Proceedings of the Edinburgh Mathematical Society (1987) **30**, 397–400 ©

STRONGLY EXPOSED POINTS AND A CHARACTERIZATION OF l_1 (Γ) BY THE SCHUR PROPERTY

by IOANNIS A. POLYRAKIS

(Received 27 January 1986, Revised 5 June 1986)

1. Introduction

In this paper we study the existence of strongly exposed points in unbounded closed and convex subsets of the positive cone of ordered Banach spaces and we prove the following characterization for the space $l_1(\Gamma)$: A Banach lattice X is order-isomorphic to $l_1(\Gamma)$ iff X has the Schur property and X^* has quasi-interior positive elements.

2. Notation

Let X be an ordered normed space. For each $A \subseteq X$ we shall denote by sep(A) the set of strongly exposed points of A. The positive cone X_+ of X is normal if there exists $a \in \mathbb{R}_+$ such that for each $x, y \in X, x \geqq y \geqq 0$ implies $a\|x\| \geqq \|y\|$. A point $x \in X$ is a quasi-interior positive element of $X, x \in Q(X)$, if the principal ideal I_x of X is (norm) dense in X. For notions not defined here see [**1**] and [**5**].

3. Strongly exposed points

Let K be an unbounded, closed and convex subset of a normed space X. For each $\rho \in \mathbb{R}_+$ we define the following subsets of K:

$$K_\rho = \{x \in K \mid \|x\| \leqq \rho\}, K_{S,\rho} = \{x \in K \mid \|x\| = \rho\}$$

whenever these sets are non-empty.

In [**3**, Prop. 3] it is proved that if X is a Banach space with Radon–Nikodým Property (R.N.P.) and K is an unbounded, closed and convex subset of X, then sep(K) $= \emptyset$ iff $\overline{co}K_{S,\rho} = K_\rho$, $\forall \rho \in \mathbb{R}_+$. If we assume that the set K has the R.N.P. we obtain the following analogous proposition:

Proposition 3.1. *Let K be an unbounded, closed and convex subset of a Banach space X and K have the* R.N.P. *Then* sep$(K) = \emptyset$ iff $K_\rho = \overline{co}K_{S,\rho}, \forall \rho \in \mathbb{R}_+$.

Proof. If sep$(K) = \emptyset$ then sep$(K_\rho) \subseteq K_{S,\rho} \forall \rho \in \mathbb{R}_+$. Since $K_\rho = \overline{co}\,\text{sep}(K_\rho)$ and $\overline{co}\,\text{sep}(K_\rho) \subseteq \overline{co}K_{S,\rho} \subseteq K_\rho$ we have that $K_\rho = \overline{co}K_{S,\rho} \forall \rho \in \mathbb{R}_+$. If $K_\rho = \overline{co}K_{S,\rho} \forall \rho \in \mathbb{R}_+$, then K is not dentable, [**4**, Prop. 2] and therefore sep$(K) = \emptyset$.

Theorem 3.1 *Let K be an unbounded, closed and convex subset of the positive cone X_+ of an ordered normed space X. If the cone X_+ is normal and the set of quasi-interior positive elements of X^* is non-empty, $Q(X^*)\neq\emptyset$, the following statements are equivalent:*

(i) $K_\rho=\overline{co}K_{S,\rho}, \forall \rho\in\mathbb{R}_+$.

(ii) *For each $x\in K$ and for each $\rho>\|x\|$, there exists a sequence (x_ν) of $K_{S,\rho}$ which converges weakly to x.*

Proof. Let $K_\rho=\overline{co}K_{S,\rho}, \forall \rho\in\mathbb{R}_+$. Let $x_0\in K$ and a constant real number $\rho>\|x_0\|$. At first we shall show that there exists $g\in Q(X^*), A\in\mathbb{R}_+$, a natural number $\nu_0\geqq\rho$ and a sequence $y_\nu\in K_{S,\nu}$ such that

$$g(y_\nu)\leqq A \qquad \forall \nu\geqq\nu_0. \tag{1}$$

To prove this we assume that there exist $g\in Q(X^*)$ and $y_0\in K$ with $g(y_0)>g(x_0)$. (If $g(x)\leqq g(x_0)$ $\forall x\in K$ then the statement (1) is true for $A=g(x_0)$). Let

$$F=\{x\in K \mid g(x)=g(y_0)\}.$$

If the set F is a face of K then $g(x)\leqq g(y_0)$ $\forall x\in K$, because $g(x_0)<g(y_0)$ and therefore the statement (1) is true for $A=g(y_0)$. If the set F is not a face of K then F is unbounded, [3, Prop. 4], hence for each $\nu\geqq\|y_0\|$ there exists $y_\nu\in K_{S,\nu}$ such that $g(y_\nu)=g(y_0)$ and therefore the statement (1) is true.

Let $g\in Q(X^*)$, $A\in\mathbb{R}_+$, $\nu_0\geqq\rho$ and $y_\nu\in K_{S,\nu}$ such that

$$g(y_\nu)\leqq A \qquad \forall \nu\geqq\nu_0.$$

For each $\nu\geqq\nu_0$ the line segment x_0y_ν cuts the set $K_{S,\rho}$ at a point $x_\nu=\lambda_\nu x_0+(1-\lambda_\nu)y_\nu$ where $\lambda_\nu\in(0,1)$. Then $x_\nu\geqq(1-\lambda_\nu)y_\nu$. Since X_+ is normal, there exists $a\in\mathbb{R}_+$ such that $a\|X_\nu\|\geqq(1-\lambda_\nu)\|y_\nu\|$ $\forall \nu\geqq\nu_0$. By definition of y_ν and x_ν we have that $\|y_\nu\|=\nu$ and $\|x_\nu\|=\rho$, hence,

$$0<1-\lambda_\nu\leqq\frac{a\rho}{\nu}, \qquad \forall \nu\geqq\nu_0.$$

We shall show that the sequence (x_ν) of $K_{S,\rho}$ converges weakly to x_0.

