial}\psi_{\mathrm{R}} + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\mathcal{S}.\end{aligned} \tag{5.36}$$

We note that the field

$$\begin{aligned}\hat{\mathcal{S}}_{\mathrm{L}}^\dagger = {} & \mathcal{S}^\dagger - \mathrm{i}\sqrt{2}\bar{\psi}\theta_{\mathrm{R}} - \mathrm{i}\bar{\theta}\theta_{\mathrm{R}}\mathcal{F}^\dagger - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}^\dagger \\ & - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\not{\partial}\psi_{\mathrm{R}} + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\mathcal{S}^\dagger\end{aligned} \tag{5.37}$$

has the form of a right-chiral scalar superfield.

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

5.6.3 The curl superfield

Let us define the field strength tensor field $F^{\mu\nu} \equiv \partial^\mu V^\nu - \partial^\nu V^\mu$. It is then straightforward to check that,

$$\delta F^{\mu\nu} = -\mathrm{i}\bar{\alpha}[\gamma^\nu\partial^\mu - \gamma^\mu\partial^\nu]\lambda, \tag{5.38a}$$

$$\delta\lambda = -\mathrm{i}\gamma_5\alpha\mathcal{D} + \frac{1}{4}[\gamma_\nu, \gamma_\mu]F^{\mu\nu}\alpha, \quad \text{and} \tag{5.38b}$$

$$\delta\mathcal{D} = \bar{\alpha}\partial\!\!\!/\gamma_5\lambda, \tag{5.38c}$$

so that the components $F^{\mu\nu}$, λ, and $\mathcal{D}$ transform into each other. Nevertheless, it is not possible to choose $\mathcal{S}$, ψ, $\mathcal{M}$, and $\mathcal{N}$ all equal to zero, since (because of (5.29b) this choice is not invariant under a SUSY transformation. In a gauge theory, however, which is where we will have need for the curl superfield, there is more freedom because of gauge invariance. We will see in the next chapter how it is possible to work with a multiplet containing only the $F^{\mu\nu}$, λ, and $\mathcal{D}$ fields. Such a gauge multiplet will be derived from a *real* superfield that contains the gauge potential V^μ.

5.7 Products of superfields

We begin by noting that the expansion,

$$\begin{aligned}\hat{\mathcal{S}}_{\mathrm{L}}(x,\theta) = \mathcal{S}(x) + \mathrm{i}\sqrt{2}\bar{\theta}\psi_{\mathrm{L}}(x) - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)\mathcal{F} + \frac{\mathrm{i}}{2}(\bar{\theta}\theta)\mathcal{F} + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}(x)\\ -\frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\partial\!\!\!/\psi_{\mathrm{L}}(x) + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\mathcal{S}(x),\end{aligned} \tag{5.39}$$

for a left-chiral scalar superfield can be succinctly written in terms of a new variable $\hat{x}_\mu = x_\mu + \frac{\mathrm{i}}{2}\bar{\theta}\gamma_5\gamma_\mu\theta$ as,

$$\hat{\mathcal{S}}_{\mathrm{L}}(x,\theta) = \mathcal{S}(\hat{x}) + \mathrm{i}\sqrt{2}\bar{\theta}\psi_{\mathrm{L}}(\hat{x}) + \mathrm{i}\bar{\theta}\theta_{\mathrm{L}}\mathcal{F}(\hat{x}). \tag{5.40}$$

To see this, we can expand each of the fields in (5.40) as power series around $\hat{x} \simeq x$. Since any term can contain at most two θs and two $\bar{\theta}$s, this expansion must terminate. We can thus write $\mathcal{S}(\hat{x})$ as

$$\begin{aligned}\mathcal{S}(\hat{x}) &= \mathcal{S}(x) + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}(x) + \frac{1}{2!}(\frac{\mathrm{i}}{2})^2(\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta)\partial^\mu\partial^\nu\mathcal{S}(x)\\ &= \mathcal{S}(x) + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}(x) + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\mathcal{S}\end{aligned} \tag{5.41}$$

where we have used identity (5.24e) to obtain the last term. Likewise, using (5.23d) we have

$$\begin{aligned}\bar{\theta}\psi_{\mathrm{L}}(\hat{x}) &= \bar{\theta}\left[\psi_{\mathrm{L}}(x) + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\psi_{\mathrm{L}}(x)\right] \\ &= \bar{\theta}\psi_{\mathrm{L}}(x) + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)\bar{\theta}\not{\partial}\psi_{\mathrm{L}}.\end{aligned} \tag{5.42}$$

Finally, from (5.24b) and (5.24c) we have

$$\bar{\theta}\theta\mathcal{F}(\hat{x}) = \bar{\theta}\theta\mathcal{F}(x) \quad \text{and} \quad \bar{\theta}\gamma_5\theta\mathcal{F}(\hat{x}) = \bar{\theta}\gamma_5\theta\mathcal{F}(x). \tag{5.43}$$

Combining these results, we arrive at Eq. (5.40).

The important point about Eq. (5.40) is that it shows that a left-chiral scalar superfield is a function of just $\hat{x}$ and θ_{L} (recall that $\bar{\theta}_{\mathrm{R}} = \theta_{\mathrm{L}}^T C$). The θ_{R} dependence of $\hat{\mathcal{S}}_{\mathrm{L}}$ enters only via $\hat{x}$. If we take the product of two (or more) left-chiral scalar superfields, it will again be a function of just $\hat{x}$ and θ_{L}, and can be written in the form of Eq. (5.40). We thus conclude that *a product of any number of left-chiral scalar superfields is itself a left-chiral scalar superfield*.

