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A note on generalised linear complementarity problems

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Given an $n \times n$ matrix A, an n-dimensional vector q, and a closed, convex cone S of R^n, the generalized linear complementarity problem considered here is the following: find a $z \in R^n$ such that

$$Az-q \in S^* , \quad z \in S ,$$
$$\langle Az-q, z \rangle = 0 ,$$

where S^* is the polar cone of S. The existence of a solution to this problem for arbitrary vector q has been established both analytically and constructively for several classes of matrices A. In this note, a new class of matrices, denoted by J, is introduced. A is a J-matrix if

$$Az \in S^* , \quad z^TAz \leq 0 , \quad z \in S \text{ imply that } z = 0 .$$

The new class can be seen to be broader than previously studied classes. We analytically show that for any A in this class, a solution to the above problem exists for arbitrary vector q. This is achieved by using a result on variational inequalities.

1. Introduction

The generalized linear complementarity problem is to find a $z \in R^n$ satisfying

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$$\text{(1.1)} \qquad \begin{aligned} &Az-q \in S^* , \quad z \in S , \\ &\langle Az-q , z \rangle = 0 , \end{aligned}$$

where A is a given $n \times n$ matrix, q is a given n-dimensional vector, S is a closed, convex cone in R^n, and S^* is the polar cone of S.

For $S = R^n_+$, the complementarity problem (1.1) has been extensively studied in the literature. The existence of a unique solution to this problem has been shown by Dantzig, Cottle [2] for P-matrices, which include all the previously studied matrices for which there is a unique solution. Karamardian [7] has solved this problem for the class of regular matrices, and thus has enlarged the class of matrices each of which guarantees a solution (but not necessarily unique).

Habetler and Price [4] have shown that the problem (1.1) has a solution when S is a pointed, closed, convex cone with nonempty interior, A is a strictly S-copositive matrix, and either

(i) $q \in \text{int } S^*$ or

(ii) $S \subset S^*$.

Karamardian [6] has generalized this result, and shown that strict S-copositiveness of A is sufficient to ensure the existence of a solution to (1.1). In [9], the authors have shown that the linear complementarity problem defined over polyhedral cones in a complex n-space possesses a solution when A is a strictly S-copositive complex matrix.

In this note, we define a new class J of matrices A such that: if $A \in J$, then

$$Az \in S^* , \quad z^T Az \le 0 , \quad z \in S \text{ imply that } z = 0 .$$

The classes of P- and regular matrices become proper subclasses of this class when S is taken as R^n_+. It is also found that the class J includes the class of strictly S-copositive matrices, and thus becomes a broader class than previously studied ones.

We show that if $A \in J$ and there exists a vector $p \in \text{int } S^*$ such that the system $0 \ne z \in S$, $Az+p \in S^*$, $z^T(Az+p) = 0$ is not consistent, then (1.1) possesses a solution for every vector $q \in R^n$.

2. Notations and definitions

Throughout this note, R^n will denote euclidean n-space with the usual inner product $\langle x, y \rangle = y^T x$ of $x, y \in R^n$ and norm $\|x\|$ of $x \in R^n$. R^n_+ denotes the nonnegative orthant of R^n. A subset S of R^n will be called a closed, convex cone if, and only if,

(i) S is closed, and

(ii) $\alpha x + \beta y \in S$ for $\alpha, \beta \geq 0$ and $x, y \in S$.

The polar of a cone S is the cone S^* defined by

$$S^* = \{x \in R^n : \langle x, y \rangle \geq 0 \text{ for all } y \in S\} .$$

The interior of S^* is given by

$$\text{int } S^* = \{x \in S^* : \langle x, y \rangle > 0 \text{ for all } 0 \neq y \in S\} .$$

A cone is said to be pointed if whenever $x \neq 0$ is in the cone, $-x$ is not in the cone. For a closed, convex cone S, int S^* is nonempty if, and only if, S is pointed.

A square matrix A is a P-matrix if all its principal minors are positive.

For every $x \geq 0$, let $I_+(x)$ and $I_0(x)$ denote the set of indices corresponding to the positive and zero components of x; that is, $I_+(x) = \{i : x_i > 0\}$ and $I_0(x) = \{i : x_i = 0\}$. A square matrix A is said to be regular if the system

$$(Ax)_i + t = 0 \quad \text{for} \quad i \in I_+(x) ,$$

$$(Ax)_i + t \geq 0 \quad \text{for} \quad i \in I_0(x) ,$$

$$0 \neq x \geq 0 , \quad t \geq 0 ,$$

is inconsistent. Here $(Ax)_i$ denotes the ith component of the vector Ax.

A square matrix A is said to be strictly S-copositive if $x^T Ax > 0$ for all $0 \neq x \in S$.

3. Preliminary results

LEMMA 3.1. *Let* A *be an* $n \times n$ *matrix, and let* S *be a closed, convex cone in* R^n .

