Proceedings of the Edinburgh Mathematical Society (1990) 33, 327–332 ©

WHEN IS THE SPECTRUM OF A CONVOLUTION OPERATOR ON L^p INDEPENDENT OF p?

by BRUCE A. BARNES

(Received 14th June 1989)

In this paper conditions are given that imply that a convolution operator has the same spectrum on all of the spaces $L^p(G)$, $1 \leqq p \leqq \infty$.

1980 *Mathematics subject classification* (1985 Revision): 47A10.

Introduction

Let G be a locally compact topological group equipped with a fixed left Haar measure. For $f \in L^1(G)$ the convolution operators

$$(T_f)_p(g) = f * g \qquad (g \in L^p(G))$$

are bounded linear operators for all $p \in [1, \infty]$. In general, $\sigma((T_f)_p)$, the spectrum of the operator $(T_f)_p$, can vary with p. In this situation, where a linear operator determines a family of bounded linear operators T_p on the L^p-spaces of some measure space, it is of interest to have information concerning how $\sigma(T_p)$ varies with p; see for example the recent interesting paper [**7**] in which T. Ransford studies the general situation. In this paper we give conditions under which the convolution operators $(T_f)_p$ have the same spectrum for all p. We prove in our main result that $\sigma((T_f)_p)$ is independent of p for all $f \in L^1(G)$ exactly when G is both amenable and symmetric (G is symmetric if the Banach $*$-algebra $L^1(G)$ is symmetric). When G is abelian, this result is well-known. In the abelian case, using a theorem of Wiener, it is shown that for all $f \in L^1(G)$

$$\sigma_{L^1}(f) = \sigma((T_f)_p) \qquad (p \in [1, \infty])$$

where $\sigma_{L^1}(f)$ denotes the spectrum of f in the algebra $L^1(G)$; a proof of this for $G = \mathbb{R}^m$ is given in [**4**, Theorem 13.3, p. 342] for example.

The results in this paper are related to some of the work of Hulanicki in [**3**] and T. Pytlik in [**6**]. Hulanicki proves that if G has polynomial growth, $f = f^* \in L^1(G)$, and f is rapidly decreasing, then $\sigma_{L^1}(f) = \sigma((T_f)_2)$. T. Pytlik shows more generally that when $f = f^*$ is in certain Beurling subalgebras of $L^1(G)$, then $\sigma_{L^1}(f) = \sigma((T_f)_2)$. Making use of some of the methods of Hulanicki and Pytlik we prove more general results.

The results

Let A be an algebra. Recall that there is an operation defined on A by

$$f \circ g = f + g - fg \qquad (f, g \in A),$$

and that $f \in A$ is quasiregular in A if $\exists g \in A$ such that $f \circ g = g \circ f = 0$ [**8**, p. 16]. In fact, the set, $Q(A)$, consisting of all quasiregular elements of A is a group with operation $\circ$ and unit 0. When A has an identity 1, $f \in Q(A)$ if and only if $1 - f$ is invertible in A.

Note 1. *Let A be a Banach algebra, and let $f \to T_f$ be a continuous and faithful representation of A into $B(X)$ (with $T_1 = I$ when A has an identity 1). The property,*

$$\textit{whenever } I - T_f \textit{ is invertible in } B(X), \textit{ then } f \in Q(A), \tag{1}$$

implies that

$$\sigma_A(f) = \sigma(T_f) \qquad (f \in A).$$

The verification of Note 1 is straightforward except for one technical detail. It is necessary to show (with the assumption that (1) holds) that if T_f is invertible in $B(X)$ for some $f \in A$, then A has an identity. Assume T_f is invertible, so $0 \notin \sigma(T_f)$. Now (1) easily implies that $\sigma_A(f) \backslash \{0\} = \sigma(T_f)$. Choose U an open set such that $\sigma(T_f) \subseteq U$ and $0 \notin U$. Let γ be a cycle in U surrounding $\sigma(T_f)$ in the usual way and such that the index of every point $z \notin U$ with respect to γ is zero. Let

$$p = \frac{1}{2\pi i} \int_\gamma (\lambda 1 - f)^{-1} d\lambda.$$

This integral makes sense in A with identity adjoined if necessary. For all λ on the image of γ,

$$(\lambda 1 - f)^{-1} = \lambda^{-1}(1 - \lambda^{-1} f)^{-1} = \lambda^{-1}(1 - g_\lambda)$$

where g_λ is the quasi-inverse of $\lambda^{-1} f$ in A. Then for all λ on the image of γ,

$$(\lambda 1 - f)^{-1} - \lambda^{-1} 1 = -\lambda^{-1} g_\lambda \in A,$$

and it follows that

$$p = \frac{1}{2\pi i} \int_\gamma ((\lambda 1 - f)^{-1} - \lambda^{-1} 1) \, d\lambda \in A.$$