In a similar fashion, a right-chiral scalar superfield $\hat{\mathcal{S}}_{\mathrm{R}}$ can be written as just a function of $\hat{x}^\dagger$ and θ_{R}:

$$\hat{\mathcal{S}}_{\mathrm{R}}(x,\theta) = \mathcal{S}(\hat{x}^\dagger) - \mathrm{i}\sqrt{2}\bar{\theta}\psi_{\mathrm{R}}(\hat{x}^\dagger) - \mathrm{i}\bar{\theta}\theta_{\mathrm{R}}\mathcal{F}(\hat{x}^\dagger). \tag{5.44}$$

This then establishes that the product of two (or more) right-chiral scalar superfields is a right-chiral scalar superfield.

Exercise *By explicit multiplication, or otherwise, convince yourself that the product of a left-chiral superfield with a right-chiral superfield is a general superfield.*

5.8 Supercovariant derivatives

Covariant derivatives are defined so that when these act on any object, they yield a new object with the same transformation properties as the original one. For instance, in gauge theories, unlike the ordinary derivative, the gauge covariant derivative acting on a field whose components transform according to a representation **R** of the gauge group, is a new field with components that transform in the same way.

Since the representation (5.25) of the generator for supersymmetry includes the second term with a θ in it, it is clear that, under SUSY, the components $\partial\hat{\Phi}/\partial\bar{\theta}$ transform differently from those of $\hat{\Phi}$. This is in contrast to spatial derivatives where, because P_μ commutes with the super-charge, the components of $\partial_\mu\hat{\Phi}$ transform the

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

same way as those of $\hat{\Phi}$. Thus ordinary spacetime derivatives are automatically covariant with respect to SUSY transformations.

To facilitate the construction of invariant functions of superfields and their derivatives with respect to θ, we want to define a supersymmetric covariant derivative D so that the components of $D\hat{\Phi}$ transform the same way as the components of $\hat{\Phi}$ under a supersymmetry transformation. We thus require,

$$\left[-\bar{\alpha}\frac{\partial}{\partial\bar{\theta}} - \mathrm{i}\bar{\alpha}\not{\partial}\theta\right] D\hat{\Phi} = D\left[-\bar{\alpha}\frac{\partial}{\partial\bar{\theta}} - \mathrm{i}\bar{\alpha}\not{\partial}\theta\right]\hat{\Phi}. \tag{5.45}$$

We will leave it to the reader to verify that the fermionic derivative operator,

$$D = \frac{\partial}{\partial\bar{\theta}} - \mathrm{i}\not{\partial}\theta, \tag{5.46}$$

anticommutes with $-\frac{\partial}{\partial\bar{\theta}} - \mathrm{i}\not{\partial}\theta$ and satisfies (5.45) because the fermionic parameter α anticommutes with θ.

Exercise *Verify that the expression for D in (5.46) satisfies (5.45).*

For later use, we will *define* a related derivative $\bar{D}$ so that $D = C\bar{D}^T$ so that D satisfies the "Majorana condition". We can readily find the explicit form for $\bar{D}$. Starting with $\bar{D} = D^T C$, we find,

$$\begin{aligned}
\bar{D}_b &= \left[\frac{\partial}{\partial\bar{\theta}_a} - \mathrm{i}(\not{\partial}\theta)_a\right] C_{ab} \\
&= \frac{\partial}{\partial\theta_c}\frac{\partial\theta_c}{\partial\bar{\theta}_a} C_{ab} - \mathrm{i}(\not{\partial}C\bar{\theta}^T)_a C^T_{ba} \\
&= -\frac{\partial}{\partial\theta_b} + \mathrm{i}(\bar{\theta}\not{\partial})_b
\end{aligned}$$

so that

$$\bar{D} = -\frac{\partial}{\partial\theta} + \mathrm{i}\bar{\theta}\not{\partial}. \tag{5.47}$$

We can also *define* left and right SUSY covariant derivatives by acting on D with the projectors P_{L} or P_{R}. To do so, we note that $\bar{\theta}_a = \bar{\theta}_{\mathrm{L}b} P_{\mathrm{R}ba} + \bar{\theta}_{\mathrm{R}b} P_{\mathrm{L}ba}$ immediately gives us,

$$\frac{\partial\bar{\theta}_a}{\partial\bar{\theta}_{\mathrm{L}b}} = P_{\mathrm{R}ba} \quad \text{and} \quad \frac{\partial\bar{\theta}_a}{\partial\bar{\theta}_{\mathrm{R}b}} = P_{\mathrm{L}ba}, \tag{5.48}$$

which implies,

$$\frac{\partial}{\partial\bar{\theta}_{\mathrm{L}a}} = \frac{\partial}{\partial\bar{\theta}_b}\frac{\partial\bar{\theta}_b}{\partial\bar{\theta}_{\mathrm{L}a}} = \frac{\partial}{\partial\bar{\theta}_b} P_{\mathrm{R}ab}. \tag{5.49}$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

Although it should be clear from the context, we clarify that we are taking the derivative with respect to the conjugates of the spinors θ_{L} or θ_{R}. A similar relation holds for $\partial/\partial\bar{\theta}_{\mathrm{R}}$, so that

$$\frac{\partial}{\partial\bar{\theta}_{\mathrm{L}}} = P_{\mathrm{R}}\frac{\partial}{\partial\bar{\theta}} \quad \text{and} \quad \frac{\partial}{\partial\bar{\theta}_{\mathrm{R}}} = P_{\mathrm{L}}\frac{\partial}{\partial\bar{\theta}}. \tag{5.50}$$

We then have

$$D_{\mathrm{L}} \equiv P_{\mathrm{L}} D = \frac{\partial}{\partial\bar{\theta}_{\mathrm{R}}} - \mathrm{i}\not{\partial}\theta_{\mathrm{R}} \tag{5.51a}$$

$$D_{\mathrm{R}} \equiv P_{\mathrm{R}} D = \frac{\partial}{\partial\bar{\theta}_{\mathrm{L}}} - \mathrm{i}\not{\partial}\theta_{\mathrm{L}}, \tag{5.51b}$$

where $D_{\mathrm{L}} + D_{\mathrm{R}} = D$. Finally, let us also *define*,

$$\bar{D}_{\mathrm{R}} \equiv D_{\mathrm{L}}^T C \text{ and } \bar{D}_{\mathrm{L}} \equiv D_{\mathrm{R}}^T C. \tag{5.52a}$$