(a) If A *is strictly* S*-copositive, then* $A \in J$.

(b) If $S = R^n_+$ *, then* A *is in* J *whenever* A *is either*

(i) a P*-matrix or*

(ii) a regular matrix.

Proof. *(a)* It immediately follows from the definitions of J- and S-copositive matrices given above.

(b) Let A be a P-matrix. The conclusion *(b)* for P-matrices follows from the following result of Fiedler and Pták [3]: if A is a P-matrix, then for each $0 \neq x \in R^n$, there is an index i for which $x_i(Ax)_i > 0$.

To prove the second part of *(b)*, we observe that when $S = R^n_+$, the system $x \in S$, $Ax \in S^*$, $x^T Ax \leq 0$, reduces to $x \geq 0$, $Ax \geq 0$, $x^T Ax \leq 0$, the consistency of which implies that $x_i(Ax)_i = 0$ for $1 \leq i \leq n$. If $x \neq 0$, we will have $(Ax)_i = 0$ for $i \in I_+(x)$ and $(Ax)_i \geq 0$ for $i \in I_0(x)$, which is a contradiction to the regularity of A .

REMARK 3.2. It is interesting to note that the class of regular matrices is properly included in J . For example, the matrix

$$\begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix}$$ is a J-matrix, but not regular.

LEMMA 3.3. *Let* C *be a closed, convex cone in* R^n *with nonempty interior,* $d \in R^n$ *, and let* $x \in \operatorname{int} C$. *Then there is a* $\lambda_0 > 0$ *such that* $\lambda x + d \in C$ *for every* $\lambda \geq \lambda_0$.

Proof. Since $x \in \operatorname{int} C$, there is a $\delta > 0$ such that $u \in C$ whenever $\|u-x\| \leq \delta$. Consider the vector $w = x + d/\mu$ for some $\mu > 0$. Now

$\|w-x\| = \|d\|/\mu \le \delta$ if $\mu \ge \|d\|/\delta$. Taking $\lambda_0 = \|d\|/\delta$, we see that $x + d/\lambda \in C$ for every $\lambda \ge \lambda_0$. Since C is a cone, $\lambda x + d = \lambda(x+(d/\lambda))$ will be in C for all $\lambda \ge \lambda_0$.

LEMMA 3.4. *Let C be a closed, convex cone in R^n with nonempty interior, $d \in C$, and let $x \in \operatorname{int} C$. Then $x + d \in \operatorname{int} C$.*

Proof. If $x, d \in C$, then $x + d \in C$ because C is a convex cone. Further, if $0 \ne y \in C^*$, then we have $\langle d, y\rangle \ge 0$, $\langle x, y\rangle > 0$, and hence $\langle x+d, y\rangle > 0$, from which it follows that $(x+d)$ is in $\operatorname{int} C$.

We shall make use of the following results.

LEMMA 3.5 [4, Lemma 5.1, p. 227]. *Let S be a pointed, closed, convex cone in R^n , and let $p \in \operatorname{int} S^*$. Then the set*

$$V = \{x : x \in S, \langle p, x\rangle = 1\}$$

is bounded.

THEOREM 3.6. *If $F : R^n \to R^n$ is a continuous mapping on the nonempty, compact, convex set C in R^n , then there is an x^0 in C such that*

$$\langle F(x^0), x-x^0\rangle \ge 0 \quad \text{for all } x \in C .$$

REMARK 3.7. Theorem 3.6 was first stated and proved in [5]. A complex version of this result has been used by the present authors to obtain some existence theorems for nonlinear complementarity problems in complex space [8].

4. Solvability of the complementarity problem

THEOREM 4.1. *Let S be a pointed, closed, convex cone in R^n . If $A \in J$ and there exists a vector $p \in \operatorname{int} S^*$ such that the system $0 \ne z \in S$, $Az + p \in S^*$, $z^T(Az+p) = 0$ is not consistent, then for each $q \in R^n$ there is a vector z^0 satisfying $Az - q \in S^*$, $z \in S$.*

Proof. Consider the set $V = \{z : z \in S, \langle p, z\rangle = 1\}$. It is clear that V is a closed, convex set, and by Lemma 3.5, it is also bounded. Thus V is a nonempty, compact, convex set in R^n . Now applying Theorem

3.6, we get a point $\bar{z}$ in V such that

(4.1) $$\langle A\bar{z}, z-\bar{z}\rangle \geq 0 \text{ for all } z \in V .$$

It follows from (4.1) that

$$\langle A\bar{z}, \bar{z}\rangle = \min_{z \in V} \langle A\bar{z}, z\rangle ,$$

and therefore, the following set of necessary conditions [1] is satisfied:

(4.2) $$\begin{gathered} A\bar{z} + \eta p \in S^* , \ \langle A\bar{z}+\eta p, \bar{z}\rangle = 0 , \\ \bar{z} \in S , \ \langle p, \bar{z}\rangle = 1 , \ \eta \in R . \end{gathered}$$