Now

https://doi.org/10.1017/S0013091500018241 Published online by Cambridge University Press

$$T_p = \frac{1}{2\pi i}\int_\gamma (\lambda I - T_f)^{-1}\, d\lambda = I.$$

Since $g \to T_g$ is $1-1$ on A, we have that p is an identity for A.

Note 2. *Assume* $f \in L^1(G)$. *Then*

$$\sigma_{L^1}(f) = \sigma((T_f)_1) = \sigma((T_f)_\infty).$$

Proof. First we verify that $\sigma_{L^1}(f) = \sigma((T_f)_1)$. By Note 1 it suffices to show that if $I-(T_f)_1$ is invertible in $B(L^1)$, then $f \in Q(L^1)$. Since $f \in L^1$, $\exists g \in L^1$ such that $(I-(T_f)_1)(g) = -f$. But this means $g - f * g = -f$. Using this last equality, a straightforward computation shows that $(I-(T_f)_1)(f * g - g * f) = 0$, so $f * g = g * f$. Therefore $g \circ f = f \circ g = f + g - f * g = 0$, so $f \in Q(L^1)$.

Now for $f \in L^1(G)$, the function

$$\check{f}(x) = f(x^{-1})\Delta(x^{-1}) \ (x \in G)$$

is in $L^1(G)$ (here Δ is the modular function of G). For $k \in L^1$ and $h \in L^\infty$ let $\langle k, h\rangle = \int_G kh\, dx$. Then for $f \in L^1$, $g \in L^1$, $h \in L^\infty$,

$$\langle (T_f)_1(g), h\rangle = \langle g, (T_{\check{f}})_\infty(h)\rangle.$$

Therefore $(T_{\check{f}})_\infty$ is the conjugate operator of $(T_f)_1$, and so $\sigma((T_f)_1) = \sigma((T_{\check{f}})_\infty)$. The relation $(f * g)\check{} = \check{g} * \check{f}$ for $f, g \in L^1$ implies $\sigma_{L^1}(f) = \sigma_{L^1}(\check{f})$. We conclude that

$$\sigma((T_f)_\infty) = \sigma((T_{\check{f}})_1) = \sigma_{L^1}(\check{f}) = \sigma_{L^1}(f).$$

If A is a Banach algebra, let $r_A(f)$ denote the spectral radius of an element $f \in A$.

A key ingredient in the proof of our results is a theorem of A. Hulanicki [3, Prop. 2.5]. We need a slightly extended form of this theorem given as follows.

Hulanicki's Theorem. *Assume A is a Banach $*$-algebra and S is a $*$-subalgebra of A. Let $f \to T_f$ be a faithful $*$-representation of A on a Hilbert space H, and assume that*

$$r_A(f) = \|T_f\| \qquad (f = f^* \in S).$$

Then

$$\sigma_A(f) = \sigma(T_f) \qquad (f \in S).$$

Proof. The conclusion in [3, Prop. 2.5] is that $\sigma_A(f) = \sigma(T_f)$ whenever $f = f^* \in S$. To extend this result to an arbitrary element $f \in S$, by Note 1 it suffices to prove that when $I - T_f$ is invertible on H, then $f \in Q(A)$. Assuming $I - T_f$ is invertible, then $(I - T_f)^* = I - T_{f^*}$ is also invertible. Therefore $I - T_g$ and $I - T_h$ are invertible where $g = f + f^* - ff^*$ and $h = f + f^* - f^*f$. By [3, Prop. 2.5] both g and h are quasiregular in A. Now

$g=f\circ f^*$ and $h=f^*\circ f$. Since $Q(A)$ is a group, and $f\circ f^*$ and $f^*\circ f\in Q(A)$, f is both left and right quasiregular in A. Therefore $f\in Q(A)$.