Clearly, $\bar{D}_{\mathrm{R}} + \bar{D}_{\mathrm{L}} = (D_{\mathrm{L}}^T + D_{\mathrm{R}}^T)C = D^T C = \bar{D}$. Note that once again our definition is consistent with the "Majorana condition" for the spinorial operator D. Notice also that,

$$\bar{D}_{\mathrm{L}} = D_{\mathrm{R}}^T C = D^T P_{\mathrm{R}}^T C = \bar{D} P_{\mathrm{R}} \ , \tag{5.52b}$$

and

$$\bar{D}_{\mathrm{R}} = D_{\mathrm{L}}^T C = D^T P_{\mathrm{L}}^T C = \bar{D} P_{\mathrm{L}} \ . \tag{5.52c}$$

We will leave it to the reader to verify that by steps very similar to those that led us to (5.51a) and (5.51b) we obtain,

$$\bar{D}_{\mathrm{L}} = -\frac{\partial}{\partial\theta_{\mathrm{R}}} + \mathrm{i}\bar{\theta}_{\mathrm{R}}\not{\partial} \ , \tag{5.53a}$$

and

$$\bar{D}_{\mathrm{R}} = -\frac{\partial}{\partial\theta_{\mathrm{L}}} + \mathrm{i}\bar{\theta}_{\mathrm{L}}\not{\partial} \ . \tag{5.53b}$$

As one more exercise in the manipulation of the supercovariant derivative, we establish an identity involving D_{L} and D_{R} to be used in the next chapter. We compute the anticommutation relation

$$\begin{aligned}
\{D_{\mathrm{L}a}, D_{\mathrm{R}b}\} &= \left\{\frac{\partial}{\partial\bar{\theta}_{\mathrm{R}a}} - \mathrm{i}(\not{\partial}\theta_{\mathrm{R}})_a, \frac{\partial}{\partial\bar{\theta}_{\mathrm{L}b}} - \mathrm{i}(\not{\partial}\theta_{\mathrm{L}})_b\right\} \\
&= -\mathrm{i}\left\{(\not{\partial}\theta_{\mathrm{R}})_a, \frac{\partial}{\partial\bar{\theta}_{\mathrm{L}b}}\right\} - \mathrm{i}\left\{\frac{\partial}{\partial\bar{\theta}_{\mathrm{R}a}}, (\not{\partial}\theta_{\mathrm{L}})_b\right\} \\
&= -\mathrm{i}\frac{\partial}{\partial\bar{\theta}_{\mathrm{L}b}}(\not{\partial}\theta_{\mathrm{R}})_a - \mathrm{i}\frac{\partial}{\partial\bar{\theta}_{\mathrm{R}a}}(\not{\partial}\theta_{\mathrm{L}})_b.
\end{aligned}$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

To obtain the last step we can explicitly act on a superfield and, since the θs anticommute, see that just the terms shown survive. Finally, since $\theta_{\mathrm{R}} = C\bar{\theta}_{\mathrm{L}}^T$ and $\theta_{\mathrm{L}} = C\bar{\theta}_{\mathrm{R}}^T$, we have $\partial\theta_{\mathrm{R}b}/\partial\bar{\theta}_{\mathrm{L}a} = C_{ba}$ and $\partial\theta_{\mathrm{L}b}/\partial\bar{\theta}_{\mathrm{R}a} = C_{ba}$, so that

$$\begin{aligned}\{D_{\mathrm{L}a}, D_{\mathrm{R}b}\} &= -\mathrm{i}(\partial\!\!\!/ C)_{ab} - \mathrm{i}(\partial\!\!\!/ C)_{ba} \\ &= -2\mathrm{i}(\partial\!\!\!/ C)_{ab}.\end{aligned} \tag{5.54}$$

Exercise *Similarly show that*

$$\{D_{\mathrm{L}a}, D_{\mathrm{L}b}\} = \{D_{\mathrm{R}a}, D_{\mathrm{R}b}\} = 0.$$

Thus any term with a product of three D_{L}s or three D_{R}s vanishes.

To conclude this section, we show that the action of D_{R} on a left-chiral superfield $\hat{\mathcal{S}}_{\mathrm{L}}(\theta_{\mathrm{L}}, \hat{x})$ gives zero. To evaluate the first term, we note that the $\bar{\theta}_{\mathrm{L}}$ dependence enters only via $\hat{x}$, and we can write,

$$\begin{aligned}\frac{\partial\hat{\mathcal{S}}_{\mathrm{L}}}{\partial\bar{\theta}_{\mathrm{L}}} &= \frac{\partial\hat{\mathcal{S}}_{\mathrm{L}}}{\partial\hat{x}^\mu}\frac{\partial\hat{x}^\mu}{\partial\bar{\theta}_{\mathrm{L}}} = \frac{\partial\hat{\mathcal{S}}_{\mathrm{L}}}{\partial\hat{x}^\mu}\frac{\mathrm{i}}{2}\frac{\partial(\bar{\theta}\gamma_5\gamma^\mu\theta)}{\partial\bar{\theta}_{\mathrm{L}}} \\ &= \frac{\partial\hat{\mathcal{S}}_{\mathrm{L}}}{\partial x^\mu}\cdot\mathrm{i}\gamma^\mu\theta_{\mathrm{L}},\end{aligned}$$

where in the last step we used $\bar{\theta}\gamma_5\gamma_\mu\theta = 2\bar{\theta}_{\mathrm{L}}\gamma_\mu\theta_{\mathrm{L}}$. We thus establish the important property,

$$D_{\mathrm{R}}\hat{\mathcal{S}}_{\mathrm{L}} = \left(\frac{\partial}{\partial\bar{\theta}_{\mathrm{L}}} - \mathrm{i}\partial\!\!\!/\theta_{\mathrm{L}}\right)\hat{\mathcal{S}}_{\mathrm{L}} = 0. \tag{5.55}$$

Working the steps backwards, we see that this is also a sufficient condition for any field to be a left-chiral superfield. The reader can similarly show,

$$D_{\mathrm{L}}\hat{\mathcal{S}}_{\mathrm{R}} = 0. \tag{5.56}$$

We remark that the result of the last exercise in the previous section follows immediately from Eq. (5.55) and (5.56).