Let $f\in X^*$ and $\varepsilon>0$. Since $g\in Q(X^*)$, there exists $n_0\in\mathbb{N}$ and $h\in X^*$ with $-n_0g\leqq h\leqq n_0g$ and $\|f-h\|<\varepsilon$. So we have that

$$|h(y_\nu)|\leqq n_0g(y_\nu)\leqq n_0A, \qquad \forall \nu\geqq\nu_0$$

and

$$|f(y_\nu)-h(y_\nu)|\leqq\varepsilon\|y_\nu\|=\varepsilon\nu, \qquad \forall\nu\geqq\nu_0.$$

Hence

$$0\leqq(1-\lambda_\nu)|f(y_\nu)|\leqq(1-\lambda_\nu)(|f(y_\nu)-h(y_\nu)|+|h(y_\nu)|)$$

$$\leqq(1-\lambda_\nu)(\varepsilon\nu+n_0A)\leqq\frac{a\rho}{\nu}(\varepsilon\nu+n_0A).$$

This shows that $\lim_{\nu\to\infty}(1-\lambda_\nu)f(y_\nu)=0$ and therefore

$$\lim_{\nu\to\infty} f(x_\nu)=\lim_{\nu\to\infty} \lambda_\nu f(x_0)=f(x_0).$$

So the sequence (x_ν) of $K_{S,\rho}$ converges weakly to x and therefore the statement (i) implies (ii).

Also the statement (ii) implies (i) because the closure of $coK_{S,\rho}$ with respect to the weak and the strong topology of X coincide.

By Proposition 3.1 and Theorem 3.1 we have the following corollary:

Corollary 3.1. *Let K be an unbounded, closed and convex subset of the positive cone X_+ of an ordered Banach space X. If X_+ is normal, $Q(X^*)\neq\emptyset$ and the set K has the* R.N.P., *the following statements are equivalent:*

(*i*) $\mathrm{sep}(K)=\emptyset$.

(*ii*) *For each $x\in K$ and for each $\rho>\|x\|$, there exists a sequence (x_ν) of K with $\|x_\nu\|=\rho$ which converges weakly to x.*

4. Characterizations of $l_1(\Gamma)$

A Banach space X has the Schur property provided each weakly convergent sequence in X is norm convergent. It is well known that l_1 has the Schur property and hence so does $l_1(\Gamma)$ for any $\Gamma\neq\emptyset$.

Theorem 4.1. *Let X be an infinite-dimensional Banach lattice. Then:*

(*i*) *X is order-isomorphic to $l_1(\Gamma)$ iff X has the Schur property and X^* has quasi-interior positive elements.*

(*ii*) *X is order-isomorphic to l_1 iff X is separable, X has the Schur property and X^* has quasi-interior positive elements.*

Proof. It is straightforward to check that if X and $l_1(\Gamma)$ are order isomorphic under T then X has the Schur property and T^* preserves quasi-interior positive elements. Since the point $x=(x(i))_{i\in\Gamma}$ with $x(i)=1\ \forall\, i\in\Gamma$ is a quasi-interior positive element of $l_\infty(\Gamma)$, $Q(X^*)\neq\emptyset$.

Let X have the Schur property and $Q(X^*)\neq\emptyset$. To show that X is order-isomorphic to $l_1(\Gamma)$ it is enough to show that X_+ has the R.N.P. and $0\in\mathrm{sep}(X_+)$, [**4**, Prop. 4.2]. X has the R.N.P. because in Banach lattices, the Schur property implies the R.N.P., [**2**, Corollary 1]. If $0\notin\mathrm{sep}(X_+)$ then $\mathrm{sep}(X_+)=\emptyset$. By Corollary 3.1, for each $\rho>0$ there exists a sequence (x_ν) of X_+ with $\|x_\nu\|=\rho$ which converges weakly and therefore strongly to 0. This is a contradiction, hence $0\in\mathrm{sep}(X_+)$ and therefore X is order-isomorphic to $l_1(\Gamma)$. So the statement (i) is true. The statement (ii) follows by (i).

REFERENCES

1. R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property* (Lecture Notes in Mathematics **993**).

2. N. Ghoussoub-M. Talagrand, Order dentability and the Radon–Nikodým property in Banach lattices, *Math. Ann.* **243** (1979), 217–225.

3. I. A. Polyrakis, Extreme points of unbounded, closed and convex sets in Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 319–323.

4. I. A. Polyrakis, Strongly exposed points in bases for the positive cone of ordered Banach spaces and characterizations of $l_1(\Gamma)$, *Proc. Edinburgh Math. Soc.*, **29,** (1986), 271–282.

5. H. H. Schaefer, *Banach Lattices and Positive Operators* (Springer-Verlag, 1974).

National Technical University
Department of Mathematics
Zografou Campus 157 73
Athens, Greece