A second major ingredient needed to prove our results is some information concerning the continuity of certain linear maps on $L^p(G)$ when G is amenable. The ideas involved are due to C. Herz; we use [**5**, Prop. 18.18].

Let $\mathcal{K}(G)$ denote the space of all continuous complex-valued functions on G with compact support.

Proposition 3. *Assume G is amenable. Fix p, $1<p<\infty$, and suppose either $p\leqq q\leqq 2$ or $2\leqq q\leqq p$. For any $f\in L^1(G)$*

$$\sigma((T_f)_q)\subseteq\sigma((T_f)_p).$$

Proof. It suffices to show that when $I-(T_f)_p$ is invertible in $B(L^p)$, then $I-(T_f)_q$ is invertible in $B(L^q)$.

First assume $f\in L^1\cap L^\infty$. If $I-(T_f)_p$ is invertible, there exists $g\in L^p$ such that $(1-f)*g=-f$. Therefore for any $h\in\mathcal{K}(G)$, $(1-f)*(h-g*h)=h$.

Let $R=(I-(T_f)_p)^{-1}\in B(L^p)$. We have $(I-(T_f)_p)[R(h)-(h-g*h)]=0$ for all $h\in\mathcal{K}(G)$. Thus

$$(R-I)(h)=g*h \qquad (h\in\mathcal{K}(G)).$$

As G is amenable, the argument in [**5**, Proposition 18.18] applies, from which we conclude that for $K=\|R-I\|$

$$\|(R-I)(h)\|_q\leqq K\|h\|_q \qquad (h\in\mathcal{K}(G)).$$

Thus, for $M=K+1$,

$$\|R(h)\|_q\leqq M\|h\|_q \qquad (h\in\mathcal{K}(G)). \tag{$*$}$$

Let $h\in L^p\cap L^q$, and choose $\{h_n\}\subseteq\mathcal{K}(G)$ such that $\|h_n-h\|_p\to 0$ and $\|h_n-h\|_q\to 0$. Then $\|R(h_n)-R(h)\|_p\to 0$, and $R(h_n)$ converges to some L_q function by ($*$). Therefore $R(h)\in L^q$, and

$$\|R(h)\|_q\leqq M\|h\|_q \qquad (h\in L^p\cap L^q). \tag{$**$}$$

This inequality proves that $R(L^p\cap L^q)\subseteq L^q$, so $R(L^p\cap L^q)\subseteq L^p\cap L^q$. From this it follows that $(I-(T_f)_q)(L^p\cap L^q)=(I-(T_f)_p)(L^p\cap L^q)=L^p\cap L^q$. Also, ($**$) implies that

$$\|h\|_q\leqq M\|(I-(T_f)_q)(h)\|_q \qquad (h\in L^p\cap L^q).$$

This inequality extends to all $h\in L^q$, so $I-(T_f)_q$ is one-to-one and has closed range on L^q. But $L^p\cap L^q$ is in the range of $I-(T_f)_q$, so in fact, $I-(T_f)_q$ maps L^q onto L^q.

Now in the general case choose $\{f_n\} \subseteq L^1 \cap L^\infty$ such that $\|f-f_n\|_1 \to 0$. We may assume that $(I-(T_{f_n})_p)^{-1}$ exists for $n \geqq 1$ and $R_n = (I-(T_{f_n})_p)^{-1}$ has norm $\|R_n\| \leqq \|R\|+1$, $n \geqq 1$. As argued above, for each $n \geqq 1$, $I-(T_{f_n})_q$ is invertible and, furthermore, for $n \geqq 1$

$$\|(I-(T_{f_n})_q)^{-1}\| \leqq (\|R\|+2). \tag{$\#$}$$

Now $I-(T_{f_n})_q \to I-(T_f)_q$, so by ($\#$) and [**8**, Theorem (1.4.7)] $I-(T_f)_q$ is invertible.

Our first result follows by applying Hulanicki's Theorem and Proposition 3.