5.9 Lagrangians for chiral scalar superfields

Our goal in this section is to present a systematic strategy to construct actions that are invariant under supersymmetric transformations. This means that the variation of the Lagrangian density can at most be a total derivative. In fact, the Lagrangian density can *never* be a SUSY invariant. This follows simply from the SUSY algebra.

By manipulations similar to (3.20) of Chapter 3 we get,

$$(\delta_2\delta_1 - \delta_1\delta_2)\mathcal{L} = -2\bar{\alpha}_2\gamma_\mu\alpha_1\partial_\mu\mathcal{L}.$$

This would, of course, have to vanish if $\mathcal{L}$ were truly a SUSY invariant. We would then be led to conclude that $\mathcal{L}$ is a constant and that the theory has no dynamics. Thus, SUSY transformations always change the Lagrangian density by a (non-vanishing) total derivative.

The first observations toward our goal stem from Eq. (5.29g) and Eq. (5.35c) which show that the D-component (the coefficient of $(\bar{\theta}\gamma_5\theta)^2$) of *any* superfield and the F-component of *chiral* superfields (the coefficient of $\bar{\theta}\theta_L$ of a left-chiral superfield, or the coefficient of $\bar{\theta}\theta_R$ of a right-chiral superfield) transform as a total derivative under a SUSY transformation. This leads us to two important conclusions:

- if we take the product of any number of chiral superfields and their Hermitian conjugates, the D-term of the product superfield will change only by a total derivative under SUSY transformations, and
- if we take the product of only left- (or only right-) chiral superfields, the F-term of the product will also change by just a total derivative. The would-be D-term (i.e. the coefficient of $(\bar{\theta}\gamma_5\theta)^2$) of this product is already a total derivative.

These D- or F-components of the composite (product) superfield are themselves products of the ordinary fields that were the components of the individual superfields in the product. Thus, *these D- and F-terms are candidates for a SUSY Lagrangian.* With just chiral scalar multiplets, we can only obtain a theory with spin 0 and spin $\frac{1}{2}$ fields.

The recipe for obtaining SUSY invariant actions (given a set of N chiral superfields) is now in hand. We start with two functions $K(\hat{\mathcal{S}}_{Li}^\dagger, \hat{\mathcal{S}}_{Lj})$ and $\hat{f}(\hat{\mathcal{S}}_{Li})$ of a set of left-chiral superfields $\hat{\mathcal{S}}_{Li}$, where $i = 1, \ldots, N$. Since $\hat{\mathcal{S}}_{Li}^\dagger$ is a right-chiral superfield, K is a general superfield, while $\hat{f}$ is a left-chiral superfield. Then, the D-term of K and the F-term of $\hat{f}$ are candidates for a SUSY Lagrangian density. The function K is called the Kähler potential and the function $\hat{f}$ is known as the superpotential. We make two clarifying remarks.

- There is no loss of generality in writing the superpotential as a function of just left-chiral superfields because every right-chiral superfield $\hat{\mathcal{S}}_{Rj}$ can, by the analogue of Eq. (5.37), be written as a left-chiral superfield $\left(\hat{\mathcal{S}}_{Rj}\right)^\dagger$.
- The reader may wonder why we do not include the D-term of the superpotential in the Lagrangian. We see from Eq. (5.34) that the coefficient of the $(\bar{\theta}\gamma_5\theta)^2$ term of any left-chiral superfield is itself a total derivative, and so does not contribute to

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

the action. For the same reason, terms in the Kähler potential that do not depend on *both* $\hat{\mathcal{S}}_{\mathrm{L}}$ and $\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}$ would also be irrelevant.

Just as the scalar potential specifies any theory of spin zero and spin half fields in usual field theory, a supersymmetric field theory with chiral superfields is specified by the Kähler potential together with the superpotential. We now compute the Lagrangian density for any SUSY theory with just spin zero and spin half fields in terms of these functions. For simplicity, we will restrict our discussion to theories that are power counting renormalizable.

5.9.1 Kähler potential contributions to the Lagrangian density

We begin with the computation of the Kähler potential contribution to the action. This requires us to compute the coefficient of the $(\bar{\theta}\gamma_5\theta)^2$ term (or the D-term) of the function K. For this reason, this contribution is frequently known as the "D-term contribution" to the Lagrangian density.

Renormalizability imposes stringent restrictions on the form of K, and also as we will see below, on the form of the superpotential. To see this, we have to do some dimensional analysis. We will denote the mass dimension of any quantity X as $[X]$. Since $[P] = 1$, from the SUSY algebra $\{Q, \bar{Q}\} = 2\gamma^{\mu}P_{\mu}$, we must have $[Q] = [\bar{Q}] = 1/2$. (Remember that $Q = C\bar{Q}^T$ implies $[Q] = [\bar{Q}]$.) Then from Eq. (5.25), we obtain $[\theta] = [\bar{\theta}] = -1/2$.

If, in our expansion (5.34) of the chiral superfield, we now choose the scalar field $\mathcal{S}$ to have the canonical dimension $[\mathcal{S}] = 1$, then $[\psi] = 3/2$ and $[\mathcal{F}] = 2$, just as for the Wess–Zumino model. Indeed the left-chiral superfield $\hat{\mathcal{S}}_{\mathrm{L}}$ can be assigned $[\hat{\mathcal{S}}_{\mathrm{L}}] = 1$. Since $[(\bar{\theta}\gamma_5\theta)^2] = -2$, then

$$[K_{D\text{-term}}] = [K] + 2. \tag{5.57}$$

If this D-term is to represent a *renormalizable* Lagrangian, then $[K_{D\text{-term}}] \leq 4$, so that

$$[K] \leq 2 \quad \text{(renormalizable theory)} \tag{5.58}$$

and the Kähler potential is at most a quadratic polynomial of $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{\dagger}$. In non-renormalizable theories (such as supergravity), higher powers may be present. In fact, then K need not even be a polynomial.