Obviously $\bar{z} \neq 0$. Now the consistency of (4.2) for a vector $0 \neq \bar{z} \in S$ and $\eta \geq 0$ will contradict the assumptions made in the statement of the theorem. So $\eta < 0$, and thus we have a $0 \neq y = \bar{z}/\eta \in S$ satisfying $Ay - p \in S^*$. Since $p \in \text{int } S^*$, it follows from Lemma 3.4 that $Ay \in \text{int } S^*$. Now, for any given vector $q \in R^n$, Lemma 3.3 will determine a $\lambda > 0$ such that $\lambda(Ay) - q \in S^*$. Since S is a cone, $\lambda y \in S$. The proof of the theorem is then completed by writing $z^0 = \lambda y$.

Now we give the following existence theorem.

THEOREM 4.2. *Let* S *be a pointed, closed, convex cone in* R^n, *and let* p *be a vector in* $\text{int } S^*$ *such that the system* $0 \neq z \in S$, $Az + p \in S^*$, $z^T(Az+p) = 0$ *is inconsistent. Then there is a solution to* (1.1) *for each* $q \in R^n$ *if* A *is a* J*-matrix.*

Proof. Consider the function

$$F(z, t) = \begin{bmatrix} Az+t(p-q) \\ t \end{bmatrix}$$

defined over the set

$$C = \{(z, t) : z \in S, t \geq 0, \langle p, z\rangle+t = 1\} .$$

From Lemma 3.5, it is clear that C is a nonempty, compact, convex set in R^{n+1}. It is also evident that $F(z, t)$ and C satisfy the conditions of Theorem 3.6, and hence, there exists a point $(\bar{z}, \bar{t})$ in C such that

$$\langle A\bar{z}+\bar{t}(p-q), z-\bar{z}\rangle + \bar{t}(t-\bar{t}) \geq 0 \text{ for all } (z, t) \in C .$$

But this means that

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$$\langle A\bar{z}+\bar{t}(p-q),\ \bar{z}\rangle + \bar{t}\cdot\bar{t} = \min_{(z,t)\in C} \left(\langle A\bar{z}+\bar{t}(p-q),\ z\rangle+\bar{t}\cdot t\right) .$$

Now using the Kuhn-Tucker necessary conditions of optimality [1] for cone domains, we have a ζ_0 in R such that

$$A\bar{z} + \bar{t}(p-q) + \zeta_0 p \in S^* ,\quad \bar{t} + \zeta_0 \geq 0 ,$$

$$\text{(4.3)} \qquad \langle A\bar{z}+\bar{t}(p-q)+\zeta_0 p,\ \bar{z}\rangle = 0 ,\quad \bar{t}\left(\bar{t}+\zeta_0\right) = 0$$

$$\bar{z} \in S ,\quad \bar{t} \geq 0 ,\quad \langle p,\ \bar{z}\rangle + \bar{t} = 1 .$$

Suppose that $\bar{t} = 0$. Since $\langle p, \bar{z}\rangle + \bar{t} = 1$ and $p \in \operatorname{int} S^*$, $\bar{z} \neq 0$. If this is the case, then (4.3) will imply that the system

$$0 \neq \bar{z} \in S ,\quad A\bar{z} + \zeta_0 p \in S^* ,$$

$$\text{(4.4)} \qquad \langle A\bar{z}+\zeta_0 p,\ \bar{z}\rangle = 0 ,\quad \zeta_0 \geq 0 ,$$

is consistent. When $\zeta_0 = 0$, the consistency of (4.4) will contradict the assumption that A is a J-matrix. Further, when $\zeta_0 > 0$, (4.4) will yield a nonzero vector $\bar{y} = \bar{z}/\zeta_0 \in S$ satisfying $A\bar{y} + p \in S^*$, $\bar{y}^T(A\bar{y}+p) = 0$, again a contradiction. Hence, $\bar{t} > 0$, and since $\bar{t}\left(\bar{t}+\zeta_0\right) = 0$, therefore, we have $\bar{t} + \zeta_0 = 0$. Now substituting $\zeta_0 = -\bar{t}$ in (4.3), and then dividing throughout the resulting relations by $\bar{t}$, we get the desired solution.

REMARK 4.3. We do not require any other assumption for the cone S , except that it is to be pointed in the statement of Theorem 4.2. But for q , we impose the restriction that when $-q = p \in \operatorname{int} S^*$, there is no nonzero solution to (1.1). For this value of q , zero is obviously a solution. This restriction is automatically satisfied when A is strictly S-copositive, whereas if A is a regular matrix and $S = R_+^n$, (1.1) has no nonzero solution for the vector $-q = et$, $t \geq 0$, with $e^T = (1, 1, \ldots, 1)$.

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