Theorem 4. *Assume G is amenable. Let S be a $*$-subalgebra of $L^1(G)$ with the property that $r_{L^1}(f) = r((T_f)_2)$ whenever $f = f^* \in S$. Then $\sigma_{L^1}(f) = \sigma((T_f)_p)$ for all $f \in S$ and all $p \in [1, \infty]$.*

Proof. Applying Hulanicki's Theorem, we have

$$\sigma_{L^1}(f) = \sigma((T_f)_2) \ (f \in S).$$

Now fix p, $1 < p < \infty$. We may assume $2 \leqq p$. By Proposition 3

$$\sigma_{L^1}(f) = \sigma((T_f)_2) \subseteq \sigma((T_f)_p) \ (f \in S).$$

Thus, $\sigma_{L^1}(f) = \sigma((T_f)_p)$ for all $f \in S$. When $p = 1, \infty$, then $\sigma_{L^1}(f) = \sigma((T_f)_p)$ as verified in Note 2.

In [**6**], T. Pytlik, using Hulanicki's Theorem, proves that certain convolution algebras, $L^1(G,\omega) \subseteq L^1(G)$, where ω is a weight function, have the property that $\sigma_{L^1}(f) = \sigma((T_f)_2)$ whenever $f = f^* \in L^1(G,\omega)$. Applying Theorem 4 yields a stronger result.

Corollary 5. *Let G be a locally compact group with polynomial growth. If $f \in L^1(G,\omega)$ for some polynomial weight ω, then*

$$\sigma_{L^1}(f) = \sigma((T_f)_p) \quad \text{for all} \quad p \in [1, \infty].$$

The main result of this paper is the following characterization of when $\sigma((T_f)_p)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(G)$.

Theorem 6. *The following are equivalent:*

(1) *G is amenable and symmetric;*

(2) *For all $f \in L^1(G)$, $\sigma((T_f)_p)$ is independent of $p \in [1, \infty]$;*

(3) *For all $f \in L^1(G)$, $\sigma_{L^1}(f) = \sigma((T_f)_p)$ for all $p \in [1, \infty]$;*

(4) *For all $f \in L^1(G)$ with $f = f^*$, $\sigma_{L^1}(f) = \sigma((T_f)_2)$;*

(5) *For all $f \in L^1(G)$ with $f = f^*$, $r_{L^1}(f) = r((T_f)_2)$.*

Proof. That (2)$\Leftrightarrow$(3) follows from Note 2, and the implications (3)$\Rightarrow$(4)$\Rightarrow$(5) are obvious.

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Let γ denote the largest C^*-norm on $L^1(G)$. The following two results hold:

$$\gamma(f)=\|(T_f)_2\| \text{ for all } f=f^*\in L^1(G) \text{ if and only if } G \text{ is amenable;} \tag{A}$$

$$\gamma(f)=r_{L^1}(f) \text{ for all } f=f^*\in L^1(G) \text{ if and only if } G \text{ is symmetric.} \tag{S}$$

(A) follows from [**5**, Theorem 8.9(i), p. 80], and (S) results from [**2**, Theorem 11, p. 227; and Theorem 5, p. 226]. Therefore if G is amenable and symmetric, then combining (A) and (S) we have that (5) holds.

Assume that (5) holds. By Hulanicki's Theorem (4) holds. This implies that for all $f=f^*\in L^1(G)$, $\sigma_{L^1}(f)\subseteq\mathbb{R}$. By Shirali's Theorem [**2**, Theorem 5, p. 226] it follows that $L^1(G)$ is symmetric. Therefore [**2**, Corollary 8, p. 227] shows that $\gamma(f)=r_{L^1}(f^* * f)^{1/2}$ for all $f\in L^1(G)$. Combining this with (5), we have for all $f=f^*\in L^1(G)$

$$\gamma(f)=r_{L^1}(f)=r((T_f)_2)=\|(T_f)_2\|.$$

Thus, by (A), G is amenable. This completes the proof that (1)⇔(5).

Finally, assume (1) holds. Then G is amenable, and as noted above, (5) is true. By Theorem 4 $\sigma_{L^1}(f)=\sigma((T_f)_p)$ for all $f\in L^1(G)$ and all $p\in[1,\infty]$.

It is known that when G is amenable, the $\sigma_{L^1}(f)=\sigma_0((T_f)_p)$ for all $f\in L^1(G)$ and all $p\in[1,\infty]$ [**1**, Theorem 3.4] (here σ_0 (T) denotes the order spectrum of T). Therefore when G is both amenable and symmetric, applying Theorem 6 we have $\sigma_0((T_f)_p)=\sigma((T_f)_p)$ for all $f\in L^1(G)$ and all $p\in[1,\infty]$.

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Department of Mathematics
University of Oregon
Eugene, Oregon 97403
U.S.A.