As already noted, chiral superfields have only gradient D-terms, so there is no point writing linear terms (or for that matter terms involving just $\hat{\mathcal{S}}$ or just $\hat{\mathcal{S}}^{\dagger}$) in the Kähler potential. In a renormalizable theory, since cubic and higher terms are not allowed in K, the most general form of K is a real function (to ensure the

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

Hermiticity of the Lagrangian density)

$$K = \sum_{i,j=1}^{N} A_{ij}\hat{\mathcal{S}}_i^{\dagger}\hat{\mathcal{S}}_j. \tag{5.59}$$

Without loss of generality, we can choose a basis so that A_{ij} is diagonal, and the fields $\hat{\mathcal{S}}_i$ can be normalized so that the $A_{ii} = 1$. Then

$$K[\hat{\mathcal{S}}^{\dagger}, \hat{\mathcal{S}}] = \sum_{i=1}^{N} \hat{\mathcal{S}}_i^{\dagger}\hat{\mathcal{S}}_i \tag{5.60}$$

is a general choice for K in a renormalizable theory.

Exercise *For the curl superfield, show that if we choose* $[V^{\mu}] = 1$*, we would have* $[\lambda] = 3/2$ *and* $[D] = 2$*. Notice that unlike the chiral supermultiplet, the mass dimension of the curl superfield vanishes, so that renormalizability considerations do not restrict the power of this multiplet in the Kähler potential. It cannot, of course, enter the superpotential since it is not a chiral superfield. We will exploit this in the next chapter when we discuss supersymmetric gauge theories.*

We now have only to compute the coefficient of the $(\bar{\theta}\gamma_5\theta)^2$ term in the product $\hat{\mathcal{S}}_i^{\dagger}\hat{\mathcal{S}}_i$. In so doing, we need only keep terms with four θ's or $\bar{\theta}$'s in any combination. Multiplying the expansion (5.37) by (5.34), four sets of terms will arise.

The first set of terms is

- $\frac{1}{8}\mathcal{S}^{\dagger}\Box\mathcal{S}(\bar{\theta}\gamma_5\theta)^2 + \frac{1}{8}\Box\mathcal{S}^{\dagger}\mathcal{S}(\bar{\theta}\gamma_5\theta)^2 + \frac{1}{4}(\bar{\theta}\gamma_5\gamma_{\mu}\theta)(\bar{\theta}\gamma_5\gamma_{\nu}\theta)\partial^{\mu}\mathcal{S}^{\dagger}\partial^{\nu}\mathcal{S}.$

We integrate by parts on each of the first two terms above, and discard the surface terms. For the third term, apply identity (5.24e). The result is that

$$\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}\hat{\mathcal{S}}_{\mathrm{L}} \ni -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2\partial_{\mu}\mathcal{S}^{\dagger}\partial^{\mu}\mathcal{S}. \tag{5.61}$$

The second set of terms is

- $\mathrm{i}\bar{\psi}\theta_{\mathrm{R}}(\bar{\theta}\gamma_5\theta)\bar{\theta}\partial\!\!\!/\psi_{\mathrm{L}} - \mathrm{i}(\bar{\theta}\gamma_5\theta)\bar{\theta}\partial\!\!\!/\psi_{\mathrm{R}}\bar{\theta}\psi_{\mathrm{L}}.$

In the first term, re-write $\bar{\psi}\theta_{\mathrm{R}} \to \bar{\psi}_{\mathrm{L}}\theta$ and apply the Fierz re-arrangement identity (5.21) to the $\theta\bar{\theta}$ product. The second and third terms of (5.21) lead to vanishing contributions via the identities (5.24a) and (5.24b), respectively, while the second term of (5.21) leads to a contribution $-\frac{\mathrm{i}}{4}(\bar{\theta}\gamma_5\theta)^2\bar{\psi}_{\mathrm{L}}\partial\!\!\!/\psi_{\mathrm{L}}$. Similarly, the second term of • above leads to a contribution $-\frac{\mathrm{i}}{4}(\bar{\theta}\gamma_5\theta)^2\bar{\psi}_{\mathrm{R}}\partial\!\!\!/\psi_{\mathrm{R}}$. Since $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}$ is Majorana, we find

$$\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}\hat{\mathcal{S}}_{\mathrm{L}} \ni -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2\frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi. \tag{5.62}$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

The third set of terms consists of

$$\bullet\ \bar{\theta}\theta_{\mathrm{R}}\bar{\theta}\theta_{\mathrm{L}}\mathcal{F}^{\dagger}\mathcal{F}.$$

By expanding the P_{L} and P_{R} projection operators and using (5.24a) and (5.24d), we find

$$\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}\hat{\mathcal{S}}_{\mathrm{L}} \ni -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2\mathcal{F}^{\dagger}\mathcal{F}. \tag{5.63}$$

A fourth set of terms

$$\bullet\ \frac{1}{2}\bar{\theta}\theta_{\mathrm{R}}(\bar{\theta}\gamma_5\gamma_\mu\theta)\mathcal{F}^{\dagger}\partial^{\mu}\mathcal{S} + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\bar{\theta}\theta_{\mathrm{L}}\partial_{\mu}\mathcal{S}^{\dagger}\mathcal{F}$$

will identically vanish due to identities (5.24b) and (5.24c).

Putting all the pieces together, we find

$$\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}\hat{\mathcal{S}}_{\mathrm{L}} \ni -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2\{\partial_{\mu}\mathcal{S}^{\dagger}\partial^{\mu}\mathcal{S} + \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi + \mathcal{F}^{\dagger}\mathcal{F}\}. \tag{5.64}$$

We will *define* the D-term to be the coefficient of the $-\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2$ term in the product $\hat{\mathcal{S}}_{\mathrm{L}}^{\dagger}\hat{\mathcal{S}}_{\mathrm{L}}$ since this gives us the canonically normalized kinetic energy terms for the scalar field $\mathcal{S}$ and the Majorana spinor field ψ. The D-term contribution to the Lagrangian density for a single chiral scalar superfield is thus

$$\mathcal{L}_D = \partial_{\mu}\mathcal{S}^{\dagger}\partial^{\mu}\mathcal{S} + \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi + \mathcal{F}^{\dagger}\mathcal{F}. \tag{5.65}$$

The field $\mathcal{F}$ enters without any derivative. It turns out to be an auxiliary field that satisfies an *algebraic* equation of motion.

5.9.2 Superpotential contributions to the Lagrangian density

We now turn to the computation of the superpotential contributions to the Lagrangian density. This is proportional to the coefficient of the $\bar{\theta}\theta_{\mathrm{L}}$, or the F-term, of the superpotential function. These contributions are therefore frequently referred to as F-term contributions. Dimensional analysis tells us that the F-term of the superpotential $\hat{f}$ has dimensions $[\hat{f}] - 1$. In a renormalizable theory, therefore, the superpotential is at most a cubic polynomial in $\hat{\mathcal{S}}_i$.

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

We can formally write any superpotential as a power series about $\hat{\mathcal{S}} = \mathcal{S}$ as,

$$\begin{aligned}\hat{f}(\hat{\mathcal{S}}) = \hat{f}(\hat{\mathcal{S}} = \mathcal{S}) &+ \sum_i \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i \\ &+ \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i (\hat{\mathcal{S}} - \mathcal{S})_j \\ &+ \frac{1}{3!}\sum_{ijk} \left.\frac{\partial^3 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j \partial \hat{\mathcal{S}}_k}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i (\hat{\mathcal{S}} - \mathcal{S})_j (\hat{\mathcal{S}} - \mathcal{S})_k \\ &+ \cdots \end{aligned} \tag{5.66}$$

Here, $\hat{\mathcal{S}} = \mathcal{S}$ means that after the derivative is evaluated, each superfield is set to be the scalar component so that these "derivative coefficients" are functions of just the scalar fields. The terms $(\hat{\mathcal{S}} - \mathcal{S})_i$ are at least linear in θ so that there can be at most four factors of this type because any product of five θs and $\bar{\theta}$s vanishes. In fact, because the superpotential is a function of only left-chiral superfields, even the product of four factors vanishes, so that there really are no terms represented by the ellipsis in the expansion above.

Let us now isolate the potential sources of the $\bar{\theta}\theta_{\mathrm{L}}$ terms in $\hat{f}(\hat{\mathcal{S}})$ whose coefficient is the item of interest to us. We see that:

1. the first term in the expansion will not contribute since it has no θs,
2. the last terms cannot contribute since they all contain at least three θs,
3. the $\sum_i \partial \hat{f}/\partial \hat{\mathcal{S}}_i|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i$ term contributes with the $\bar{\theta}\theta_{\mathrm{L}}$ coefficient from $\hat{\mathcal{S}} - \mathcal{S}$, and
4. the $\frac{1}{2}\sum_{ij} \partial^2 \hat{f}/\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i (\hat{\mathcal{S}} - \mathcal{S})_j$ term contributes when $(\hat{\mathcal{S}} - \mathcal{S})_i$ and $(\hat{\mathcal{S}} - \mathcal{S})_j$ each contribute a term linear in θ.

The form of the term from item 3 above is easy to write down; using (5.34) it is just

$$\left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i = \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \left(\mathrm{i}\mathcal{F}_i \bar{\theta}\theta_{\mathrm{L}}\right). \tag{5.67}$$

The term from item 4 can be written as

$$\frac{1}{2}\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\hat{\mathcal{S}} - \mathcal{S})_i (\hat{\mathcal{S}} - \mathcal{S})_j = \frac{1}{2}\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} (\mathrm{i}\sqrt{2}\bar{\psi}_i P_{\mathrm{L}}\theta)(\mathrm{i}\sqrt{2}\bar{\theta}\psi_{j\mathrm{L}})$$

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

$$= \frac{1}{4} \left. \frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{L}} \left[\bar{\theta}\theta \mathbf{1} + \bar{\theta}\gamma_5\theta \cdot \gamma_5 - \bar{\theta}\gamma_\mu\gamma_5\theta \cdot \gamma^\mu \gamma_5 \right] P_{\mathrm{L}} \psi_j$$

$$= \frac{1}{2} \left. \frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\theta}\theta_{\mathrm{L}} \bar{\psi}_i P_{\mathrm{L}} \psi_j, \tag{5.68}$$

where we have used identity (5.21).

The coefficient of $\bar{\theta}\theta_{\mathrm{L}}$ in $\hat{f}(\hat{\mathcal{S}})$ is thus

$$\mathrm{i} \left. \frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i + \frac{1}{2} \left. \frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{L}} \psi_j, \tag{5.69}$$

where a sum over the various fields is implied. This term is not Hermitian, since $\hat{f}$ is intrinsically complex. However, we note that the F-term of the right-chiral superfield $[\hat{f}(\hat{\mathcal{S}})]^\dagger$ which also leads to a SUSY-invariant action gives just the Hermitian conjugate of the expression (5.69). We will add this to obtain a Hermitian Lagrangian density.

In defining the F-terms, we will actually take the coefficient of $-\bar{\theta}\theta_{\mathrm{L}}$ as the choice for a Lagrangian. This is purely conventional. We will choose the size of the terms in the superpotential to give mass and interaction terms with usual normalizations in the Lagrangian density. Thus,

$$\begin{aligned}\mathcal{L}_F = &-\mathrm{i} \sum_i \left. \frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i - \frac{1}{2} \sum_{i,j} \left. \frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{L}} \psi_j \\ &+ \mathrm{i} \sum_i \left. \left(\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} \right)^\dagger \right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i^\dagger - \frac{1}{2} \sum_{i,j} \left. \left(\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j} \right)^\dagger \right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{R}} \psi_j .\end{aligned} \tag{5.70}$$

We remark that nowhere in our derivation of (5.70) did we need to assume the dimensionality of the superpotential.

5.9.3 A technical aside

The careful reader may have noticed that we did not allow the superpotential to contain terms involving supercovariant derivatives of the superfield. This is because the supercovariant derivative of a chiral superfield is *not*, in general, a chiral superfield. However, by the exercise immediately following (5.54), we see that the product of any three right (or left) supercovariant derivatives vanishes. Hence, even for a general superfield $\hat{\Phi}$, $\bar{D}_{\mathrm{L}} D_{\mathrm{R}} \hat{\Phi}$ must be a left-chiral superfield (since D_{R} acting on this vanishes). This raises the question whether such terms (or functions thereof)

https://doi.org/10.1017/9781009289801.006 Published online by Cambridge University Press

may be included in the superpotential of a more general theory not involving just chiral superfields.

First, we note that up to total derivatives, this term is just $-\partial^2\hat{\Phi}/\partial\theta_{\mathrm{R}}\partial\bar{\theta}_{\mathrm{L}}$ so that it just removes one θ_{R} and one $\bar{\theta}_{\mathrm{L}}$ from the general expansion (5.4) of $\hat{\Phi}$. Aside from total derivatives, this then leaves only terms with $\mathcal{M}$ or $\mathcal{N}$ (with no θ or $\bar{\theta}$), a term with λ (with one $\bar{\theta}$) and a term with $\mathcal{D}$ (with a $\bar{\theta}\gamma_5\theta$). Up to total derivatives, the F-term (which is proportional to the coefficient of $\bar{\theta}\theta_{\mathrm{L}}$) of $\bar{D}_{\mathrm{L}}D_{\mathrm{R}}\hat{\Phi}$ is then just a multiple of the $\mathcal{D}$ component of $\hat{\Phi}$ and would be included in our general list of contributions from the Kähler potential.

Next, the reader may worry about terms like $\hat{\mathcal{S}}_{\mathrm{L}}\bar{D}_{\mathrm{L}}D_{\mathrm{R}}\hat{\Phi}$ since this is also a left-chiral superfield. However, since $D_{\mathrm{R}}\hat{\mathcal{S}} = \bar{D}_{\mathrm{L}}\hat{\mathcal{S}} = 0$, this can be written as $\bar{D}_{\mathrm{L}}D_{\mathrm{R}}(\hat{\mathcal{S}}\hat{\Phi})$ which we just argued that we do not need to include. Powers of $\bar{D}_{\mathrm{L}}D_{\mathrm{R}}\hat{\Phi}$ are just a special case of this. We thus see that there is no loss of generality in not including supercovariant derivatives of superfields in the superpotential *as long as we allow for a general Kähler potential* (which can include terms involving these derivatives). In a renormalizable theory, however, the choice of Kähler potential is greatly restricted as we have already noted. Finally, we remark that our analysis above shows that certain F-terms (which lead to non-renormalizable interactions in four dimensions) can be rewritten as D-terms.

5.9.4 A master Lagrangian for chiral scalar superfields

We can now combine the D- and F-term Lagrangian candidates above to arrive at the general Lagrangian for renormalizable theories involving only chiral scalar superfields:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_D + \mathcal{L}_F \\
&= \sum_i \left[\partial_\mu \mathcal{S}_i^\dagger \partial^\mu \mathcal{S}_i + \frac{\mathrm{i}}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + \mathcal{F}_i^\dagger \mathcal{F}_i \right] \\
&\quad - \mathrm{i}\sum_i \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i - \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{L}} \psi_j \\
&\quad + \mathrm{i}\sum_i \left.\left(\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right)^\dagger\right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i^\dagger - \frac{1}{2}\sum_{i,j} \left.\left(\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right)^\dagger\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i P_{\mathrm{R}} \psi_j .
\end{aligned} \quad (5.71)$$

We see that while the fields $\mathcal{S}_i$ and ψ_i have conventional kinetic energy terms, the fields $\mathcal{F}_i$ have no kinetic energy term, and so are not dynamical fields.

At this stage we eliminate these auxiliary fields from the Lagrangian by using their (algebraic) Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_i^\dagger} = 0 \Rightarrow \mathcal{F}_i + \mathrm{i}\left(\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right)^\dagger = 0, \tag{5.72a}$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}_i} = 0 \Rightarrow \mathcal{F}_i^\dagger - \mathrm{i}\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} = 0. \tag{5.72b}$$

We thus obtain the general supersymmetric Lagrangian for theories with just scalars and spinors to be

$$\begin{aligned}\mathcal{L} = {} & \sum_i (\partial_\mu \mathcal{S}_i)^\dagger (\partial^\mu \mathcal{S}_i) + \frac{\mathrm{i}}{2}\sum_i \bar{\psi}_i \partial\!\!\!/ \psi_i - \sum_i \left|\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i}\right|^2_{\hat{\mathcal{S}}=\mathcal{S}} \\ & - \frac{1}{2}\sum_{i,j}\left[\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i \frac{1-\gamma_5}{2}\psi_j + \quad \mathrm{h.c.}\right].\end{aligned} \tag{5.73}$$

The third term yields the scalar potential (which is quartic if the superpotential is cubic). The masses and Yukawa interactions of fermions are all included in the last term. The model dependence of the theory enters via the choice of the superpotential which can be an arbitrary function (at most a cubic polynomial for renormalizable theories) of left-chiral superfields, but not their Hermitian conjugates.

Exercise (Recovering the Wess–Zumino model) *To recover the Wess–Zumino model, complete with interactions, create a theory with a single left-chiral scalar superfield* $\hat{\mathcal{S}}_\mathrm{L} \ni (\mathcal{S},\ \psi_\mathrm{L},\ \mathcal{F})$. *Let* $\mathcal{S} = \frac{A+\mathrm{i}B}{\sqrt{2}}$ *and* $\mathcal{F} = \frac{F+\mathrm{i}G}{\sqrt{2}}$, *where* A, B, F, *and* G *are real scalar fields. Assume a superpotential of the form* $\hat{f} = \frac{1}{2}m\hat{\mathcal{S}}^2 + \frac{1}{3}g\hat{\mathcal{S}}^3$. *Recover the Lagrangian terms given in Eq. (3.1b), (3.1c), and (3.43). This exercise completes the proof that the WZ model interaction terms Eq. (3.43) are, in fact, supersymmetric.*

5.10 The action as an integral over superspace

Supersymmetric actions are commonly expressed as integrals over superspace. To understand how this is accomplished, we must first define integration over Grassmann numbers. Consider the integral over the entire range of η of a function $f(\eta)$ of a single Grassmann variable η:

$$\int f(\eta)\mathrm{d}\eta = \int (A + B\eta)\mathrm{d}\eta,$$

where we have expanded f as a power series in η. Following Berezin,[3] we *define*

$$\int \mathrm{d}\eta = 0 \tag{5.74a}$$

$$\int \mathrm{d}\eta \cdot \eta = 1. \tag{5.74b}$$

Notice that (5.74b) implies that the dimension of η is the negative of that of $\mathrm{d}\eta$ – hence $\mathrm{d}\eta$ should not be thought of as an increment of η. This then gives

$$\int f(\eta)\mathrm{d}\eta = \int (A + B\eta)\mathrm{d}\eta = B.$$

Exercise *Verify that with this definition, Berezin integration is a linear operation, i.e. that* $\int \mathrm{d}\eta[af(\eta) + bg(\eta)] = a\int \mathrm{d}\eta f(\eta) + b\int \mathrm{d}\eta g(\eta)$*, where a and b are (commuting) constants and f and g are functions. Show also that the integral* $\int \mathrm{d}\eta f(\eta)$ *over the entire range of* η *is invariant under finite shifts* $\eta \to \eta + \eta'$ *of the integration variable by a Grassmann-valued constant.*

For integrals over several Grassmann variables, there is a sign ambiguity. We define the integral over several variables by requiring that the variable to be integrated first be moved to the extreme left: we thus have,

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \cdot \eta_1\eta_2 = -\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \cdot \eta_2\eta_1 = -1. \tag{5.75}$$

We are now ready to see how to write the D- and F-term contributions to the action as integrals over superspace. The D-term contribution to the Lagrangian density was defined as the coefficient of $-\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2$ when the Kähler potential is expanded in the canonical form. Since $(\bar{\theta}\gamma_5\theta)^2$ is quartic in the θs, it must be proportional to $\theta_1\theta_2\theta_3\theta_4$. Plugging in an explicit representation for the Dirac γ matrices shows that $(\bar{\theta}\gamma_5\theta)^2 = 8\,\theta_4\theta_3\theta_2\theta_1$. A look at (5.4) then tells us that

$$\int \mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3\mathrm{d}\theta_4 K\left(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}\right) \equiv \int \mathrm{d}^4\theta K\left(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}\right)$$

equals 8 times the coefficient of $(\bar{\theta}\gamma_5\theta)^2$ in the expansion of K. Since we have *defined* the D-term as the coefficient of $-\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2$, we can write the D-term part of the action as,

$$\int \mathrm{d}^4x\mathcal{L}_D = -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^4\theta K\left(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}\right). \tag{5.76}$$

[3] See *The Method of Second Quantization,* F. A. Berezin, Academic Press (1966).

To see how to write the F-term action as a superspace integral, it is most straightforward to work in the chiral representation where, as in (4.9), the upper (lower) components of the spinor correspond to the two left-(right-)chiral components; i.e. $(\theta_1, \theta_2, \theta_3, \theta_4) = (\theta_{L1}, \theta_{L2}, \theta_{R1}, \theta_{R2})$. It is then easy to check that $\bar{\theta}\theta_L = 2\theta_{L2}\theta_{L1}$. From the form (5.34) for the expansion of an (elementary or composite) left-chiral superfield, we see that

$$\int d\theta_{L1} d\theta_{L2} \hat{f}(\hat{\mathcal{S}}_L) \equiv \int d^2\theta_L \hat{f}(\hat{\mathcal{S}}_L)$$

is exactly twice the coefficient of $\bar{\theta}\theta_L$ in the expansion of the superpotential. Since the F-term contribution to the Lagrangian density was *defined* to be the coefficient of $-\bar{\theta}\theta_L$ in this expansion, we see that the F-term part of the action can be expressed as

$$\int d^4x \mathcal{L}_F = -\frac{1}{2}\left[\int d^4x d^2\theta_L \hat{f}(\hat{\mathcal{S}}) + \quad \text{h.c.}\right]. \tag{5.77}$$

While the F-term of left-chiral superfields involves integration over just the two Grassmann co-ordinates θ_{L1} and θ_{L2}, the D-term involves an integration over θ_{R1} and θ_{R2} as well. In the literature, it is instead common to see integrations over $d^2\theta$ and $d^2\bar{\theta}$, where θ and $\bar{\theta}$ are two-component spinors. Eq. (4.9) provides the connection. The two undotted components in (4.9) of the Majorana spinor θ (i.e. the two components of our θ_L) are frequently denoted by θ_i while the two dotted components are denoted by $\bar{\theta}^i$ $(i = 1, 2)$. The analogue of our integration over the two θ_L (θ_R) co-ordinates is then integration over the two components of θ $(\bar{\theta})$.